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Chapter 63

Structural alterations

One of the daunting things about stochastic calculus is the amount of preliminary material required before one can approach the main theorems, let alone the interesting applications. Before proceeding to Chapter 64 we must do some more work more or less at the level of Chapters 61-62, and this is what I will try to deal with in the present chapter.

I start with a quick run through the properties of near-simple processes (§631) which got pushed out of Chapter 61 due to shortage of space. The real work of the chapter begins in §632, with 'right-continuous' filtrations $\langle\mathfrak{A}_t\rangle_{t\in T}$. For such stochastic integration structures, which include the most important examples (632D), there are useful simplifications in the theory (632C, 632F, 632I, 632J), so it is not surprising that most presentations of this material take right-continuity of the filtration as a standard hypothesis.

The integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$, as I have defined it, depends on an elaborate structure: a probability algebra $(\mathfrak{A},\bar\mu)$, a filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ and the sublattice $\mathcal{S}$. We anticipate that changing any of these will change the value of the integral. But there are many cases in which this doesn't happen. The simplest of these is 'change of law'. As long as we have a strictly positive countably additive measure on the given algebra $\mathfrak{A}$, we shall have the same integrals, as I pointed out right at the beginning in 613I. Next, there are important classes of pairs $\mathcal{S}'$, $\mathcal{S}$ of lattices for which we can expect equality of the integrals $\int_{\mathcal{S}'}$ and $\int_{\mathcal{S}}$. For these we have to work fairly hard, since it is certainly not enough just to have $\min\mathcal{S}'=\min\mathcal{S}$ and $\max\mathcal{S}'=\max\mathcal{S}$. In §633 I explore sufficient conditions to make a sublattice $\mathcal{S}'$ of $\mathcal{S}$ behave as if it had full outer (Riemann) measure, so that an integral over $\mathcal{S}$ will be the same when taken over $\mathcal{S}'$ (633K).

We are now in a position to look at the effect of replacing $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\in T})$ with $(\mathfrak{B},\bar\mu\restriction\mathfrak{B},\langle\mathfrak{B}\cap\mathfrak{A}_t\rangle_{t\in T})$ where $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. As long as we are just looking at integrals, we need only quote from §633; but if we want to understand martingales (634I), we need the theory of relative independence from Chapter 48. And there is a yet more radical change which we can consider, where the filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ is replaced by $\langle\mathfrak{A}_{\pi_r}\rangle_{r\in R}$ for some family $\langle\pi_r\rangle_{r\in R}$ of stopping times. This is what I do in §635.

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631 Near-simple processes

My presentation so far has focused on 'moderately oscillatory' integrands, with regular mentions of 'simple' processes, and an excursion into 'jump-free' processes in §§618-619. Later on, however, there will be many important results applying to an intermediate class, the 'near-simple' processes.

631B Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) A fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is **near-simple** if it is in the closure of $M_{\mathrm{simp}}(\mathcal{S})$ for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$; that is, it is order-bounded and for every $\epsilon>0$ there is a simple process $\boldsymbol{v}$ with domain $\mathcal{S}$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)\le\epsilon$.

(b) A fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is **locally near-simple** if $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is near-simple for every $\tau\in\mathcal{S}$.

631C Proposition (a) (Locally) near-simple processes are (locally) moderately oscillatory.

(b) (Locally) jump-free processes are (locally) near-simple.

631D Where near-simple processes come from: Theorem Let (Ω,Σ,μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\ge 0}$ a filtration of σ-subalgebras of Σ, all containing every negligible subset of Ω. Suppose that we are given a family $\langle X_t\rangle_{t\ge 0}$ of real-valued functions on Ω such that X_t is Σ_t-measurable for every t and $t\mapsto X_t(\omega):[0,\infty[\,\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$.

In this case, $\langle X_t\rangle_{t\ge 0}$ is progressively measurable, and if $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge 0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ and $\langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ are defined as in 612H, then $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ is locally near-simple.

631E Proposition (a) If $T=[0,\infty[$, then the identity process on $\mathcal{T}_f$ is locally near-simple.

(b) Brownian motion is locally near-simple.

(c) The Poisson process is locally near-simple.

631F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) Write $M_{\text{n-s}}$ for the set of near-simple processes with domain $\mathcal{S}$.

(i) If $h:\mathbb{R}\to\mathbb{R}$ is continuous, then $\bar h\boldsymbol{u}\in M_{\text{n-s}}$ for every $\boldsymbol{u}\in M_{\text{n-s}}$.

(ii) $M_{\text{n-s}}$ is an f-subalgebra of $M_{\text{o-b}}=M_{\text{o-b}}(\mathcal{S})$.

(iii) $M_{\text{n-s}}$ is complete for the ucp uniformity.

(iv) If $\tau\in\mathcal{S}$ and $\boldsymbol{u}\in M_{\text{fa}}(\mathcal{S})$, then $\boldsymbol{u}$ is near-simple iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are both near-simple.

(v) If $\boldsymbol{u}\in M_{\text{n-s}}$ and $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$, then $z\boldsymbol{u}$ belongs to $M_{\text{n-s}}$.

(vi) If $\boldsymbol{u}$ is near-simple it is locally near-simple.

(b) Write $M_{\text{ln-s}}$ for the set of locally near-simple processes with domain $\mathcal{S}$.

(i) If $h:\mathbb{R}\to\mathbb{R}$ is continuous, then $\bar h\boldsymbol{u}\in M_{\text{ln-s}}$ for every $\boldsymbol{u}\in M_{\text{ln-s}}$.

(ii) $M_{\text{ln-s}}$ is an f-subalgebra of the space $M_{\text{lo-b}}=M_{\text{lo-b}}(\mathcal{S})$ of locally order-bounded processes with domain $\mathcal{S}$.

(iii) If $\boldsymbol{u}\in M_{\text{fa}}(\mathcal{S})$ and $\tau\in\mathcal{S}$, then $\boldsymbol{u}$ is locally near-simple iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are both locally near-simple.

(iv) If $\boldsymbol{u}\in M_{\text{ln-s}}$ and $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$, then $z\boldsymbol{u}\in M_{\text{ln-s}}$.

(v) If $\boldsymbol{u}\in M_{\text{fa}}(\mathcal{S})$ and $\{\sigma:\sigma\in\mathcal{S},\,\boldsymbol{u}\restriction\mathcal{S}\wedge\sigma$ is near-simple$\}$ covers $\mathcal{S}$, then $\boldsymbol{u}\in M_{\text{ln-s}}$.

(c) Suppose that $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$.

(i) If $\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']$ is near-simple whenever $\tau\le\tau'$ in $\mathcal{S}$, then $\boldsymbol{u}$ is near-simple.

(ii) If $\boldsymbol{u}$ is locally near-simple it is near-simple.

631G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\hat{\mathcal{S}}$ its covered envelope, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process, and $\hat{\boldsymbol{u}}$ its fully adapted extension to $\hat{\mathcal{S}}$.

(a) $\boldsymbol{u}$ is near-simple iff $\hat{\boldsymbol{u}}$ is near-simple.

(b) $\boldsymbol{u}$ is locally near-simple iff $\hat{\boldsymbol{u}}$ is locally near-simple.

631H Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a process of bounded variation with domain $\mathcal{S}$.

(a)(i) $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined for every $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$.

(ii) $\boldsymbol{v}\mapsto\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}:M_{\text{n-s}}(\mathcal{S})\to L^0$ is continuous for the ucp topology on $M_{\text{n-s}}(\mathcal{S})$ and the topology of convergence in measure on L^0.

(b)(i) The indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple for every $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$.

(ii) $\boldsymbol{v}\mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}):M_{\text{n-s}}(\mathcal{S})\to M_{\text{n-s}}(\mathcal{S})$ is continuous for the ucp topology.

631I Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a moderately oscillatory process and $\boldsymbol{v}$ a near-simple integrator, both with domain $\mathcal{S}$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple.

631J Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) If $\boldsymbol{v}$, $\boldsymbol{w}$ are near-simple integrators with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ and $\boldsymbol{v}^*$ are near-simple.

(b) If $\boldsymbol{v}$, $\boldsymbol{w}$ are locally near-simple local integrators with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ and $\boldsymbol{v}^*$ are locally near-simple.

631K Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a near-simple process of bounded variation. Then its cumulative variation $\boldsymbol{v}^\uparrow$ is near-simple.

631L Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$. Then $\boldsymbol{v}$ is near-simple and of bounded variation iff it is expressible as the difference of two non-negative non-decreasing near-simple processes.

631M Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which is coinitial with $\mathcal{S}$.

(a)(i) There is a unique function $\Phi : M_{\mathrm{simp}}(\mathcal{S}') \to M_{\mathrm{simp}}(\mathcal{S})$ such that, for every $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$, $\Phi(\boldsymbol{u})$ extends $\boldsymbol{u}$ and has a breakpoint string in $\mathcal{S}'$.

(ii) $\Phi(\bar{h}\boldsymbol{u}) = \bar{h}\Phi(\boldsymbol{u})$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$ and every $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$.

(iii) Φ is a multiplicative Riesz homomorphism.

