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Chapter 56

Choice and determinacy

Nearly everyone reading this book will have been taking the axiom of choice for granted nearly all the time. This is the home territory of twentieth-century abstract analysis, and the one in which the great majority of the results have been developed. But I hope that everyone is aware that there are other ways of doing things. In this chapter I want to explore what seem to me to be the most interesting alternatives. In one sense they are minor variations on the standard approach, since I keep strictly to ideas expressible within the framework of Zermelo-Fraenkel set theory; but in other ways they are dramatic enough to rearrange our prejudices. The arguments I will present in this chapter are mostly not especially difficult by the standards of this volume, but they do depend on intuitions for which familiar results which are likely to remain valid under the new rules being considered.

Let me say straight away that the real aim of the chapter is §567, on the axiom of determinacy. The significance of this axiom is that it is (so far) the most striking rival to the axiom of choice, in that it leads us quickly to a large number of propositions directly contradicting familiar theorems; for instance, every subset of the real line is now Lebesgue measurable (567G). But we need also to know which theorems are still true, and the first six sections of the chapter are devoted to a discussion of what can be done in ZF alone (§§561-565) and with countable or dependent choice (§566). Actually §§562-565 are rather off the straight line to §567, because they examine parts of real analysis in which the standard proofs depend only on countable choice or less; but a great deal more can be done than most of us would expect, and the methods are instructive.

Going into details, §561 looks at basic facts from real analysis, functional analysis and general topology which can be proved in ZF. §562 deals with 'codable' Borel sets and functions, using Borel codes to keep track of constructions for objects, so that if we know a sequence of codes we can avoid having to make a sequence of choices. A 'Borel-coded measure' (§563) is now one which behaves well with respect to codable sequences of measurable sets; for such a measure we have an integral with versions of the convergence theorems (§564), and Lebesgue measure fits naturally into the structure (§565). In §566, with ZF + AC(ω), we are back in familiar territory, and most of the results of Volumes 1 and 2 can be proved if we are willing to re-examine some definitions and hypotheses. Finally, in §567, I look at infinite games and half a dozen of the consequences of AD, with a postscript on determinacy in the context of ZF + AC.

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561 Analysis without choice

Elementary courses in analysis are often casual about uses of weak forms of choice; a typical argument runs 'for every $\epsilon > 0$ there is an $a \in A$ such that $|a - x| \le \epsilon$, so there is a sequence in A converging to x'. This is a direct call on the countable axiom of choice: setting $A_n = \{a : a \in A,\ |a - x| \le 2^{-n}\}$, we are told that every A_n is non-empty, and conclude that $\prod_{n\in\mathbb{N}} A_n$ is non-empty. In the present section I will abjure such methods and investigate what can still be done with the ideas important in measure theory. We have useful partial versions of Tychonoff's theorem (561D), Baire's theorem (561E), Stone's theorem (561F) and Kakutani's theorem on the representation of L-spaces (561H); moreover, there is a direct construction of Haar measures, regarded as linear functionals (561G).

Unless explicitly stated otherwise, throughout this section (and the next four) I am working entirely without any form of the axiom of choice.

561A Set theory without choice The most obvious point is that in the absence of choice

the union of a sequence of countable sets need not be countable.

The elementary arguments of 1A1E still give

$$\mathbb{N} \simeq \mathbb{Z} \simeq \mathbb{N} \times \mathbb{N} \simeq \mathbb{Q};$$

$$\mathbb{N} \simeq [\mathbb{N}]^{<\omega} \simeq \bigcup_{n\ge 1} \mathbb{N}^n \simeq \mathbb{Q}^r \times \mathbb{Q}^r$$

for every integer $r \ge 1$. The Schröder-Bernstein theorem survives. Consequently

$$\mathbb{R} \simeq \mathcal{P}\mathbb{N} \simeq \{0,1\}^{\mathbb{N}} \simeq \mathcal{P}(\mathbb{N} \times \mathbb{N}) \simeq (\mathcal{P}\mathbb{N})^{\mathbb{N}} \simeq \mathbb{R}^{\mathbb{N}} \simeq \mathbb{N}^{\mathbb{N}}.$$

$X \not\simeq \mathcal{P}X$, so $\mathbb{R}$ is not countable.

We can still use transfinite recursion. We still have a class On of von Neumann ordinals such that every well-ordered set is isomorphic to exactly one ordinal and equipollent with exactly one initial ordinal. I will say that a set X is **well-orderable** if there is a well-ordering of X. The standard arguments for Zermelo's Well-Ordering Theorem now tell us that for any set X the following are equiveridical:

(i) X is well-orderable;

(ii) X is equipollent with some ordinal;

(iii) there is an injective function from X into a well-orderable set;

(iv) there is a choice function for $\mathcal{P}X \setminus \{\emptyset\}$

(that is, a function f such that $f(A) \in A$ for every non-empty $A \subseteq X$). What this means is that if we are given a family $\langle A_i \rangle_{i\in I}$ of non-empty sets, and $X = \bigcup_{i\in I} A_i$ is well-orderable, then $\prod_{i\in I} A_i$ is not empty.

Note that while we still have a first uncountable ordinal ω_1, it can have countable cofinality. The union of a sequence of finite sets need not be countable; but the union of a sequence of finite subsets of a given totally ordered set *is* countable. Consequently, if $\gamma : \omega_1 \to \mathbb{R}$ is a monotonic function there is a $\xi < \omega_1$ such that $\gamma(\xi + 1) = \gamma(\xi)$.

561C Lemma Let $\mathcal{E}$ be the set of non-empty closed subsets of $\mathbb{N}^{\mathbb{N}}$. Then there is a family $\langle f_F \rangle_{F\in\mathcal{E}}$ such that, for each $F \in \mathcal{E}$, f_F is a continuous function from $\mathbb{N}^{\mathbb{N}}$ to F and $f_F(\alpha) = \alpha$ for every $\alpha \in F$.

561D Tychonoff's theorem Let $\langle X_i \rangle_{i\in I}$ be a family of compact topological spaces such that I is well-orderable. For each $i \in I$ let $\mathcal{E}_i$ be the family of non-empty closed subsets of X_i, and suppose that there is a choice function for $\bigcup_{i\in I} \mathcal{E}_i$. Then $X = \prod_{i\in I} X_i$ is compact.

561E Baire's theorem (a) Let (X, ρ) be a complete metric space with a well-orderable dense subset. Then X is a Baire space.

(b) Let X be a compact Hausdorff space with a well-orderable π-base. Then X is a Baire space.

561F Stone's Theorem Let $\mathfrak{A}$ be a well-orderable Boolean algebra. Then there is a compact Hausdorff Baire space Z such that $\mathfrak{A}$ is isomorphic to the algebra of open-and-closed subsets of Z.

561G Haar measure: Theorem Let X be a completely regular locally compact Hausdorff topological group.

(i) There is a non-zero left-translation-invariant positive linear functional on $C_k(X)$.

(ii) If ϕ, ϕ' are non-zero left-translation-invariant positive linear functionals on $C_k(X)$ then each is a scalar multiple of the other.