(iv) If $\mathcal{S}' \neq \emptyset$ then $\boldsymbol{u}$ and $\Phi(\boldsymbol{u})$ have the same starting value for every $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$.

(v) $\sup|\Phi(\boldsymbol{u})| = \sup|\boldsymbol{u}|$ for every $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$.

(b)(i) There is a unique function $\Psi : M_{\text{n-s}}(\mathcal{S}') \to M_{\text{n-s}}(\mathcal{S})$ such that Ψ extends Φ and is continuous with respect to the ucp topologies on $M_{\text{n-s}}(\mathcal{S}')$ and $M_{\text{n-s}}(\mathcal{S})$.

(ii) $\Psi(\boldsymbol{u})\restriction\mathcal{S}' = \boldsymbol{u}$ and $\sup|\Psi(\boldsymbol{u})| = \sup|\boldsymbol{u}|$ for every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$.

(iii) Ψ is a multiplicative Riesz homomorphism and $\Psi(\bar{h}\boldsymbol{u}) = \bar{h}\Psi(\boldsymbol{u})$ for every continuous $h : \mathbb{R} \to \mathbb{R}$ and every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$.

(iv) For $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$, $\Psi(\boldsymbol{u})$ is non-decreasing iff $\boldsymbol{u}$ is non-decreasing.

(v) For $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$, $\sup|\Psi(\boldsymbol{u})| = \sup|\boldsymbol{u}|$, so $[\![\Psi(\boldsymbol{u}) \neq \mathbf{0}]\!] = [\![\boldsymbol{u} \neq \mathbf{0}]\!]$.

(c) Now suppose that $\sup_{\tau\in\mathcal{S}'}[\![\sigma \le \tau]\!] = 1$ for every $\sigma \in \mathcal{S}$.

(i) If $\boldsymbol{v}$ is an integrator with domain $\mathcal{S}$, then $\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}}\Psi(\boldsymbol{u})\,d\boldsymbol{v}$ for every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$.

(ii) There is a unique function $\Psi^* : M_{\text{ln-s}}(\mathcal{S}') \to M_{\text{ln-s}}(\mathcal{S})$ extending the map $\Phi : M_{\mathrm{simp}}(\mathcal{S}') \to M_{\mathrm{simp}}(\mathcal{S})$ and such that $\sup|\Psi^*(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau| = \sup|\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau|$ whenever $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S})$ and $\tau \in \mathcal{S}'$.

(iii) $\Psi^*(\boldsymbol{u})\restriction\mathcal{S}' = \boldsymbol{u}$ for every $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S}')$, Ψ^* is a multiplicative Riesz homomorphism, $\Psi^*(\bar{h}\boldsymbol{u}) = \bar{h}\Psi^*(\boldsymbol{u})$ for every continuous $h : \mathbb{R} \to \mathbb{R}$ and every $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S}')$, and $\Psi^*(\boldsymbol{u})$ is non-decreasing whenever $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S}')$ is non-decreasing.

631N Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally near-simple process, and $A \subseteq \mathcal{S}$ a non-empty set such that $\inf A \in \mathcal{S}$. If $u_\rho = 0$ for every $\rho \in A$, then $u_{\inf A} = 0$.

631O Witnessing sequences Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with greatest and least elements, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) $\mathrm{SL}_1(\boldsymbol{u})$ is the statement

for every $\delta > 0$ there is a non-decreasing sequence $\langle\tau_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{S}$ such that

(α) $\tau_0 = \min\mathcal{S}$,

(β) $[\![|u_{\tau_{i+1}} - u_{\tau_i}| < \delta]\!] \subseteq [\![\tau_{i+1} = \max\mathcal{S}]\!]$ for every $i \in \mathbb{N}$,

(γ) $\inf_{i\in\mathbb{N}}[\![\tau_i < \max\mathcal{S}]\!] = 0$,

(δ) $[\![\sigma < \tau_{i+1}]\!] \subseteq [\![|u_\sigma - u_{\tau_i}| < \delta]\!]$ whenever $i \in \mathbb{N}$ and $\sigma \in \mathcal{S}\cap[\tau_i, \tau_{i+1}]$.

(b) $\mathrm{SL}_2(\boldsymbol{u})$ is the statement

for every $\delta > 0$ there is a non-decreasing sequence $\langle\tau_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{S}$ such that

(α) $\tau_0 = \min\mathcal{S}$,

(β) $[\![|u_{\tau_{i+1}} - u_{\tau_i}| < \delta]\!] \subseteq [\![\tau_{i+1} = \max\mathcal{S}]\!]$ for every $i \in \mathbb{N}$,

(γ) $\inf_{i\in\mathbb{N}}[\![\tau_i < \max\mathcal{S}]\!] = 0$,

(δ) $[\![\sigma < \tau_{i+1}]\!] \subseteq [\![|u_\sigma - u_{\tau_i}| < \delta]\!]$ whenever $i \in \mathbb{N}$ and $\sigma \in \mathcal{S}\cap[\tau_i, \tau_{i+1}]$,

(ϵ) $|u_{\tau_{i+1}} - u_{\tau_i}| \le \delta$ for every $i \in \mathbb{N}$.

631P Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with greatest and least elements and $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\mathrm{SL}_1(\boldsymbol{u})$ is true, then $\boldsymbol{u}$ is near-simple.

(b) If $\mathrm{SL}_2(\boldsymbol{u})$ is true, then $\boldsymbol{u}$ is jump-free.

631Q Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with a greatest member such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$, and $\boldsymbol{u}$ a near-simple process with domain $\mathcal{S}$. Take $\delta > 0$ and construct $\langle D_i\rangle_{i\in\mathbb{N}}$ and $\langle y_i\rangle_{i\in\mathbb{N}}$ from $\boldsymbol{u}$ and δ as in 615M. Then $\inf D_i \in D_i$ and $u_{\inf D_i} = y_i$ for every $i \in \mathbb{N}$.

631R Stopping Lemmas: Theorem Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with greatest and least members such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$, and $\boldsymbol{u}$ a moderately oscillatory process with domain $\mathcal{S}$.

(a) $\boldsymbol{u}$ is near-simple iff $\mathrm{SL}_1(\boldsymbol{u})$ is true.

(b) $\boldsymbol{u}$ is jump-free iff $\mathrm{SL}_2(\boldsymbol{u})$ is true.

631S Proposition Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with greatest and least members and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process. If $\inf A \in \mathcal{S}$ and $u_{\inf A} = \lim_{\sigma\downarrow A} u_\sigma$ for every non-empty downwards-directed $A \subseteq \mathcal{S}$, then $\boldsymbol{u}$ is near-simple.

631T Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and C the set of non-negative non-decreasing order-bounded near-simple processes with domain $\mathcal{S}$. If $\boldsymbol{u}$ belongs to C and $\mathrm{Osclln}(\boldsymbol{u}) \neq 0$, there is a non-zero simple process $\boldsymbol{v} \in C$ such that $\boldsymbol{u} - \boldsymbol{v} \in C$.

631U Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a non-negative, order-bounded, non-decreasing near-simple process with domain $\mathcal{S}$. Then for any $\epsilon > 0$ there are non-negative, non-decreasing processes $\boldsymbol{v}$, $\boldsymbol{w}$ with domain $\mathcal{S}$ such that $\boldsymbol{v}$ is simple, $\boldsymbol{w}$ is jump-free, $\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{w}$ is non-negative and non-decreasing, and $\theta(\sup|\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{w}|) \le \epsilon$.

631V Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a near-simple process of bounded variation with domain $\mathcal{S}$. Then for any $\epsilon > 0$ there are processes $\boldsymbol{v}$, $\boldsymbol{w}$ with domain $\mathcal{S}$ such that $\boldsymbol{v}$ is simple, $\boldsymbol{w}$ is jump-free and of bounded variation and $\theta(\int_{\mathcal{S}} |d(\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{w})|) \le \epsilon$.

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632 Right-continuous filtrations

Up to this point, we have been able (with some effort) to work in the full generality of stochastic integration structures $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T})$ as described in §§611-613. We are now approaching territory in which we shall need to have filtrations which are 'right-continuous' in the sense of 632B. These include the standard examples (632D). The results I present here are a quick run through new features of the structures developed in §§611-612 (632C) and an important characterization of near-simple processes on infimum-closed full sublattices (632F). With this in hand, we see that in the most familiar contexts local martingales will be locally near-simple (632I) and we have a useful test for being a martingale (632J). In 632N I describe a classic example of a local martingale.

632B Definition I will say that $\langle\mathfrak{A}_t\rangle_{t\in T}$ or $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T})$ or $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is **right-continuous** if $\mathfrak{A}_t = \bigcap_{s>t} \mathfrak{A}_s$ whenever $t \in T$ is not isolated on the right.

If P and Q are partially ordered sets, an order-preserving function $f : P \to Q$ is **right-continuous** if $\inf f[C]$ is defined and equal to $f(\inf C)$ whenever $C \subseteq P$ is non-empty and downwards-directed and has an infimum in P.

632C Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous.

(a) Suppose that $C \subseteq \mathcal{T}$ is non-empty.

(i)

$$\begin{aligned}[\![\inf C > t]\!] &= \inf_{\tau\in C} [\![\tau > t]\!] \text{ if } t \text{ is isolated on the right,}\\ &= \sup_{s>t} \inf_{\tau\in C} [\![\tau > s]\!] \text{ otherwise.}\end{aligned}$$

(ii) $[\![\inf C < \tau]\!] = \sup_{\upsilon\in C} [\![\upsilon < \tau]\!]$ for every $\tau \in \mathcal{T}$.

(iii) $\mathfrak{A}_{\inf C} = \bigcap_{\tau\in C} \mathfrak{A}_\tau$.

(b) If C, $D \subseteq \mathcal{T}$ are non-empty, then $\inf C \vee \inf D = \inf\{\sigma \vee \tau : \sigma \in C, \tau \in D\}$.

(c) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process and $A \subseteq \mathcal{S}$ is a non-empty downwards-directed set such that $u = \lim_{\sigma\downarrow A} u_\sigma$ is defined in L^0, then $u \in L^0(\mathfrak{A}_{\inf A})$.

632D Examples (a) In the construction of Brownian motion in 612T, $\langle\mathfrak{C}_t\rangle_{t\ge 0}$ is right-continuous.

(b) In the construction of the standard Poisson process in 612U, $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is right-continuous.

632E Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally near-simple process, and $A\subseteq\mathcal{S}$ a non-empty downwards-directed set such that $\tau=\inf A$ belongs to $\mathcal{S}$. Then $u_\tau=\lim_{\sigma\downarrow A}u_\sigma$, and in fact for every $\epsilon>0$ there is a $\sigma\in A$ such that $\theta(\sup_{\rho\in\mathcal{S}\cap[\tau,\sigma]}|u_\rho-u_\tau|)\le\epsilon$.

632F Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ whenever $A\subseteq\mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process, then $\boldsymbol{u}$ is locally near-simple iff it is locally moderately oscillatory and

$$(\dagger)\ u_{\inf A}=\lim_{\sigma\downarrow A}u_\sigma \text{ for every non-empty downwards-directed } A\subseteq\mathcal{S}$$

with a lower bound in $\mathcal{S}$.

632G Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ for every non-empty $A\subseteq\mathcal{S}$ with a lower bound in $\mathcal{S}$, and $a\in\mathfrak{A}$. Set $u_\sigma=\chi(\mathrm{upr}(a,\mathfrak{A}_\sigma))$ for $\sigma\in\mathcal{S}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. Then $\boldsymbol{u}$ is near-simple.

632H Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ for every non-empty $A\subseteq\mathcal{S}$ with a lower bound in $\mathcal{S}$, and $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}}$ a locally jump-free non-decreasing process. Then $\boldsymbol{v}:\mathcal{S}\to L^0$ is an order-continuous lattice homomorphism.

632I Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ for every non-empty $A\subseteq\mathcal{S}$. Then a virtually local martingale with domain $\mathcal{S}$ is a locally near-simple local martingale.

632J Where martingales come from: Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\langle u_t\rangle_{t\in T}$ be a martingale in the sense that $u_t\in L^1_{\bar\mu}$ and u_s is the conditional expectation of u_t on $\mathfrak{A}_s$ whenever $s\le t$ in T. If $\boldsymbol{w}=\langle w_\tau\rangle_{\tau\in\mathcal{T}_b}$ is a locally near-simple process such that $w_{\check t}=u_t$ for every $t\in T$, then $\boldsymbol{w}$ is a martingale.

632K Lemma Let (Ω,Σ,μ) be a probability space with measure algebra $(\mathfrak{A},\bar\mu)$, and $\langle\Sigma_t\rangle_{t\ge 0}$ a right-continuous filtration of σ-subalgebras of Σ. If we set $\mathfrak{A}_t=\{E^\bullet:E\in\Sigma_t\}$ for $t\ge 0$, then $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is a right-continuous filtration.

632L Proposition Let (Ω,Σ,μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\in[0,\infty[}$ a right-continuous filtration of σ-subalgebras of Σ, all containing every negligible subset of Ω. Suppose that we are given a family $\langle X_t\rangle_{t\ge 0}$ of measurable real-valued functions on Ω such that $t\mapsto X_t(\omega)$ is càdlàg for every ω and X_s is a conditional expectation of X_t on Σ_s whenever $0\le s\le t$. Define $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\ge 0})$ and $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\mathcal{T}_f}$ as in 612H and 631D. Then $\boldsymbol{u}\restriction\mathcal{T}_b$ is a martingale and $\boldsymbol{u}$ is a local martingale.

632M Proposition Let $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ be the Poisson process.
(a) If $\boldsymbol{\iota}$ is the identity process, $(\boldsymbol{v}-\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale, so that $\boldsymbol{v}-\boldsymbol{\iota}$ is a local martingale.
(b) The previsible variation of $\boldsymbol{v}\restriction\mathcal{T}_b$ is $\boldsymbol{\iota}\restriction\mathcal{T}_b$.

***632N Example (a)** Let $\mu=\mu_W$ be three-dimensional Wiener measure on $\Omega=C([0,\infty[\,;\mathbb{R}^3)_0$, Σ its domain, and $(\mathfrak{A},\bar\mu)$ its measure algebra. For $t\ge 0$ set

$$\Sigma'_t=\{F:F\in\Sigma,\ \omega'\in F \text{ whenever } \omega\in F,\ \omega'\in\Omega \text{ and } \omega'\restriction[0,t]=\omega\restriction[0,t]\},$$

$$\Sigma_t=\{F\triangle H:F\in\Sigma'_t,\ \mu H=0\},$$

$$\mathfrak{A}_t=\{F^\bullet:F\in\Sigma_t\}.$$

$\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is right-continuous.

(b) Let e be a unit vector in $\mathbb{R}^3$. Set $\Omega' = \{\omega : \omega \in \Omega,\ e \text{ is not a value of } \omega\}$. For $t \geq 0$ and $\omega \in \Omega$ set

$$\begin{aligned} Y_t(\omega) &= \tfrac{1}{\|\omega(t)-e\|} \text{ if } \omega \in \Omega', \\ &= 0 \text{ otherwise .} \end{aligned}$$

Then Y_t is Σ_t-measurable for every t and $t \mapsto Y_t(\omega)$ is continuous for every ω, so we have a corresponding locally jump-free process $\boldsymbol{v}$.

(c) $\lim_{t\to\infty} \mathbb{E}(Y_t) = 0$. $\boldsymbol{v}$ is not a martingale.

(d) If $n \in \mathbb{N}$ and h_n is the Brownian hitting time to the ball $B(e, 2^{-n})$, then h_n represents a stopping time τ_n adapted to $\langle \mathfrak{A}_t \rangle_{t \geq 0}$. Set $\mathcal{S} = \bigcup_{n\in\mathbb{N}} \{\tau : \tau \in \mathcal{T}_f,\ \tau \leq \tau_n\}$. $\mathcal{S}$ is a covering ideal.

(e) At the same time, for each $n \in \mathbb{N}$, $\langle v_\sigma \rangle_{\sigma \leq \tau_n}$ is a martingale. $\boldsymbol{v}$ is a local martingale.

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633 Separating sublattices

At various points, I have looked at relations between a process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ and its fully adapted extension to the covered envelope of $\mathcal{S}$; turning these round, we find connections between the properties of $\boldsymbol{u}$ and its restriction to a covering sublattice of $\mathcal{S}$. When the filtration is right-continuous and we have a near-simple process we can go much farther, and an effective concept is that of 'separating' sublattice (633B). Once again we have a useful result on equality of integrals (633K) and many correspondences between properties of $\boldsymbol{u}$ and $\boldsymbol{u}\restriction\mathcal{S}'$ (633O, 633P).

633B Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and A, B subsets of $\mathcal{S}$.

(a) I will say that A **separates** B if $[\![\sigma < \tau]\!] = \sup_{\rho\in A}([\![\sigma \leq \rho]\!] \cap [\![\sigma < \tau]\!])$ for all σ, $\tau \in B$.

(b) If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process, I will say that A ***v*-separates** B if whenever σ, $\tau \in B$ then $[\![\sigma < \tau]\!] \cap [\![v_\sigma \neq v_\tau]\!] \subseteq \sup_{\rho\in A}([\![\sigma \leq \rho]\!] \cap [\![\rho < \tau]\!])$.

633C Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and A, B, C, D subsets of $\mathcal{S}$.
(a) If $A \subseteq B$, $C \subseteq D$ and A separates D then B separates C.
(b) A separates its covered envelope.
(c) If A separates B and B separates C then A separates C.
(d) If A separates B and τ^* is an upper bound of B in $\mathcal{T}$ then $A \wedge \tau^* = \{\sigma \wedge \tau^* : \sigma \in A\}$ separates B.
(e) A separates B iff A $\boldsymbol{v}$-separates B for every fully adapted process $\boldsymbol{v}$ with domain $\mathcal{S}$.