561H Kakutani's theorem (a) Let U be an Archimedean Riesz space with a weak order unit. Then there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and an order-dense Riesz subspace of $L^0(\mathfrak{A})$, containing $\chi 1$, which is isomorphic to U.

(b) Let U be an L-space with a weak order unit e. Then there is a totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$ such that U is isomorphic, as normed Riesz space, to $L^1(\mathfrak{A}, \bar{\mu})$, and we can choose the isomorphism to match e with $\chi 1$.

561I Hilbert spaces: Proposition Let U be a Hilbert space.

(a) If $C \subseteq U$ is a non-empty closed convex set then for any $u \in U$ there is a unique $v \in C$ such that $\|u - v\| = \min\{\|u - w\| : w \in C\}$.

(b) Every closed linear subspace of U is the image of an orthogonal projection, that is, has an orthogonal complement.

(c) Every member of U^* is of the form $u \mapsto (u|v)$ for some $v \in U$.

(d) U is reflexive.

(e) If $C \subseteq U$ is a norm-closed convex set then it is weakly closed.

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562 Borel codes

The concept of 'Borel set', either in the real line or in general topological spaces, has been fundamental in measure theory since before the modern subject existed. It is at this point that the character of the subject changes if we do not allow ourselves even the countable axiom of choice. I have already mentioned the Feferman-Lévy model in which $\mathbb{R}$ is a countable union of countable sets; immediately, every subset of $\mathbb{R}$ is a countable union of countable sets and is 'Borel' on the definition of 111G. In these circumstances that definition becomes unhelpful.

An alternative which leads to a non-trivial theory, coinciding with the usual theory in the presence of AC, is the algebra of 'codable Borel sets' (562B). This is not necessarily a σ-algebra, but is closed under unions and intersections of 'codable sequences' (562K). When we come to look for measurable functions, the corresponding concept is that of 'codable Borel function' (562L); again, we do not expect the limit of an arbitrary sequence of codable Borel functions to be measurable in any useful sense, but the limit of a codable sequence of codable Borel functions is again a codable Borel function (562Ne). The same ideas can be used to give a theory of 'codable Baire sets' in any topological space (562T).

562A Trees (a) Set $S^* = \bigcup_{n\ge1} \mathbb{N}^n$. For $\sigma \in S^*$ and $T \subseteq S^*$, write T_σ for $\{\tau : \tau \in S^*, \sigma^\frown\tau \in T\}$ (notation: 5A1C).

(b) Let $\mathcal{T}_0$ be the family of sets $T \subseteq S^*$ such that $\sigma\restriction n \in T$ whenever $\sigma \in T$ and $n \ge 1$. Recall from 421N[1] that we have a derivation $\partial : \mathcal{T}_0 \to \mathcal{T}_0$ defined by setting

$$\partial T = \{\sigma : \sigma \in S^*, T_\sigma \ne \emptyset\},$$

with iterates ∂^ξ, for $\xi < \omega_1$, defined by setting

$$\partial^0 T = T, \quad \partial^\xi T = \bigcap_{\eta<\xi} \partial(\partial^\eta T) \text{ for } \xi \ge 1.$$

Now for any $T \in \mathcal{T}_0$ there is a $\xi < \omega_1$ such that $\partial^\xi T = \partial^\eta T$ whenever $\xi \le \eta < \omega_1$.

(c) We therefore still have a rank function $r : \mathcal{T}_0 \to \omega_1$ defined by saying that $r(T)$ is the least ordinal such that $\partial^{r(T)} T = \partial^{r(T)+1} T$. Now $\partial^{r(T)} T$ is empty iff there is no $\alpha \in \mathbb{N}^\mathbb{N}$ such that $\alpha\restriction n \in T$ for every $n \ge 1$.

Let $\mathcal{T}$ be the set of those $T \in \mathcal{T}_0$ with no infinite branch, that is, such that $\partial^{r(T)} T = \emptyset$.

(d) For $T \in \mathcal{T}$, set $A_T = \{i : {<}i{>} \in T\}$. We need a fact not covered in §421: for any $T \in \mathcal{T}$, $r(T) = \sup\{r(T_{<i>}) + 1 : i \in A_T\}$.

562B Coding sets with trees (a) Let X be a set and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of X. Define $\phi : \mathcal{T} \to \mathcal{P}X$ inductively by saying that

$$\begin{aligned}\phi(T) &= \bigcup_{i\in A_T} E_i \text{ if } r(T) \le 1,\\ &= \bigcup_{i\in A_T} X \setminus \phi(T_{<i>}) \text{ if } r(T) > 1.\end{aligned}$$

[1] Early editions of Volume 4 used a slightly different definition of iterated derivations, so that the 'rank' of a tree was not quite the same.

I will call ϕ the **interpretation of Borel codes** defined by X and $\langle E_n\rangle_{n\in\mathbb{N}}$.

(b) Of course $\phi(\emptyset)=\emptyset$. If we set

$$T^*=\{<0>,<0>^\frown<0>,<0>^\frown<0>^\frown<0>,<1>,<1>^\frown<0>\}$$

and

$$T=\{<0>\}\cup\{<0>^\frown\sigma:\sigma\in T^*\},$$

then

$$\phi(T^*)=X,\quad \phi(T)=\emptyset,$$

while $T\ne\emptyset$.

(c) Now suppose that X is a second-countable topological space and that $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ are two sequences running over bases for the topology of X. Let $\phi:\mathcal{T}\to\mathcal{P}X$ and $\phi':\mathcal{T}\to\mathcal{P}X$ be the interpretations of Borel codes defined by $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ respectively. Then there is a function $\Theta:\mathcal{T}\to\mathcal{T}\setminus\{\emptyset\}$ such that $\phi'\Theta=\phi$.

(d) Now say that a **codable Borel set** in X is one expressible as $\phi(T)$ for some $T\in\mathcal{T}$, starting from some sequence running over a base for the topology of X. I will write $\mathcal{B}_c(X)$ for the family of codable Borel sets of X.

The definition of 'interpretation of Borel codes' makes it plain that any σ-algebra of subsets of X containing every open set will also contain every codable Borel set; every codable Borel set is a 'Borel set' on the definition of 111G or 4A3A.

562C (a) For instance, there are functions $\Theta_0:\mathcal{T}\to\mathcal{T}$, $\Theta_1:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$, $\Theta_2:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$, $\Theta_3:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ such that, for any interpretation ϕ of Borel codes,

$$\phi(\Theta_0(T))=X\setminus\phi(T),\quad \phi(\Theta_1(T,T'))=\phi(T)\cup\phi(T'),$$

$$\phi(\Theta_2(T,T'))=\phi(T)\cap\phi(T'),\quad \phi(\Theta_3(T,T'))=\phi(T)\setminus\phi(T')$$

for all T, $T'\in\mathcal{T}$.

(b) For any countable set K we have functions $\tilde\Theta_1$, $\tilde\Theta_2:\bigcup_{J\subseteq K}\mathcal{T}^J\to\mathcal{T}$ such that whenever X is a set, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X and ϕ is the corresponding interpretation of Borel codes, then $\phi(\tilde\Theta_1(\tau))=\bigcup_{j\in J}\phi(\tau(j))$ and $\phi(\tilde\Theta_2(\tau))=X\cap\bigcap_{j\in J}\phi(\tau(j))$ whenever $J\subseteq K$ and $\tau\in\mathcal{T}^J$.