633D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.
(a) If $T_0 \subseteq T$ is dense for the order topology of T and contains every point of T which is isolated on the right in T, then $\check{T}_0 = \{\check{t} : t \in T_0\}$ separates $\mathcal{S}$.
(b)(i) If $A \subseteq \mathcal{T}$ separates $\mathcal{S}$, $B \subseteq \mathcal{S}$ is coinitial with $\mathcal{S}$ and $C \subseteq \mathcal{S}$ is cofinal with $\mathcal{S}$, then $A' = \{\operatorname{med}(\tau, \sigma, \tau') : \tau \in B,\ \sigma \in A,\ \tau' \in C\}$ separates $\mathcal{S}$.
(ii) If $\boldsymbol{v}$ is a fully adapted process with domain $\mathcal{S}$, $A \subseteq \mathcal{T}$ $\boldsymbol{v}$-separates $\mathcal{S}$, $B \subseteq \mathcal{S}$ is coinitial with $\mathcal{S}$ and $C \subseteq \mathcal{S}$ is cofinal with $\mathcal{S}$, then $\mathcal{S}$ is $\boldsymbol{v}$-separated by $A' = \{\operatorname{med}(\tau, \sigma, \tau') : \tau \in B,\ \sigma \in A,\ \tau' \in C\}$.

633E Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$, $\mathcal{S}'$ a sublattice of $\mathcal{S}$, $\hat{\mathcal{S}}'_f$ the finitely-covered envelope of $\mathcal{S}'$, and τ an element of $\bigcup_{\sigma\in\mathcal{S}'} \mathcal{S} \wedge \sigma$.
(a) $A = \{\sigma : \tau \leq \sigma \in \hat{\mathcal{S}}'_f\}$ is non-empty and downwards-directed, and $\inf A \in \mathcal{S}$.
(b) If $\mathcal{S}'$ separates $\mathcal{S}$ then $\inf A = \tau$.
(c) If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is fully adapted and $\mathcal{S}'$ $\boldsymbol{v}$-separates $\mathcal{S}$, then $v_{\inf A} = v_\tau$.

633F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ locally near-simple processes with domain $\mathcal{S}$. Suppose that $C \subseteq \mathcal{S}$ separates $\mathcal{S}$ and that $\inf_{\sigma\in C} [\![\sigma < \tau]\!] = 0$ for every $\tau \in \mathcal{S}$. If $\boldsymbol{u}\restriction C = \boldsymbol{v}\restriction C$ then $\boldsymbol{u} = \boldsymbol{v}$.

633G Lemma Let $\mathcal{S} \subseteq \mathcal{T}$ be a sublattice and $D \subseteq \mathcal{S}$ a cofinal finitely full set which separates $\mathcal{S}$ and is such that $\inf A \in D$ for every non-empty downwards-directed $A \subseteq D$ with a lower bound in $\mathcal{S}$. Then $D = \mathcal{S}$.

633H Lemma Suppose that I, $J \in \mathcal{I}(\mathcal{T})$, with $J \subseteq I$. Then there are totally ordered sets $J_0 \subseteq J$ and $I_0 \subseteq I$ such that J_0 covers J, I_0 covers I and $J_0 \subseteq I_0$.

633I Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\mathcal{S}'$ a finitely full sublattice of $\mathcal{S}$ and ψ a strictly adapted interval function defined on $\mathcal{S}^{2\uparrow}$. Suppose that $J \in \mathcal{I}(\mathcal{S})$ and $\langle\sigma_\tau\rangle_{\tau\in J}$ are such that $\tau \le \sigma_\tau \in \mathcal{S}'$ and

$$\bar u_\tau = \sup_{\sigma\in\mathcal{S}\wedge\tau,\tau'\in\mathcal{S}'\cap[\tau,\sigma_\tau]} |\psi(\sigma,\tau') - \psi(\sigma,\tau)|$$

is defined in L^0 for each $\tau \in J$. Let $I \in \mathcal{I}(\mathcal{S}')$ be such that $\sigma_\tau \in I$ for every $\tau \in J$. Then

$$|S_{I\sqcup J}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| \le 2\textstyle\sum_{\tau\in J} \bar u_\tau.$$

633J Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\mathcal{S}'$ a finitely full cofinal sublattice of $\mathcal{S}$ and ψ an order-bounded strictly adapted interval function defined on $\mathcal{S}^{2\uparrow}$. For $\tau \in \mathcal{S}$ set $A_\tau = \{\sigma : \tau \le \sigma \in \mathcal{S}'\}$, and for $\tau \in \mathcal{S}$, $\tau' \in A_\tau$ set

$$u_{\tau\tau'} = \sup_{\sigma\in\mathcal{S}\wedge\tau,\rho\in A_\tau\wedge\tau'} |\psi(\sigma,\rho) - \psi(\sigma,\tau)|.$$

Suppose that $z = \int_{\mathcal{S}} d\psi$ is defined and that $\inf_{\tau'\in A_\tau} u_{\tau\tau'} = 0$ for every $\tau \in \mathcal{S}$. Then $\int_{\mathcal{S}'} d\psi$ is defined and equal to z.

633K Theorem (a) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ for every non-empty subset A of $\mathcal{S}$ with a lower bound in $\mathcal{S}$, and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$ such that $\boldsymbol{u}$ is order-bounded and $\boldsymbol{v}$ is locally near-simple. Let $\mathcal{S}'$ be a sublattice of $\mathcal{S}$, cofinal with $\mathcal{S}$, which $\boldsymbol{v}$-separates $\mathcal{S}$. If $z = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined then $\int_{\mathcal{S}'} \boldsymbol{u}\,d\boldsymbol{v}$ is defined and equal to z.

(b) Suppose that $\mathcal{S}$, $\mathcal{S}'$ are sublattices of $\mathcal{T}$ such that $\mathcal{S}'$ is finitely full and is included in $\mathcal{S}$. Let $\boldsymbol{u}$, $\boldsymbol{v}$ be fully adapted processes defined on $\mathcal{S}$. For $\tau \in \mathcal{S}$ set $A_\tau = \{\sigma : \tau \le \sigma \in \mathcal{S}'\}$. Suppose that A_τ is non-empty and

$$u_\tau = \lim_{\sigma\downarrow A_\tau} u_\sigma, \quad v_\tau = \lim_{\sigma\downarrow A_\tau} v_\sigma$$

for every $\tau \in \mathcal{S}$. If $z = \int_{\mathcal{S}'} \boldsymbol{u}\,d\boldsymbol{v}$ is defined then $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined and equal to z.

633L Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and $\tau \le \tau'$ in $\mathcal{T}$. Let $\boldsymbol{u}$ be a near-simple process and $\boldsymbol{v}$ a near-simple integrator, both defined on $[\tau,\tau']$. Suppose that $T_0 \subseteq T$ is a dense set for the order topology containing every point of T which is isolated on the right. Set $\mathcal{S}' = \{\tau'\} \cup \{\operatorname{med}(\tau,\check t,\tau') : t \in T_0\}$. Then $\int_{\mathcal{S}'} \boldsymbol{u}\,d\boldsymbol{v}$ is defined and equal to $\int_{[\tau,\tau']} \boldsymbol{u}\,d\boldsymbol{v}$.

***633M Lemma** Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process with domain $\mathcal{S}$.

(a) For every $\epsilon > 0$ and $\beta > 0$, there are a $b \in \mathfrak{A}$ and a $\gamma \ge 0$ such that $\bar\mu b \ge 1 - \epsilon$ and

$$\textstyle\sum_{i=0}^{n-1} \bar\mu(b \cap [\![|u_{\tau_{i+1}} - u_{\tau_i}| \ge \beta]\!]) \le \gamma$$

whenever $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$,

(b)(i) For $\sigma \le \tau$ in $\mathcal{S}$, set

$$\psi(\sigma,\tau) = \operatorname{med}(-\chi 1, u_\tau - u_\sigma, \chi 1), \quad \psi'(\sigma,\tau) = u_\tau - u_\sigma - \psi(\sigma,\tau).$$

Then ψ and ψ' are strictly adapted interval functions on $\mathcal{S}$.

(ii) If $\sigma \le \tau$ and $\sigma' \le \tau'$ in $\mathcal{S}$, then $|\psi(\sigma,\tau) - \psi(\sigma',\tau')|$ and $|\psi'(\sigma,\tau) - \psi'(\sigma',\tau')|$ are both at most $|u_\sigma - u_{\sigma'}| + |u_\tau - u_{\tau'}|$.

(c) $\int_{\mathcal{S}} d\psi'$ and $\int_{\mathcal{S}} d\psi$ are defined.

(d) $\boldsymbol{w}' = ii_{\psi'}(\mathbf{1})$ is of bounded variation and $\boldsymbol{w} = ii_\psi(\mathbf{1})$ is moderately oscillatory.

(e) $\mathrm{Osclln}(\boldsymbol{w}) \le \chi 1$.

(f) Express $\boldsymbol{w}$ as $\langle w_\tau\rangle_{\tau\in\mathcal{S}}$. If $\tau \le \tau'$ in $\mathcal{S}$, then $w_{\tau'} - w_\tau \in L^0(\mathfrak{D}_\tau)$, where $\mathfrak{D}_\tau$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{u_{\sigma'} - u_\sigma : \sigma, \sigma' \in \mathcal{S}\cap[\tau,\tau'], \sigma \le \sigma'\}$.

(g) If $\boldsymbol{u}$ is near-simple, $\boldsymbol{w}$ is near-simple.

***633N Lemma** Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple process. As in 633M, set

$$\psi(\sigma,\tau) = \mathrm{med}(-\chi 1, u_\tau - u_\sigma, \chi 1)$$

when $\sigma \le \tau$ in $\mathcal{S}$. Let $\mathcal{S}'$ be a cofinal sublattice of $\mathcal{S}$ which separates $\mathcal{S}$. Then $\int_{\mathcal{S}'} d\psi = \int_{\mathcal{S}} d\psi$.

633O Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$ with a lower bound in $\mathcal{S}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an order-bounded locally near-simple process, and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which is cofinal and coinitial with $\mathcal{S}$ and $\boldsymbol{u}$-separates $\mathcal{S}$. Write $\boldsymbol{u}'$ for $\boldsymbol{u}\restriction\mathcal{S}'$.

(a) $\mathrm{Osclln}(\boldsymbol{u}) = \mathrm{Osclln}(\boldsymbol{u}')$.

(b) $\boldsymbol{u}$ is jump-free iff $\boldsymbol{u}'$ is jump-free.

633P Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$ with a lower bound in $\mathcal{S}$, $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which is cofinal and coinitial with $\mathcal{S}$ and separates $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally near-simple fully adapted process. Write $\boldsymbol{u}'$ for $\boldsymbol{u}\restriction\mathcal{S}'$.

(a) If $\boldsymbol{u}'$ is simple then $\boldsymbol{u}$ is simple.

(b) If $\boldsymbol{u}'$ is near-simple then $\boldsymbol{u}$ is near-simple.

(c) $\boldsymbol{u}$ is order-bounded iff $\boldsymbol{u}'$ is order-bounded.

(d) $\boldsymbol{u}$ is (locally) of bounded variation iff $\boldsymbol{u}'$ is (locally) of bounded variation.

(e) $\boldsymbol{u}$ is an integrator iff $\boldsymbol{u}'$ is an integrator.

(f) $\boldsymbol{u}$ is a martingale iff $\boldsymbol{u}'$ is a martingale.

(g) If $\boldsymbol{u}'$ is a local martingale then $\boldsymbol{u}$ is a local martingale.

(h) Suppose that $\boldsymbol{u}'$ is a local integrator. Then $\boldsymbol{u}$ is a local integrator and the quadratic variation of $\boldsymbol{u}'$ is $\boldsymbol{u}^*\restriction\mathcal{S}'$, where $\boldsymbol{u}^*$ is the quadratic variation of $\boldsymbol{u}$.

633S Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S} \subseteq \mathcal{T}$ be a finitely full sublattice such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$, and A a subset of $\mathcal{S}$.

(a) If $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted and whenever $\sigma \le \tau$ in $\mathcal{S}$ and $v_\sigma \ne v_\tau$ there is a $\rho \in A$ such that $[\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!] \ne 0$, then A $\boldsymbol{v}$-separates $\mathcal{S}$.

(b) If whenever $\sigma \le \tau$ in $\mathcal{S}$ and $[\![\sigma < \tau]\!] \ne 0$ there is a $\rho \in A$ such that $[\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!] \ne 0$, then A separates $\mathcal{S}$.

633Q Continuous time: Proposition Let $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous stochastic integration structure. Then there are a right-continuous stochastic integration structure $(\mathfrak{A},\bar\mu,T',\langle\mathfrak{A}_r\rangle_{r\in T'},\mathcal{T}',\langle\mathfrak{A}_\rho\rangle_{\rho\in\mathcal{T}'})$, based on the same probability algebra $(\mathfrak{A},\bar\mu)$, and a lattice homomorphism $\sigma \mapsto \sigma' : \mathcal{T} \to \mathcal{T}'$ such that

T' has no points isolated on the right,

$\mathfrak{A}_{\sigma'} = \mathfrak{A}_\sigma$ and $[\![\sigma' < \tau']\!] = [\![\sigma < \tau]\!]$ for all $\sigma, \tau \in \mathcal{T}$,

for every $\rho \in \mathcal{T}'$ there is a $\sigma \in \mathcal{T}$ such that $\sigma' \le \rho$ and $\mathfrak{A}_\sigma = \mathfrak{A}_\rho$.

633R Theorem If $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge 0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a real-time stochastic integration structure and $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{T}_f)$, then $\boldsymbol{u}$ can be represented by a process with càdlàg sample paths as in Theorem 631D.

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634 Changing the algebra

If $(\mathfrak{A},\bar{\mu})$ is a probability algebra with a filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then we have a probability algebra $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ with a filtration $\langle\mathfrak{B}\cap\mathfrak{A}_t\rangle_{t\in T}$. In this section I examine elementary connexions between stochastic calculus in the two structures, with notes on lattices of stopping times (634C) and stochastic processes (634E). The case in which $\mathfrak{B}$ and $\mathfrak{A}_t$ are relatively independent over their intersection for every t is particularly important (634F-634I). I end the section with a product construction adapted to the representation of families of independent stochastic processes (634K-634M) and a worked example on independent Poisson processes (634N).

634A Notation For a family $\langle\mathfrak{C}_k\rangle_{k\in K}$ of closed subalgebras of $\mathfrak{A}$, I write $\bigvee_{k\in K}\mathfrak{C}_k$ for the closed subalgebra generated by $\bigcup_{k\in K}\mathfrak{C}_k$. Similarly, for closed subalgebras $\mathfrak{C}$, $\mathfrak{C}'$ I will write $\mathfrak{C}\vee\mathfrak{C}'$ for the closed subalgebra generated by $\mathfrak{C}\cup\mathfrak{C}'$. Note that a subalgebra of $\mathfrak{A}$ is closed iff it is a σ-subalgebra.

634B Proposition Let $\mathfrak{B}$ be a Dedekind complete algebra and $\langle\mathfrak{B}_t\rangle_{t\in T}$ a filtration of order-closed subalgebras of $\mathfrak{B}$; write $\mathcal{T}_{\mathbb{B}}$ and $\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}}$ for the associated lattice of stopping times and family of closed subalgebras of $\mathfrak{B}$. Suppose that $\phi:\mathfrak{B}\to\mathfrak{A}$ is an order-continuous Boolean homomorphism such that $\phi[\mathfrak{B}_t]\subseteq\mathfrak{A}_t$ for every $t\in T$.

(a) We have a lattice homomorphism $\hat{\phi}:\mathcal{T}_{\mathbb{B}}\to\mathcal{T}_{\mathbb{A}}$ defined by saying that $[\![\hat{\phi}(\sigma)>t]\!]=\phi[\![\sigma>t]\!]$ for every $\sigma\in\mathcal{T}_{\mathbb{B}}$.

(b)(i) $\hat{\phi}(\min\mathcal{T}_{\mathbb{B}})=\min\mathcal{T}_{\mathbb{A}}$, $\hat{\phi}(\max\mathcal{T}_{\mathbb{B}})=\max\mathcal{T}_{\mathbb{A}}$. If $t\in T$ and $\check{t}$ is the constant stopping time at t in $\mathcal{T}_{\mathbb{B}}$, then $\hat{\phi}(\check{t})$ is the constant stopping time at t in $\mathcal{T}_{\mathbb{A}}$.

(ii) If $C\subseteq\mathcal{T}_{\mathbb{B}}$ then $\hat{\phi}(\sup C)=\sup\hat{\phi}[C]$.

(iii) $\hat{\phi}[\mathcal{T}_{\mathbb{B}b}]\subseteq\mathcal{T}_{\mathbb{A}b}$ and $\hat{\phi}[\mathcal{T}_{\mathbb{B}f}]\subseteq\mathcal{T}_{\mathbb{A}f}$.

(c)(i) If σ, $\sigma'\in\mathcal{T}_{\mathbb{B}}$ then

$$[\![\hat{\phi}(\sigma)<\hat{\phi}(\sigma')]\!]=\phi[\![\sigma<\sigma']\!],\quad [\![\hat{\phi}(\sigma)\le\hat{\phi}(\sigma')]\!]=\phi[\![\sigma\le\sigma']\!],$$

$$[\![\hat{\phi}(\sigma)=\hat{\phi}(\sigma')]\!]=\phi[\![\sigma=\sigma']\!].$$

(ii) If ϕ is injective, so is $\hat{\phi}$.

(d) $\phi[\mathfrak{B}_\sigma]\subseteq\mathfrak{A}_{\hat{\phi}(\sigma)}$ for every $\sigma\in\mathcal{T}_{\mathbb{B}}$.

(e) If $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous, then $\hat{\phi}$ is order-continuous,

(f) If $T=[0,\infty[$ and we define the identity processes $\langle\iota_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}f}}$, $\langle\iota_\tau\rangle_{\tau\in\mathcal{T}_{\mathbb{A}f}}$ as in 612F, then $\iota_{\hat{\phi}(\sigma)}=T_\phi\iota_\sigma$ for every $\sigma\in\mathcal{T}_{\mathbb{B}f}$, where $T_\phi:L^0(\mathfrak{B})\to L^0(\mathfrak{A})$ is the f-algebra homomorphism associated with ϕ.