(c) Let X be a regular second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X containing $\emptyset$, and $\phi:\mathcal{T}\to\mathcal{P}X$ the associated interpretation of Borel codes. Then there are functions Θ_1', $\Theta_2':\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ such that

$$\phi(\Theta_1'(T,T'))=\phi(T)\cup\phi(T'),\quad \phi(\Theta_2'(T,T'))=\phi(T)\cap\phi(T'),$$

$$r(\Theta_1'(T,T'))=r(\Theta_2'(T,T'))=\max(r(T),r(T'))$$

for all T, $T'\in\mathcal{T}$.

562D Proposition (a) If X is a second-countable space, then the family of codable Borel subsets of X is an algebra of subsets of X containing every G_δ set and every F_σ set.

(b) [AC(ω)] Every Borel set is a codable Borel set.

562E Proposition Let X be a second-countable space and $Y\subseteq X$ a subspace of X. Then $\mathcal{B}_c(Y)=\{Y\cap E:E\in\mathcal{B}_c(X)\}$.

***562F Theorem** (a) If X is a Hausdorff second-countable space and A, B are disjoint analytic subsets of X, there is a codable Borel set $E\subseteq X$ such that $A\subseteq E$ and $B\cap E=\emptyset$.

(b) Let X be a Polish space. Then a subset E of X is a codable Borel set iff E and $X\setminus E$ are analytic.

562G Resolvable sets: Definition I will say that a subset E of a topological space X is **resolvable** if there is no non-empty set $A \subseteq X$ such that $A \subseteq \overline{A \cap E} \cap \overline{A \setminus E}$.

562H Proposition Let X be a topological space, and $\mathcal{E}$ the set of resolvable subsets of X. Then $\mathcal{E}$ is an algebra of sets containing every open subset of X.

562I Theorem Let X be a second-countable space, $\langle U_n \rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ the associated interpretation of Borel codes. Let $\mathcal{E}$ be the algebra of resolvable subsets of X. Then there is a function $\psi : \mathcal{E} \to \mathcal{T}$ such that $\phi(\psi(E)) = E$ for every $E \in \mathcal{E}$.

562J Codable families of sets Let X be a second-countable space and $\mathcal{B}_c(X)$ the algebra of codable Borel subsets of X. Let $\langle U_n \rangle_{n\in\mathbb{N}}$, $\langle V_n \rangle_{n\in\mathbb{N}}$ be sequences running over bases for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$, $\phi' : \mathcal{T} \to \mathcal{B}_c(X)$ the corresponding interpretations of Borel codes. Let us say that a family $\langle E_i \rangle_{i\in I}$ is **ϕ-codable** if there is a family $\langle T^{(i)} \rangle_{i\in I}$ in $\mathcal{T}$ such that $\phi(T^{(i)}) = E_i$ for every $i \in I$. Then $\langle E_i \rangle_{i\in I}$ is ϕ-codable iff it is ϕ'-codable.

We may therefore say that a family $\langle E_i \rangle_{i\in I}$ in $\mathcal{B}_c(X)$ is **codable** if it is ϕ-codable for some interpretation of Borel codes defined by the procedure of 562B from a sequence running over a base for the topology of X.

Note that any finite family in $\mathcal{B}_c(X)$ is codable, and that any family of resolvable sets is codable; also any subfamily of a codable family is codable. Slightly more generally, if $\langle E_i \rangle_{i\in I}$ is a codable family in $\mathcal{B}_c(X)$, J is a set, and $f : J \to I$ is a function, then $\langle E_{f(j)} \rangle_{j\in J}$ is codable. If $\langle E_i \rangle_{i\in I}$ and $\langle F_i \rangle_{i\in I}$ are codable families in $\mathcal{B}_c(X)$, then so are $\langle X \setminus E_i \rangle_{i\in I}$, $\langle E_i \cup F_i \rangle_{i\in I}$, $\langle E_i \cap F_i \rangle_{i\in I}$ and $\langle E_i \setminus F_i \rangle_{i\in I}$.

562K Proposition Let X be a second-countable space and $\langle E_n \rangle_{n\in\mathbb{N}}$ a codable sequence in $\mathcal{B}_c(X)$. Then

(a) $\bigcup_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{B}_c(X)$;

(b) $\langle \bigcup_{i<n} E_i \rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$;

(c) $\langle E_n \setminus \bigcup_{i<n} E_i \rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$.

562L Codable Borel functions Let X and Y be second-countable spaces. A function $f : X \to Y$ is a **codable Borel function** if $\langle f^{-1}[H] \rangle_{H\subseteq Y \text{ is open}}$ is a codable family in $\mathcal{B}_c(X)$.

562M Theorem Let X be a second-countable space, $\langle U_n \rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ the corresponding interpretation of Borel codes.

(a) If Y is another second-countable space, $\langle V_n \rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of Y containing $\emptyset$, $\phi_Y : \mathcal{T} \to \mathcal{B}_c(Y)$ the corresponding interpretation of Borel codes, and $f : X \to Y$ is a function, then the following are equiveridical:

(i) f is a codable Borel function;

(ii) $\langle f^{-1}[V_n] \rangle_{n\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}_c(X)$;

(iii) there is a function $\Theta : \mathcal{T} \to \mathcal{T}$ such that $\phi(\Theta(T)) = f^{-1}[\phi_Y(T)]$ for every $T \in \mathcal{T}$.

(b) If Y and Z are second-countable spaces and $f : X \to Y$, $g : Y \to Z$ are codable Borel functions then $gf : X \to Z$ is a codable Borel function.

(c) If Y and Z are second-countable spaces and $f : X \to Y$, $g : X \to Z$ are codable Borel functions then $x \mapsto (f(x), g(x))$ is a codable Borel function from X to $Y \times Z$.

(d) If Y is a second-countable space then any continuous function from X to Y is a codable Borel function.

562N Proposition Let X be a second-countable space, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ the interpretation of Borel codes associated with some sequence running over a base for the topology of X.

(a) If $f : X \to \mathbb{R}$ is a function, the following are equiveridical:

(i) f is a codable Borel function;

(ii) the family $\langle \{x : f(x) > \alpha\} \rangle_{\alpha\in\mathbb{R}}$ is codable;

(iii) $\langle \{x : f(x) > q\} \rangle_{q\in\mathbb{Q}}$ is codable.

(b) Write $\tilde{\mathcal{T}}$ for the set of functions $\tau : \mathbb{R} \to \mathcal{T}$ such that

$$\phi(\tau(\alpha)) = \bigcup_{\beta>\alpha} \phi(\tau(\beta)) \text{ for every } \alpha \in \mathbb{R},$$

$$\bigcap_{n\in\mathbb{N}}\phi(\tau(n))=\emptyset,\quad \bigcup_{n\in\mathbb{N}}\phi(\tau(-n))=X.$$

Then

(i) for every $\tau\in\tilde{\mathcal{T}}$ there is a unique codable Borel function $\tilde\phi(\tau):X\to\mathbb{R}$ such that $\phi(\tau(\alpha))=\{x:\tilde\phi(\tau)(x)>\alpha\}$ for every $\alpha\in\mathbb{R}$;

(ii) every codable Borel function from X to $\mathbb{R}$ is expressible as $\tilde\phi(\tau)$ for some $\tau\in\tilde{\mathcal{T}}$.