634C Proposition Suppose that $\mathfrak{B}$ is an (order-)closed subalgebra of $\mathfrak{A}$.

(a) Set $\mathfrak{B}_t=\mathfrak{B}\cap\mathfrak{A}_t$ for $t\in T$. Then $\langle\mathfrak{B}_t\rangle_{t\in T}$ is a filtration of closed subalgebras of $\mathfrak{B}$.

(b) Let $\mathcal{T}_{\mathbb{B}}$ be the set of stopping times defined from $(\mathfrak{B},\langle\mathfrak{B}_t\rangle_{t\in T})$ by the formula of 611A(b-i).

(i) $\mathcal{T}_{\mathbb{B}}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$ containing $\min\mathcal{T}_{\mathbb{A}}$, $\max\mathcal{T}_{\mathbb{A}}$ and all constant stopping times.

(ii) If $C\subseteq\mathcal{T}_{\mathbb{B}}$ is non-empty then its supremum is the same whether calculated in $\mathcal{T}_{\mathbb{B}}$ or in $\mathcal{T}_{\mathbb{A}}$.

(iii) We can identify the order-ideals $\mathcal{T}_{\mathbb{B}b}$ and $\mathcal{T}_{\mathbb{B}f}$ of bounded and finite stopping times in $\mathcal{T}_{\mathbb{B}}$ with $\mathcal{T}_{\mathbb{B}}\cap\mathcal{T}_{\mathbb{A}b}$ and $\mathcal{T}_{\mathbb{B}}\cap\mathcal{T}_{\mathbb{A}f}$ respectively.

(c) If σ, $\tau\in\mathcal{T}_{\mathbb{B}}$ then the regions $[\![\sigma<\tau]\!]$, $[\![\sigma\le\tau]\!]$ and $[\![\sigma=\tau]\!]$, when defined by the formulae of 611D interpreted in either $(\mathfrak{A},\langle\mathfrak{A}_t\rangle_{t\in T})$ or $(\mathfrak{B},\langle\mathfrak{B}_t\rangle_{t\in T})$, are the same, and belong to $\mathfrak{B}$.

(d) If $\tau\in\mathcal{T}_{\mathbb{B}}$, and we define corresponding algebras $\mathfrak{A}_\tau$ and $\mathfrak{B}_\tau$ by the formula of 611G interpreted in $(\mathfrak{A},\langle\mathfrak{A}_t\rangle_{t\in T})$, $(\mathfrak{B},\langle\mathfrak{B}_t\rangle_{t\in T})$ respectively, then $\mathfrak{B}_\tau=\mathfrak{B}\cap\mathfrak{A}_\tau$.

(e) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$.

(i) If $\hat{\mathcal{S}}_{\mathbb{A}}$ is the covered envelope of $\mathcal{S}$ in $\mathcal{T}_{\mathbb{A}}$, then $\hat{\mathcal{S}}_{\mathbb{A}}\cap\mathcal{T}_{\mathbb{B}}$ is the covered envelope $\hat{\mathcal{S}}_{\mathbb{B}}$ of $\mathcal{S}$ in $\mathcal{T}_{\mathbb{B}}$.

(ii) A family $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ in $L^0(\mathfrak{B})$ is fully adapted to $\langle\mathfrak{B}_t\rangle_{t\in T}$ iff it is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$.

(iii) If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted to $\langle\mathfrak{B}_t\rangle_{t\in T}$ and $\hat{\boldsymbol{u}}$ is the extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}_{\mathbb{A}}$ which is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$, then $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}_{\mathbb{B}}$ is the extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}_{\mathbb{B}}$ which is fully adapted to $\langle\mathfrak{B}_t\rangle_{t\in T}$.

(f) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous.

(i) $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous.

(ii) $\mathcal{T}_{\mathbb{B}}$ is order-closed in $\mathcal{T}_{\mathbb{A}}$.

(g) If $T=[0,\infty[$ and we write $\boldsymbol{\iota}=\langle\iota_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{A}f}}$ for the identity process in the structure $(\mathfrak{A},\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{A}})$, then $\boldsymbol{\iota}\restriction\mathcal{T}_{\mathbb{B}f}$ is the identity process in the structure $(\mathfrak{B},\langle\mathfrak{B}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{B}})$.

634D Notation In the context of 634Ce, we shall be able to regard a process $\boldsymbol{u}\in L^0(\mathfrak{B})^{\mathcal{S}}$ as being fully adapted either in the structure $\mathbb{A}=(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{A}},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}_{\mathbb{A}}})$ $(\mathfrak{A},\langle\mathfrak{A}_t\rangle_{t\in T})$ or in the structure $\mathbb{B}=(\mathfrak{B},\bar\mu\restriction\mathfrak{B},T,\langle\mathfrak{B}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{B}},\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$, where $\mathfrak{B}_t=\mathfrak{B}\cap\mathfrak{A}_t$ for $t\in T$. Consequently we shall be able to test $\boldsymbol{u}$ against the definitions in this volume in two different ways. We have to check which properties are 'absolute', in the sense that $\boldsymbol{u}$ will have them in one structure iff it has them in the other, and which are not. While working through the list, it will save a great many words if I use abbreviated expressions of the type '$\boldsymbol{u}$ is $\mathbb{A}$-simple', '$\boldsymbol{u}$ is a $\mathbb{B}$-integrator' to mean 'interpreted in the structure $\mathbb{A}$, $\boldsymbol{u}$ is a simple process', 'interpreted in the structure $\mathbb{B}$, $\boldsymbol{u}$ is an integrator', and so forth.

634E Proposition Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$, and $\mathbb{B}=(\mathfrak{B},\bar\mu\restriction\mathfrak{B},T,\langle\mathfrak{B}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{B}},\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t=\mathfrak{B}\cap\mathfrak{A}_t$ for $t\in T$. Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$ and that $\boldsymbol{u}\in L^0(\mathfrak{B})^{\mathcal{S}}$ is fully adapted.

(a) $\boldsymbol{u}$ is $\mathbb{B}$-simple iff it is $\mathbb{A}$-simple.

(b) $\boldsymbol{u}$ is $\mathbb{B}$-(locally)-near-simple iff it is $\mathbb{A}$-(locally)-near-simple.

(c) $\boldsymbol{u}$ is $\mathbb{B}$-order-bounded iff it is $\mathbb{A}$-order-bounded.

(d) $\boldsymbol{u}$ is of $\mathbb{B}$-bounded variation iff it is of $\mathbb{A}$-bounded variation.

(e) $\boldsymbol{u}$ is $\mathbb{B}$-(locally)-moderately-oscillatory iff it is $\mathbb{A}$-(locally)-moderately-oscillatory.

(f) $\boldsymbol{u}$ is $\mathbb{A}$-jump-free iff it is $\mathbb{B}$-jump-free.

(g) If $\boldsymbol{v}\in L^0(\mathfrak{B})^{\mathcal{S}}$ is another fully adapted process, then the integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined for $(\mathfrak{B},\bar\mu\restriction\mathfrak{B},\langle\mathfrak{B}_t\rangle_{t\in T})$ iff it is defined for $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\in T})$, with the same value; that is, $\mathbb{B}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\mathbb{A}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ if either is defined.

634F Relative independence: Definitions (a) If $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ are closed subalgebras of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are **relatively (stochastically) independent** over $\mathfrak{D}$ if $P_{\mathfrak{D}}\chi(b\cap c)=P_{\mathfrak{D}}\chi b\times P_{\mathfrak{D}}\chi c$ for all $b\in\mathfrak{B}$ and $c\in\mathfrak{C}$.

(b) I say that a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is **coordinated** with the filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ if $\mathfrak{B}$ and $\mathfrak{A}_t$ are relatively independent over $\mathfrak{B}\cap\mathfrak{A}_t$ for every $t\in T$.

634G Proposition If $\mathfrak{B}$, $\mathfrak{C}$ are closed subalgebras of $\mathfrak{A}$, the following are equiveridical:

(i) $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B}\cap\mathfrak{C}$;

(ii) $P_{\mathfrak{B}\cap\mathfrak{C}}(u\times v)=P_{\mathfrak{B}\cap\mathfrak{C}}u\times P_{\mathfrak{B}\cap\mathfrak{C}}v$ for all $u\in L^\infty(\mathfrak{B})$, $v\in L^\infty(\mathfrak{C})$;

(iii) $P_{\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{B}\cap\mathfrak{C}}$;

(iv) $P_{\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{C}}P_{\mathfrak{B}}$;

(v) $P_{\mathfrak{B}}u\in L^0(\mathfrak{C})$ whenever $u\in L^1_{\bar\mu}\cap L^0(\mathfrak{C})$.

634H Lemma Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B}=(\mathfrak{B},\bar\mu\restriction\mathfrak{B},T,\langle\mathfrak{B}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{B}},\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t=\mathfrak{B}\cap\mathfrak{A}_t$ for $t\in T$. Then $\mathfrak{B}$ and $\mathfrak{A}_\sigma$ are relatively independent over $\mathfrak{B}_\sigma$ for every $\sigma\in\mathcal{T}_{\mathbb{B}}$.

634I Theorem Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B}=(\mathfrak{B},\bar\mu\restriction\mathfrak{B},T,\langle\mathfrak{B}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{B}},\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t=\mathfrak{B}\cap\mathfrak{A}_t$ for $t\in T$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $u_\sigma\in L^0(\mathfrak{B})$ for every $\sigma\in\mathcal{S}$.

(a) $\boldsymbol{u}$ is a (local) $\mathbb{B}$-martingale iff it is a (local) $\mathbb{A}$-martingale.

(b) $\boldsymbol{u}$ is a (local) $\mathbb{B}$-integrator iff it is a (local) $\mathbb{A}$-integrator, and in this case its $\mathbb{B}$-quadratic variation is the same as its $\mathbb{A}$-quadratic variation.

634J Lemma Let (Ω,Σ,μ) and (Ω',Σ',μ') be probability spaces, and $(\Xi,\le)$ a non-empty downwards-directed partially ordered set. Let $\langle\Sigma_\xi\rangle_{\xi\in\Xi}$ and $\langle\Sigma'_\xi\rangle_{\xi\in\Xi}$ be families of σ-subalgebras of Σ, Σ' respectively such that $\Sigma_\xi\subseteq\Sigma_\eta$ and $\Sigma'_\xi\subseteq\Sigma'_\eta$ whenever $\xi\le\eta$, while every Σ_ξ contains every μ-negligible set and every Σ'_ξ contains every μ'-negligible set. Write $\mathrm{T}=\bigcap_{\xi\in\Xi}\Sigma_\xi$ and $\mathrm{T}'=\bigcap_{\xi\in\Xi}\Sigma'_\xi$. Let λ be the product probability measure on $\Omega\times\Omega'$, and Λ its domain.

(a) Suppose that $W\in\Lambda$ is such that for every $\xi\in\Xi$ there is a $W_\xi\in\Sigma_\xi\widehat{\otimes}\Sigma'$ such that $W\triangle W_\xi$ is λ-negligible. Then there is a $W'\in\mathrm{T}\widehat{\otimes}\Sigma'$ such that $W\triangle W'$ is λ-negligible.

(b) Suppose that $W\in\Lambda$ is such that for every $\xi\in\Xi$ there is a $W_\xi\in\Sigma_\xi\widehat{\otimes}\Sigma'_\xi$ such that $W\triangle W_\xi$ is λ-negligible. Then there is a $W'\in\mathrm{T}\widehat{\otimes}\mathrm{T}'$ such that $W\triangle W'$ is λ-negligible.

(c) Suppose that $W\in\bigcap_{\xi\in\Xi}\Sigma_\xi\widehat{\otimes}\Sigma'$, $E\in\Sigma$ and $\epsilon\ge 0$ are such that $\lambda(W\triangle(E\times\Omega'))\le\epsilon$. Then there is an $E_1\in\mathrm{T}$ such that $\lambda(W\triangle(E_1\times\Omega'))\le 3\epsilon$.

634K Theorem Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a stochastically independent family of closed subalgebras of $\mathfrak{A}$. Suppose that $(\Xi,\le)$ is a non-empty downwards-directed partially ordered set and that for each $i\in I$ we have a non-decreasing family $\langle\mathfrak{B}_{i\xi}\rangle_{\xi\in\Xi}$ of closed subalgebras of $\mathfrak{A}_i$ with intersection $\mathfrak{B}_i$. Set $\mathfrak{D}=\bigvee_{i\in I}\mathfrak{B}_i$, and for $\xi\in\Xi$ set $\mathfrak{D}_\xi=\bigvee_{i\in I}\mathfrak{B}_{i\xi}$. Then $\mathfrak{D}=\bigcap_{\xi\in\Xi}\mathfrak{D}_\xi$.

634L Theorem Let $\langle\mathfrak{B}_i\rangle_{i\in I}$ be a stochastically independent family of closed subalgebras of $\mathfrak{A}$. Suppose that for each $i\in I$ we have a filtration $\langle\mathfrak{B}_{it}\rangle_{t\in T}$ of closed subalgebras of $\mathfrak{B}_i$. For each $t\in T$ set $\mathfrak{C}_t=\bigvee_{i\in I}\mathfrak{B}_{it}$.

(a) $\langle\mathfrak{C}_t\rangle_{t\in T}$ is a filtration.

(b) For $i\in I$ and $t\in T$, $\mathfrak{B}_i\cap\mathfrak{C}_t=\mathfrak{B}_{it}$ and $\mathfrak{B}_i$ and $\mathfrak{C}_t$ are relatively independent over $\mathfrak{B}_{it}$.

(c) If $\langle\mathfrak{B}_{it}\rangle_{t\in T}$ is right-continuous for every $i\in I$, then $\langle\mathfrak{C}_t\rangle_{t\in T}$ is right-continuous.

634M Corollary Suppose that $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ is a family of probability algebras, and that $\langle\mathfrak{A}_{it}\rangle_{t\in T}$ is a filtration in $\mathfrak{A}_i$ for each i. Then there are a probability algebra $(\mathfrak{C},\bar\lambda)$ with a filtration $\langle\mathfrak{C}_t\rangle_{t\in T}$ and a stochastically independent family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of closed subalgebras such that $\mathfrak{B}_i$ is coordinated with $\langle\mathfrak{C}_t\rangle_{t\in T}$ and $(\mathfrak{B}_i,\bar\lambda\restriction\mathfrak{B}_i,\langle\mathfrak{B}_i\cap\mathfrak{C}_t\rangle_{t\in T})$ is isomorphic to $(\mathfrak{A}_i,\bar\mu_i,\langle\mathfrak{A}_{it}\rangle_{t\in T})$ for every $i\in I$. If every $\langle\mathfrak{A}_{it}\rangle_{t\in T}$ is right-continuous, we can arrange that $\langle\mathfrak{C}_t\rangle_{t\in T}$ should be right-continuous.

634N Example: independent Poisson processes: (a) Let $(\mathfrak{B},\bar\nu,\langle\mathfrak{B}_t\rangle_{t\ge0},\mathcal{T}_{\mathbb{B}},\langle u_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}f}})$ be the standard Poisson process in its measure-algebra form. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra free product of $(\mathfrak{B},\bar\nu)$ with itself, with associated embeddings $\varepsilon_1:\mathfrak{B}\to\mathfrak{A}$, $\varepsilon_2:\mathfrak{B}\to\mathfrak{A}$; write $\mathfrak{B}^{(i)}$ for $\varepsilon_i[\mathfrak{B}]$ for each i. Set $\mathfrak{A}_t=\varepsilon_1[\mathfrak{B}_t]\vee\varepsilon_2[\mathfrak{B}_t]$ for $t\ge0$, so that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is a right-continuous filtration, while $\varepsilon_i[\mathfrak{B}_t]=\mathfrak{B}^{(i)}\cap\mathfrak{A}_t$ for both i and every t, and each $\mathfrak{B}^{(i)}$ is coordinated with $\langle\mathfrak{A}_t\rangle_{t\ge0}$.

(b) For each i, ε_i is an isomorphism between $(\mathfrak{B},\bar\nu,\langle\mathfrak{B}_t\rangle_{t\ge0})$ and $(\mathfrak{B}^{(i)},\bar\mu\restriction\mathfrak{B}^{(i)},\langle\mathfrak{B}^{(i)}\cap\mathfrak{A}_t\rangle_{t\in[0,\infty[})$, so matches $\mathcal{T}_{\mathbb{B}f}$ with $\mathcal{T}_{\mathbb{B}^{(i)}f}=\mathcal{T}_{\mathbb{B}^{(i)}}\cap\mathcal{T}_{\mathbb{A}f}$. Now $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}f}}$ is matched with $\boldsymbol{u}_i=\langle u_{i\sigma}\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}^{(i)}f}}$, and $\boldsymbol{u}_i$ is locally near-simple. Because $\mathcal{T}_{\mathbb{B}^{(i)}f}$ contains all the constant processes, $\sup\{[\![\tau\le\sigma]\!]:\sigma\in\mathcal{T}_{\mathbb{B}^{(i)}f}\}=1$ for every $\tau\in\mathcal{T}_{\mathbb{A}f}$, $\boldsymbol{u}_i$ has an extension to a locally near-simple process defined on $\mathcal{T}_{\mathbb{A}f}$. Because $\mathcal{T}_{\mathbb{B}^{(i)}f}$ contains the constant processes, the extension is unique; I will call it $\boldsymbol{v}_i$.

(c) For each i, $\boldsymbol{v}_i-\boldsymbol{\iota}_{\mathfrak{A}}$ is a local martingale. $\boldsymbol{w}=\boldsymbol{v}_1-\boldsymbol{v}_2$ is a local martingale.

(d)(i) $\boldsymbol{v}_i^*=\boldsymbol{v}_i$.