(c) If $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$ such that $f(x)=\sup_{n\in\mathbb{N}}\tilde\phi(\tau_n)(x)$ is finite for every $x\in X$, then f is a codable Borel function.

(d) If f, $g:X\to\mathbb{R}$ are codable Borel functions and $\gamma\in\mathbb{R}$, then $f+g$, γf, $|f|$ and $f\times g$ are codable Borel functions.

(e) If $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$, then there is a codable Borel function f such that $\liminf_{n\to\infty}\tilde\phi(\tau_n)(x)=f(x)$ whenever the lim inf is finite.

(f) A subset E of X belongs to $\mathcal{B}_c(X)$ iff $\chi E:X\to\mathbb{R}$ is a codable Borel function.

562O Remarks (a) For some purposes there are advantages in coding real-valued functions by functions from $\mathbb{Q}$ to $\mathcal{T}$ rather than by functions from $\mathbb{R}$ to $\mathcal{T}$.

(b) As in 562C, the constructions here are largely determinate. For instance, the function Θ of 562M(a-iii) can be built by a definite rule from the sequence $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$ provided by the hypothesis (a-ii) there. What this means is that if we have a family $\langle(Y_i,\langle V_{in}\rangle_{n\in\mathbb{N}},f_i)\rangle_{i\in I}$ such that Y_i is a second-countable space, $\langle V_{in}\rangle_{n\in\mathbb{N}}$ is a sequence running over a base for the topology of Y_i, and $f_i:X\to Y_i$ is a continuous function for each $i\in I$, then there will be a function $\tilde\Theta:\mathcal{T}\times I\to\mathcal{T}$ such that $\phi(\tilde\Theta(T,i))=f_i^{-1}[\phi_i(T)]$ for every $i\in I$ and $T\in\mathcal{T}$, where $\phi_i:\mathcal{T}\to\mathcal{B}_c(Y_i)$ is the interpretation of Borel codes corresponding to the sequence $\langle V_{in}\rangle_{n\in\mathbb{N}}$.

(c) Similarly, we have a function $\tilde\Theta_1:\tilde{\mathcal{T}}\times\tilde{\mathcal{T}}\to\tilde{\mathcal{T}}$ such that $\tilde\phi(\tilde\Theta_1(\tau,\tau'))$ will always be $\tilde\phi(\tau)-\tilde\phi(\tau')$ for τ, $\tau'\in\tilde{\mathcal{T}}$. Equally, we have a function $\tilde\Theta_1^*:\tilde{\mathcal{T}}^{\mathbb{N}}\to\tilde{\mathcal{T}}^{\mathbb{N}}$ such that

$$\tilde\phi(\tilde\Theta_1^*(\langle\tau_n\rangle_{n\in\mathbb{N}})(m))=\inf_{n\ge m}\tilde\phi(\tau_n)$$

for every m whenever $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$ such that $\inf_{n\in\mathbb{N}}\tilde\phi(\tau_n)$ is defined as a real-valued function on X.

562P Codable Borel equivalence (a) If X is a set, we can say that two second-countable topologies $\mathfrak{S}$, $\mathfrak{T}$ on X are **codably Borel equivalent** if the identity functions $(X,\mathfrak{S})\to(X,\mathfrak{T})$ and $(X,\mathfrak{T})\to(X,\mathfrak{S})$ are codable Borel functions. In this case, $\mathfrak{S}$ and $\mathfrak{T}$ give the same families of codable Borel functions and the same algebra $\mathcal{B}_c(X)$.

(b) If $(X,\mathfrak{T})$ is a second-countable space and $\langle E_n\rangle_{n\in\mathbb{N}}$ is any codable sequence in $\mathcal{B}_c(X)$, there is a topology $\mathfrak{S}$ on X, generated by a countable algebra of subsets of X, such that $\mathfrak{S}$ and $\mathfrak{T}$ are codably Borel equivalent and every E_n belongs to $\mathfrak{S}$.

562Q Resolvable functions Let X be a topological space. I will say that a function $f:X\to[-\infty,\infty]$ is **resolvable** if whenever $\alpha<\beta$ in $\mathbb{R}$ and $A\subseteq X$ is a non-empty set, then one of $\{x:x\in A,\,f(x)\le\alpha\}$, $\{x:x\in A,\,f(x)\ge\beta\}$ is not dense in A.

Examples (a) Any semi-continuous function from X to $[-\infty,\infty]$ is resolvable.

(b) If $f:X\to\mathbb{R}$ is such that $\{x:f(x)>\alpha\}$ is resolvable for every α, then f is resolvable. In particular, the indicator function of a resolvable set is resolvable.

(c) A function $f:\mathbb{R}\to\mathbb{R}$ which has bounded variation on every bounded set is resolvable.

562R Theorem Let X be a second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ the associated interpretation of Borel codes. Let $\mathcal{R}$ be the family of resolvable real-valued functions on X. Then there is a function $\tilde\psi:\mathcal{R}\to\mathcal{T}^{\mathbb{R}}$ such that

$$\phi(\tilde\psi(f)(\alpha))=\{x:f(x)>\alpha\}$$

for every $f\in\mathcal{R}$ and $\alpha\in\mathbb{R}$.

562S Codable families of codable functions (a) If X and Y are second-countable spaces, a family $\langle f_i\rangle_{i\in I}$ of functions from X to Y is a **codable family of codable Borel functions** if $\langle f_i^{-1}[H]\rangle_{i\in I,H\subseteq Y\text{ is open}}$ is a codable family in $\mathcal{B}_c(X)$.

(b) Uniformizing the arguments of 562N, it is easy to check that a family $\langle f_i\rangle_{i\in I}$ of real-valued functions on X is a codable family of codable Borel functions iff there is a family $\langle\tau_i\rangle_{i\in I}$ in $\tilde{\mathcal{T}}$ such that $f_i=\tilde\phi(\tau_i)$ for every i.

(c) 562Ne can be rephrased as

if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of real-valued codable Borel functions on X, there is a codable Borel function f such that $f(x)=\liminf_{n\to\infty}f_n(x)$ whenever the lim inf is finite,

and 562R implies that

the family of resolvable real-valued functions on X is a codable family of codable Borel functions.

(d) If X, Y and Z are second-countable spaces, $\langle f_i\rangle_{i\in I}$ is a codable family of codable Borel functions from X to Y, and $\langle g_i\rangle_{i\in I}$ is a codable family of codable Borel functions from Y to Z, then $\langle g_if_i\rangle_{i\in I}$ is a codable family of codable functions from X to Z.

562T Codable Baire sets Start by settling on a sequence running over a base for the topology of $\mathbb{R}^{\mathbb{N}}$, with the associated interpretation $\phi:\mathcal{T}\to\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ of Borel codes. Let X be a topological space.

(a) A subset E of X is a **codable Baire set** if it is of the form $f^{-1}[F]$ for some continuous $f:X\to\mathbb{R}^{\mathbb{N}}$ and $F\in\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$; write $\mathcal{B}\mathfrak{a}_c(X)$ for the family of such sets. If $E\in\mathcal{B}\mathfrak{a}_c(X)$, then a **code** for E will be a pair (f,T) where $f:X\to\mathbb{R}^{\mathbb{N}}$ is continuous, $T\in\mathcal{T}$ and $E=f^{-1}[\phi(T)]$. A family $\langle E_i\rangle_{i\in I}$ in $\mathcal{B}\mathfrak{a}_c(X)$ is a **codable** family if there is a family $\langle(f_i,T^{(i)})\rangle_{i\in I}$ such that $(f_i,T^{(i)})$ codes E_i for every i.

(b)(i) Suppose that $\langle f_i\rangle_{i\in I}$ is a countable family of continuous functions from X to $\mathbb{R}^{\mathbb{N}}$, and $\langle T^{(i)}\rangle_{i\in I}$ a family in $\mathcal{T}$. Then there are a continuous function $f:X\to\mathbb{R}^{\mathbb{N}}$ and a sequence $\langle\hat T^{(i)}\rangle_{i\in\mathbb{N}}$ in $\mathcal{T}$ such that $(f,\hat T^{(i)})$ codes the same Baire set as $(f_i,T^{(i)})$ for every $i\in I$.

(ii) It follows that if $\langle E_i\rangle_{i\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}\mathfrak{a}_c(X)$ then $\bigcup_{i\in\mathbb{N}}E_i$ and $\bigcap_{i\in\mathbb{N}}E_i$ belong to $\mathcal{B}\mathfrak{a}_c(X)$.

(iii) $\mathcal{B}\mathfrak{a}_c(X)$ is an algebra of subsets of X. Every zero set belongs to $\mathcal{B}\mathfrak{a}_c(X)$.

(iv) If Y is another topological space and $g:X\to Y$ is continuous, then $\langle g^{-1}[F_i]\rangle_{i\in I}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$ for every codable family $\langle F_i\rangle_{i\in I}$ in $\mathcal{B}\mathfrak{a}_c(Y)$.

(c)(i) A function $f:X\to\mathbb{R}$ is a **codable Baire function** if there are a continuous $g:X\to\mathbb{R}^{\mathbb{N}}$ and a codable Borel function $h:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ such that $f=hg$. A family $\langle f_i\rangle_{i\in I}$ of codable Baire functions is a **codable** family if there is a family $\langle(g_i,h_i)\rangle_{i\in I}$ such that $g_i:X\to\mathbb{R}^{\mathbb{N}}$ is a continuous function for every $i\in I$ and $\langle h_i\rangle_{i\in I}$ is a codable family of codable Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$.

(ii) Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions from X to $\mathbb{R}$. Then there are a continuous function $g:X\to\mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of codable Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$ such that $f_n=h_ng$ for every $n\in\mathbb{N}$.

(iii) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions, there is a codable Baire function f such that $f(x)=\liminf_{n\to\infty}f_n(x)$ whenever the lim inf is finite.

(iv) The family of codable Baire functions is a Riesz subspace of $\mathbb{R}^X$ containing all continuous functions and closed under multiplication.

(v) The family of continuous real-valued functions on X is a codable family of codable Baire functions.

(vi) If $E\subseteq X$, then $E\in\mathcal{B}\mathfrak{a}_c(X)$ iff $\chi E:X\to\mathbb{R}$ is a codable Baire function.

(d) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions from X to $\mathbb{R}$, then $\langle f_n^{-1}[H]\rangle_{n\in\mathbb{N},H\subseteq\mathbb{R}\text{ is open}}$ is codable.

562U Proposition Let $(X,\mathfrak{T})$ be a second-countable space. Then there is a second-countable topology $\mathfrak{S}$ on X, codably Borel equivalent to $\mathfrak{T}$, such that $\mathcal{B}_c(X)=\mathcal{B}\mathfrak{a}_c(X,\mathfrak{S})$ and the codable families in $\mathcal{B}_c(X)$ are exactly the codable families in $\mathcal{B}\mathfrak{a}_c(X,\mathfrak{S})$.

562V Theorem (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Then we have an interpretation $\phi:\mathcal{T}\to\mathfrak{A}$ of Borel codes such that

$$\begin{aligned}\phi(T)&=\sup_{i\in A_T} a_i \text{ if } r(T)\le 1,\\ &=\sup_{i\in A_T} 1\setminus\phi(T_{<i>}) \text{ if } r(T)>1,\end{aligned}$$

where $A_T=\{i:<i>\in T\}$ as usual.

(b) For $n\in\mathbb{N}$, set $E_n=\{x:x\in\{0,1\}^{\mathbb{N}},\,x(n)=1\}$. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Let $\phi:\mathcal{T}\to\mathfrak{A}$ and $\psi:\mathcal{T}\to\mathcal{P}(\{0,1\}^{\mathbb{N}})$ be the interpretations of Borel codes corresponding to $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle E_n\rangle_{n\in\mathbb{N}}$. If T, $T'\in\mathcal{T}$ are such that $\phi(T)\not\subseteq\phi(T')$, then $\psi(T)\not\subseteq\psi(T')$.

Version of 3.12.13

563 Borel measures without choice

Having decided that a 'Borel set' is to be one obtainable by a series of operations described by a Borel code, it is a natural step to say that a 'Borel measure' should be one which respects these operations (563A). In regular spaces, such measures have strong inner and outer regularity properties also based on the Borel coding (563D-563F), and we have effective methods of constructing such measures (563H). Analytic sets are universally measurable (563I). We can use similar ideas to give a theory of Baire measures on general topological spaces (563J-563K). In the basic case, of a second-countable space with a codably σ-finite measure, we have a measure algebra with many of the same properties as in the standard theory (563M-563N).

The theory would not be very significant if there were no interesting Borel-coded measures, so you may wish to glance ahead to §565 to confirm that Lebesgue measure can be brought into the framework developed here.

563A Definitions (a) Let X be a second-countable space and $\mathcal{B}_c(X)$ the algebra of codable Borel subsets of X. I will say that a **Borel-coded measure** on X is a functional $\mu:\mathcal{B}_c(X)\to[0,\infty]$ such that $\mu\emptyset=0$ and $\mu(\bigcup_{n\in\mathbb{N}}E_n)=\sum_{n=0}^{\infty}\mu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint codable family in $\mathcal{B}_c(X)$.

(b) I will say that a subset of X is **negligible** if it is included in a set of measure 0. We can now define the **completion** of μ to be the natural extension of μ to the algebra $\{E\triangle A:E\in\mathcal{B}_c(X),\,A \text{ is } \mu\text{-negligible}\}$.

(c) I will say that a Borel-coded measure μ is **semi-finite** if $\sup\{\mu F:F\subseteq E,\,\mu F<\infty\}=\infty$ whenever $\mu E=\infty$.