(ii) The covariation $[\boldsymbol{v}_1\overset{*}{\restriction}\boldsymbol{v}_2]$ is zero.

(iii) $\boldsymbol{w}^*=\boldsymbol{v}_1+\boldsymbol{v}_2$.

(iv) The previsible variation of $\boldsymbol{w}^2\restriction\mathcal{T}_{\mathbb{A}b}$ is $2\boldsymbol{\iota}\restriction\mathcal{T}_{\mathbb{A}b}$.

(e) $\boldsymbol{w}$ corresponds to a Lévy process derived from the family $\langle\lambda'_t\rangle_{t\ge0}$ where

$$\lambda'_t(\{n\})=e^{-2t}t^n\sum_{k=\max(-n,0)}^{\infty}\frac{t^{2k}}{k!(k+n)!}$$

for $n\in\mathbb{Z}$.

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635 Changing the filtration

In this section I introduce the elementary theory of 'local times'. In the principal applications, we have a process which is easier to handle if we replace the standard clock T with a variable-speed clock $\langle\pi_r\rangle_{r\in R}$ where the clock-times are now a totally ordered family of stopping times. I will come to such applications in Chapter 65. Here I want to set up a language to discuss the transformation in which a process $\langle u_\tau\rangle_{\tau\in\mathcal{T}}$ is mapped to $\langle u_{\pi(\rho)}\rangle_{\rho\in\mathcal{R}}$, where $\mathcal{R}$ is the lattice of R-based stopping times and $\pi(\rho)\in\mathcal{T}$ corresponds to $\rho\in\mathcal{R}$. Starting from the construction in 635B, we have basic algebraic properties (corresponding to ideas in §611) in 635C and can then follow a programme along the same lines as elsewhere, looking at the usual kinds of process and Riemann-sum integrals.

635B Construction For the whole of this section, $(R,\le)$ will be a new (non-empty) totally ordered set, and $\langle\pi_r\rangle_{r\in R}$ a non-decreasing family in $\mathcal{T}$. For $r\in R$, I will write $\mathfrak{B}_r$ for $\mathfrak{A}_{\pi_r}$, so that $\langle\mathfrak{B}_r\rangle_{r\in R}$ is a filtration of closed subalgebras of $\mathfrak{A}$, and we shall have a corresponding stochastic integration structure $(\mathfrak{A},\bar\mu,R,R,\langle\mathfrak{B}_r\rangle_{r\in R},\mathcal{R},\langle\mathfrak{B}_\rho\rangle_{\rho\in\mathcal{R}})$.

For $\rho\in\mathcal{R}$, $\pi(\rho)\in L^0(\mathfrak{A})^T$ will be defined by saying that

$$\begin{aligned}[\![\pi(\rho)>t]\!]&=\inf_{r\in R}([\![\rho>r]\!]\cup[\![\pi_r>t]\!])\text{ if }t\in T\text{ is isolated on the right in }T,\\&=\sup_{s>t}\inf_{r\in R}([\![\rho>r]\!]\cup[\![\pi_r>s]\!])\text{ for other }t\in T.\end{aligned}$$

635C Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous.

(a) For every $\rho\in\mathcal{R}$, $\pi(\rho)$, as defined in 635B, belongs to $\mathcal{T}$, and $\mathfrak{B}_\rho=\mathfrak{A}_{\pi(\rho)}$.

(b)(i) The map $\pi:\mathcal{R}\to\mathcal{T}$ is a lattice homomorphism.

(ii) $\pi(\min\mathcal{R})=\inf_{r\in R}\pi_r$ in $\mathcal{T}$, $\pi(\max\mathcal{R})=\max\mathcal{T}$.

(iii) If $r\in R$ and $\check r\in\mathcal{R}$ is the corresponding constant stopping time, then $\pi(\check r)=\pi_r$.

(iv) If $\rho\in\mathcal{R}_f$ then $\pi(\rho)\le\sup_{r\in R}\pi_r$.

(c) $[\![\pi(\rho)<\pi(\rho')]\!]\subseteq[\![\rho<\rho']\!]$, $[\![\pi(\rho)\le\pi(\rho')]\!]\supseteq[\![\rho\le\rho']\!]$ and $[\![\rho=\rho']\!]\subseteq[\![\pi(\rho)=\pi(\rho')]\!]$ for all ρ, $\rho'\in\mathcal{R}$.

(d) Suppose that $\langle\pi_r\rangle_{r\in R}$ is **right-continuous** in the sense that $\pi_r=\inf_{q\in R,q>r}\pi_q$ in $\mathcal{T}$ whenever $r\in R$ is not isolated on the right in R. Then

(i) $\langle\mathfrak{B}_r\rangle_{r\in R}$ is right-continuous;

(ii) π is right-continuous.

635D Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{Q}$ be a sublattice of $\mathcal{R}$.

(a)(i) $\pi[\mathcal{Q}]$ is a sublattice of $\mathcal{T}$.

(ii) If $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\pi[\mathcal{Q}]}$ is a process fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$, then $\boldsymbol{u}\pi=\langle u_{\pi(\rho)}\rangle_{\rho\in\mathcal{Q}}$ is fully adapted to $\langle\mathfrak{B}_r\rangle_{r\in R}$.

(iii) Let $\psi:\pi[\mathcal{Q}]^{2\uparrow}\to L^0(\mathfrak{A})$ be an adapted interval function, and set $\psi_\pi(\rho,\rho')=\psi(\pi(\rho),\pi(\rho'))$ when $\rho\le\rho'$ in $\mathcal{Q}$. Then ψ_π is an adapted interval function.

(iv) In (iii), if ψ is strictly adapted then ψ_π is strictly adapted.

(b) Now suppose that $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\pi[\mathcal{Q}]}$ is fully adapted and that ψ is an adapted interval function on $\pi[\mathcal{Q}]$. Then $\int_{\mathcal{Q}}\boldsymbol{u}\pi\,d\psi_\pi=\int_{\pi[\mathcal{Q}]}\boldsymbol{u}\,d\psi$ if either is defined.

635E Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}\subseteq\mathcal{T}$ be a sublattice and $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\mathcal{S}}$ a fully adapted process. Set $\mathcal{Q}=\pi^{-1}[\mathcal{S}]=\operatorname{dom}(\pi\boldsymbol{u})$.

(a) $\boldsymbol{u}\pi$ is order-bounded iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is order-bounded.

(b) $\boldsymbol{u}\pi$ is of bounded variation iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is of bounded variation.

(c) $\boldsymbol{u}\pi$ is an integrator for the structure $(\mathfrak{A},\bar\mu,R,\langle\mathfrak{B}_r\rangle_{r\in R})$ iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is an integrator for the structure $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T})$.

(d) $\boldsymbol{u}\pi$ is a martingale for the structure $(\mathfrak{A},\bar\mu,R,\langle\mathfrak{B}_r\rangle_{r\in R})$ iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is a martingale for the structure $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T})$.

(e) $\boldsymbol{u}\pi$ is jump-free iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is jump-free.

(f) Let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ in $\mathcal{T}$, $\hat{\mathcal{Q}}$ the covered envelope of $\mathcal{Q}$ in $\mathcal{R}$ and $\hat{\boldsymbol{u}}$ the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$. Then $\pi[\hat{\mathcal{Q}}] \subseteq \hat{\mathcal{S}}$ and $\hat{\boldsymbol{u}}\pi\restriction\hat{\mathcal{Q}}$ is the fully adapted extension of $\boldsymbol{u}\pi$ to $\hat{\mathcal{Q}}$.

(g) If $\boldsymbol{u}$ is moderately oscillatory then $\boldsymbol{u}\pi$ is moderately oscillatory.

(h)(i) If $\mathcal{S}$ is order-convex in $\mathcal{T}$ then $\mathcal{Q}$ is order-convex in $\mathcal{R}$.

(ii) Suppose that $\langle\pi_r\rangle_{r\in R}$ is right-continuous. If $\mathcal{S}$ is order-convex and $\boldsymbol{u}$ is near-simple, then $\boldsymbol{u}\pi$ is near-simple.

635F Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple integrator; let $\mathcal{Q}$ be a sublattice of $\mathcal{R}$ such that $\pi[\mathcal{Q}]$ is a cofinal sublattice of $\mathcal{S}$ which $\boldsymbol{v}$-separates $\mathcal{S}$. If $\boldsymbol{u} = \langle u_\tau\rangle_{\tau\in\mathcal{S}}$ is a moderately oscillatory process, then $\int_{\mathcal{Q}}\boldsymbol{u}\pi\,d(\boldsymbol{v}\pi)$ is defined and equal to $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

635G Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple integrator, with quadratic variation $\boldsymbol{v}^*$. If $\mathcal{Q}$ is a sublattice of $\pi^{-1}[\mathcal{S}]$ such that $\pi[\mathcal{Q}]$ $\boldsymbol{v}$-separates $\mathcal{S}$, then the quadratic variation of $\boldsymbol{v}\pi\restriction\mathcal{Q}$ is $\boldsymbol{v}^*\pi\restriction\mathcal{Q}$.