(d) A Borel-coded measure on X is **codably σ-finite** if there is a codable sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(X)$ such that $X=\bigcup_{n\in\mathbb{N}}E_n$ and μE_n is finite for every n.

563B Proposition Let $(X,\mathfrak{T})$ be a second-countable space and μ a Borel-coded measure on X.

(a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in $\mathcal{B}_c(X)$.

(i) $\mu(\bigcup_{n\in\mathbb{N}}E_n)\le\sum_{n=0}^{\infty}\mu E_n$.

(ii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}}E_n)=\lim_{n\to\infty}\mu E_n$.

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and μE_0 is finite, then $\mu(\bigcap_{n\in\mathbb{N}}E_n)=\lim_{n\to\infty}\mu E_n$.

(b) μ is τ-additive.

(c) Suppose that $\mathfrak{T}$ is T_1. If $\mathcal{E}$ is the algebra of resolvable subsets of X, then $\mu\restriction\mathcal{E}$ is countably additive in the sense that $\mu E = \sum_{n=0}^{\infty} \mu E_n$ for any disjoint family $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $E = \sup_{n\in\mathbb{N}} E_n$ is defined in $\mathcal{E}$.

563C Corollary Let X be a second-countable space, μ a Borel-coded measure on X and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of resolvable sets in X.

(a)(i) $\bigcup_{n\in\mathbb{N}} E_n$ is measurable;

(ii) $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n$;

(iii) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$;

(iv) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$.

(b)(i) $\bigcap_{n\in\mathbb{N}} E_n$ is measurable;

(ii) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\inf_{n\in\mathbb{N}} \mu E_n$ is finite, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$.

563D Lemma Let $(X,\mathfrak{T})$ be a regular second-countable space and $\mu : \mathfrak{T} \to [0,\infty]$ a functional such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G \subseteq H$,

μ is modular,

$\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$,

$\bigcup\{G : G \in \mathfrak{T},\ \mu G < \infty\} = X$.

(a) $\mu(\bigcup_{i\in I} G_i) \le \sum_{i\in I} \mu G_i$ for every countable family $\langle G_i\rangle_{i\in I}$ in $\mathfrak{T}$.

(b) There is a function $\pi^* : \mathfrak{T} \times \mathbb{N} \to \mathfrak{T}$ such that

$$X \setminus G \subseteq \pi^*(G,k), \quad \mu(G \cap \pi^*(G,k)) \le 2^{-k}$$

whenever $G \in \mathfrak{T}$ and $k \in \mathbb{N}$.

(c) Let $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ be an interpretation of Borel codes defined from a sequence running over $\mathfrak{T}$. Then there are functions π, $\pi' : \mathcal{T} \times \mathbb{N} \to \mathfrak{T}$ such that

$$\phi(T) \subseteq \pi(T,n), \quad X \setminus \phi(T) \subseteq \pi'(T,n), \quad \mu(\pi(T,n) \cap \pi'(T,n)) \le 2^{-n}$$

for every $T \in \mathcal{T}$ and $n \in \mathbb{N}$.

563E Lemma Let X be a second-countable space and M a non-empty upwards-directed set of Borel-coded measures on X. For each codable Borel set $E \subseteq X$, set $\nu E = \sup_{\mu\in\mathrm{M}} \mu E$. Then ν is a Borel-coded measure on X.

563F Proposition Let $(X,\mathfrak{T})$ be a second-countable space and μ a Borel-coded measure on X.

(a) For any $F \in \mathcal{B}_c(X)$, we have a Borel-coded measure μ_F on X defined by saying that $\mu_F E = \mu(E \cap F)$ for every $E \in \mathcal{B}_c(X)$.

(b) We have a semi-finite Borel-coded measure μ_{sf} defined by saying that

$$\mu_{\mathrm{sf}}(E) = \sup\{\mu F : F \in \mathcal{B}_c(X),\ F \subseteq E,\ \mu F < \infty\}$$

for every $E \in \mathcal{B}_c(X)$.

(c)(i) If μ is locally finite it is codably σ-finite.

(ii) If μ is codably σ-finite, it is semi-finite and there is a totally finite Borel-coded measure ν on X with the same null ideal as μ.

(iii) If μ is codably σ-finite, there is a non-decreasing codable sequence of codable Borel sets of finite measure which covers X.

(d) If X is regular then the following are equiveridical:

(i) μ is locally finite;

(ii) μ is semi-finite, outer regular with respect to the open sets and inner regular with respect to the closed sets;

(iii) μ is semi-finite and outer regular with respect to the open sets.

(e) If X is regular and μ is semi-finite, then μ is inner regular with respect to the closed sets of finite measure.

(f) If X is Polish and μ is semi-finite, then μ is inner regular with respect to the compact sets.

(g) If μ is locally finite, and ν is another Borel-coded measure on X agreeing with μ on the open sets, then $\nu = \mu$.

563G Proposition Let X be a set and $\theta : \mathcal{P}X \to [0, \infty]$ a submeasure.

(a)

$$\Sigma = \{E : E \subseteq X,\, \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}$$

is an algebra of subsets of X, and $\theta\restriction\Sigma$ is additive in the sense that $\theta(E \cup F) = \theta E + \theta F$ in $[0, \infty]$ whenever E, $F \in \Sigma$ are disjoint.

(b) If $E \subseteq X$ and for every $\epsilon > 0$ there is an $F \in \Sigma$ such that $E \subseteq F$ and $\theta(F \setminus E) \le \epsilon$, then $E \in \Sigma$.

563H Theorem Let $(X, \mathfrak{T})$ be a regular second-countable space and $\mu : \mathfrak{T} \to [0, \infty]$ a functional such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G \subseteq H$,

mu is modular,

$\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$,

$\bigcup\{G : G \in \mathfrak{T},\, \mu G < \infty\} = X$.

Then μ has a unique extension to a Borel-coded measure on X.

563I Theorem Let X be a Hausdorff second-countable space, μ a codably σ-finite Borel-coded measure on X, and $A \subseteq X$ an analytic set. Then there are a codable Borel set $E \supseteq A$ and a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact subsets of A such that $E \setminus \bigcup_{n\in\mathbb{N}} K_n$ is negligible. Consequently A is measured by the completion of μ.

563J Baire-coded measures If X is a topological space, and $\mathcal{B}\mathfrak{a}_c(X)$ its algebra of codable Baire sets, a **Baire-coded measure** on X will be a function $\mu : \mathcal{B}\mathfrak{a}_c(X) \to [0, \infty]$ such that $\mu\emptyset = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ for every disjoint codable sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}\mathfrak{a}_c(X)$.

563K Proposition (a) If X and Y are topological spaces, $f : X \to Y$ is a continuous function and μ is a Baire-coded measure on X, then $F \mapsto \mu f^{-1}[F] : \mathcal{B}\mathfrak{a}_c(Y) \to [0, \infty]$ is a Baire-coded measure on Y.

(b) Suppose that μ is a Baire-coded measure on a topological space X, and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$. Then

(i) $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n$;

(ii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$;

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and μE_0 is finite, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$.

(c) Let X be a topological space and M a non-empty upwards-directed family of Baire-coded measures on X. Set $\nu E = \sup_{\mu\in\mathrm{M}} \mu E$ for every codable Baire set $E \subseteq X$. Then ν is a Baire-coded measure on X.

563L Proposition Suppose that X is a topological space; write $\mathcal{G}$ for the lattice of cozero subsets of X. Let $\mu : \mathcal{G} \to [0, \infty]$ be such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G \subseteq H$,

μ is modular,

$\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n$ whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{G}$ and there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of continuous functions from X to $\mathbb{R}$ such that $G_n = \{x : f_n(x) \ne 0\}$ for every n,

$\mu G = \sup\{\mu H : H \in \mathcal{G},\, H \subseteq G,\, \mu H < \infty\}$ for every $G \in \mathcal{G}$.

Then there is a Baire-coded measure on X extending μ; if μX is finite, then the extension is unique.

563M Measure algebras If μ is either a Borel-coded measure or a Baire-coded measure, we can form the quotient Boolean algebra $\mathfrak{A} = \operatorname{dom}\mu/\{E : \mu E = 0\}$ and the functional $\bar\mu : \mathfrak{A} \to [0, \infty]$ defined by setting

$\bar\mu E^\bullet = \mu E$ for every $E \in \operatorname{dom}\mu$; $\bar\mu$ is a strictly positive additive functional from $\mathfrak{A}$ to $[0,\infty]$. As in §323, we have a topology and uniformity on $\mathfrak{A}$ defined by the pseudometrics $(a,b) \mapsto \bar\mu(c \cap (a \triangle b))$ for $c \in \mathfrak{A}$ of finite measure; if μ is semi-finite, the topology is Hausdorff.

563N Theorem Let X be a second-countable space, and μ a codably σ-finite Borel-coded measure on X. Let $\mathfrak{A}$ and $\bar\mu$ be as in 563M. Then $\mathfrak{A}$ is complete for its measure-algebra uniformity, therefore Dedekind complete.

563Z Problem Suppose we define 'probability space' in the conventional way, following literally the formulations in 111A, 112A and 211B. Is it relatively consistent with ZF to suppose that every probability space is purely atomic in the sense of 211K?

Version of 9.2.14

564 Integration without choice

I come now to the problem of defining an integral with respect to a Borel- or Baire-coded measure. Since a Borel-coded measure can be regarded as a Baire-coded measure on a second-countable space, I will give the basic results in terms of the wider class. I seek to follow the general plan of Chapter 12, starting from simple functions and taking integrable functions to be almost-everywhere limits of sequences of simple functions (564A); the concept of 'virtually measurable' function has to be re-negotiated (564Ab). The basic convergence theorems from §123 are restricted but recognisable (564F). We also have versions of two of the representation theorems from §436 (564H, 564I).

There is a significant change when we come to the completeness of L^p spaces (564K) and the Radon-Nikodým theorem (564L), where it becomes necessary to choose sequences, and we need a well-orderable dense set of functions to pick from. Subject to this, we have workable notions of conditional expectation operator (564Mc) and product measures (564N, 564O).

564A Definitions (a) Given a topological space X and a Baire-coded measure μ on X, I will write $\mathcal{B}\mathfrak{a}_c(X)^f$ for the ring of codable Baire sets of finite measure; $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ will be the linear subspace of $\mathbb{R}^X$ generated by $\{\chi E : E \in \mathcal{B}\mathfrak{a}_c(X)^f\}$. S is a Riesz subspace of $\mathbb{R}^X$, and also an f-algebra.

(b) I will write $\mathcal{L}^0$ for the space of real-valued functions f defined almost everywhere in X such that there is a codable Baire function $g : X \to \mathbb{R}$ such that $f =_{\text{a.e.}} g$.

(c) Let $\int : S \to \mathbb{R}$ be the positive linear functional defined by saying that $\int \chi E = \mu E$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)^f$.

(d) $\mathcal{L}^1$ will be the set of those real-valued functions f defined almost everywhere in X for which there is a codable sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in S converging to f almost everywhere and such that $\sum_{n=0}^\infty \int |h_{n+1} - h_n| < \infty$; I will call such functions **integrable**.

564B Lemma Let X be a topological space and μ a Baire-coded measure on X.

(a) $\mathcal{L}^1 \subseteq \mathcal{L}^0$.

(b) If $\langle h_n \rangle_{n\in\mathbb{N}}$ is a non-increasing codable sequence in $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ and $\lim_{n\to\infty} h_n(x) = 0$ for almost every x, then $\lim_{n\to\infty} \int h_n = 0$.

(c) If $\langle h_n \rangle_{n\in\mathbb{N}}$ and $\langle h'_n \rangle_{n\in\mathbb{N}}$ are two codable sequences in S such that $\lim_{n\to\infty} h_n$ and $\lim_{n\to\infty} h'_n$ are defined and equal almost everywhere, and $\sum_{n=0}^\infty \int |h_{n+1} - h_n|$ and $\sum_{n=0}^\infty \int |h'_{n+1} - h'_n|$ are both finite, then $\lim_{n\to\infty} \int h_n$ and $\lim_{n\to\infty} \int h'_n$ are defined and equal.

(d) If $\langle h_n \rangle_{n\in\mathbb{N}}$ is a codable sequence in S and $\sum_{n=0}^\infty \int |h_{n+1} - h_n|$ is finite, then $\langle h_n \rangle_{n\in\mathbb{N}}$ converges almost everywhere. In particular, if $\langle h_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in S and $\sup_{n\in\mathbb{N}} \int h_n$ is finite, $\langle h_n \rangle_{n\in\mathbb{N}}$ converges a.e.

(e) If $\langle h_n \rangle_{n\in\mathbb{N}}$ is a codable sequence in S^+ and $\liminf_{n\to\infty} \int h_n = 0$, then $\liminf_{n\to\infty} h_n = 0$ a.e.

564C Definition Let X be a topological space and μ a Baire-coded measure on X. For $f \in \mathcal{L}^1$, define its integral $\int f$ by saying that $\int f = \lim_{n\to\infty} \int h_n$ whenever $\langle h_n \rangle_{n\in\mathbb{N}}$ is a codable sequence in $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ converging to f almost everywhere and $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n|$ is finite.

564D Lemma Let X be a topological space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a codable sequence of codable Baire functions on X. Let $\langle q_i \rangle_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0, \infty[$, starting with $q_0 = 0$. Set

$$f'_n(x) = \max\{q_i : i \le n,\ q_i \le \max(0, f_n(x))\}$$

for $n \in \mathbb{N}$ and $x \in X$. Then $\langle f'_n \rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions.

564E Theorem Let X be a topological space and μ a Baire-coded measure on X.

(a)(i) If f, $g \in \mathcal{L}^0$ and $\alpha \in \mathbb{R}$, then $f + g$, αf, $|f|$ and $f \times g$ belong to $\mathcal{L}^0$.

(ii) If $h : \mathbb{R} \to \mathbb{R}$ is a codable Borel function, $hf \in \mathcal{L}^0$ for every $f \in \mathcal{L}^0$.

(b) If f, $g \in \mathcal{L}^1$ and $\alpha \in \mathbb{R}$, then

(i) $f + g$, αf and $|f|$ belong to $\mathcal{L}^1$;

(ii) $\int f + g = \int f + \int g$, $\int \alpha f = \alpha \int f$;

(iii) if $f \le_{\text{a.e.}} g$ then $\int f \le \int g$.

(c)(i) If $f \in \mathcal{L}^0$, $g \in \mathcal{L}^1$ and $|f| \le_{\text{a.e.}} g$, then $f \in \mathcal{L}^1$.

(ii) If $E \in \mathcal{B}\mathfrak{a}_c(X)$ and $\chi E \in \mathcal{L}^1$ then μE is finite.

564F Theorem Let X be a topological space and μ a Baire-coded measure on X. Suppose that $\langle f_n \rangle_{n\in\mathbb{N}}$ is a codable sequence of integrable codable Baire functions on X.

(a) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is non-decreasing and $\gamma = \sup_{n\in\mathbb{N}} \int f_n$ is finite, then $f = \lim_{n\to\infty} f_n$ is defined a.e. and is integrable, and $\int f = \gamma$.

(b) If every f_n is non-negative and $\liminf_{n\to\infty} \int f_n$ is finite, then $f = \liminf_{n\to\infty} f_n$ is defined a.e. and is integrable, and $\int f \le \liminf_{n\to\infty} \int f_n$.

(c) Suppose that there is a $g \in \mathcal{L}^1$ such that $|f_n| \le_{\text{a.e.}} g$ for every n, and $f = \lim_{n\to\infty} f_n$ is defined a.e. Then $\int f$ and $\lim_{n\to\infty} \int f_n$ are defined and equal.

(d) If $\sum_{n=0}^{\infty} \int |f_{n+1} - f_n|$ is finite, then $f = \lim_{n\to\infty} f_n$ is defined a.e., and $\int f$ and $\lim_{n\to\infty} \int f_n$ are defined and equal.

(e) If $\sum_{n=0}^{\infty} \int |f_n|$ is finite, then $f = \sum_{n=0}^{\infty} f_n$ is defined a.e., and $\int f$ and $\sum_{n=0}^{\infty} \int f_n$ are defined and equal.

564G Integration over subsets: Proposition Let X be a topological space and μ a Baire-coded measure on X.

(a) If $f \in \mathcal{L}^1$, the functional $E \mapsto \int f \times \chi E : \mathcal{B}\mathfrak{a}_c(X) \to \mathbb{R}$ is additive and truly continuous with respect to μ.

(c) If f, $g \in \mathcal{L}^1$, then $f \le_{\text{a.e.}} g$ iff $\int f \times \chi E \le \int g \times \chi E$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)$. $f =_{\text{a.e.}} g$ iff $\int f \times \chi E = \int g \times \chi E$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)$.

564H Theorem Let X be a topological space, and $f : C_b(X) \to \mathbb{R}$ a sequentially smooth positive linear functional. Then there is a totally finite Baire-coded measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_b(X)$.

564I Riesz Representation Theorem Let X be a completely regular locally compact space, and $f : C_k(X) \to \mathbb{R}$ a positive linear functional. Then there is a Baire-coded measure μ on X such that $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in C_k(X)$.

564J The space L^1 Let X be a topological space and μ a Baire-coded measure on X.

(a) If f, $g \in \mathcal{L}^1$ then $f =_{\text{a.e.}} g$ iff $\int |f - g| = 0$.

(b) As in §242, we have an equivalence relation $\sim$ on $\mathcal{L}^1$ defined by saying that $f \sim g$ if $f =_{\text{a.e.}} g$. The set L^1 of equivalence classes has a Riesz space structure and a Riesz norm inherited from the addition, scalar multiplication, ordering and integral on $\mathcal{L}^1$.

(c) I define $\int : L^1 \to \mathbb{R}$ by saying that $\int f^\bullet = \int f$ for every $f \in \mathcal{L}^1$. Similarly, we can define $\int_E u$, for $u \in L^1$ and $E \in \mathcal{B}\mathfrak{a}_c(X)$, by saying that $\int_E f^\bullet = \int f \times \chi E$ for $f \in \mathcal{L}^1$.

564K Theorem Let X be a second-countable space and μ a codably σ-finite Borel-coded measure on X. Then $L^1(\mu)$ is a separable L-space.

564L Radon-Nikodým theorem Let X be a second-countable space with a codably σ-finite Borel-coded measure μ. Let $\nu : \mathcal{B}_c(X) \to \mathbb{R}$ be a truly continuous additive functional. Then there is an $f \in \mathcal{L}^1(\mu)$ such that $\nu E = \int f \times \chi E$ for every $E \in \mathcal{B}_c(X)$.

564M Inverse-measure-preserving functions (a) Let X and Y be second-countable spaces, with Borel-coded measures μ and ν. Suppose that $\varphi : X \to Y$ is a codable Borel function such that $\mu\varphi^{-1}[F] = \nu F$ for every $F \in \mathcal{B}_c(Y)$. Then $h\varphi \in S_X$ and $\int h\varphi\,d\mu = \int h\,d\nu$ for every $h \in S_Y$, writing $S_X = S(\mathcal{B}_c(X)^f)$, S_Y for the spaces of simple functions. $f\varphi \in \mathcal{L}^0(\mu)$ for every $f \in \mathcal{L}^0(\nu)$. $\langle h_n\varphi\rangle_{n\in\mathbb{N}}$ is a codable sequence in S_X whenever $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in S_Y; $f\varphi \in \mathcal{L}^1(\mu)$ whenever $f \in \mathcal{L}^1(\nu)$, and we have a norm-preserving Riesz homomorphism $T : L^1(\nu) \to L^1(\mu)$ defined by setting $Tf^\bullet = (f\varphi)^\bullet$ for $f \in \mathcal{L}^1(\mu)$.

(b) If ν is codably σ-finite, we have a conditional expectation operator in the reverse direction. For any $f \in \mathcal{L}^1(\mu)$, consider the functional λ_f defined by setting $\lambda_f F = \int f \times \chi(\varphi^{-1}[F])$ for $F \in \mathcal{B}_c(Y)$. This is additive and truly continuous.

There is a unique $v_f \in L^1(\nu)$ such that $\int_F v_f = \lambda_f F$ for every $F \in \mathcal{B}_c(Y)$.

We may call v_f the **conditional expectation** of f with respect to the inverse-measure-preserving function φ.

(c) Still supposing that ν is codably σ-finite, $\lambda_f = \lambda_{f'}$ whenever f, $f' \in \mathcal{L}^1(\mu)$ are equal almost everywhere, so that we have an operator $P : L^1(\mu) \to L^1(\nu)$ defined by saying that $Pf^\bullet = v_f$ for every $f \in \mathcal{L}^1(\mu)$; $\int_F Pu = \int_{\varphi^{-1}[F]} u$ for every $u \in L^1(\mu)$ and $F \in \mathcal{B}_c(Y)$. P is linear. It is positive. It is elementary to check that if T is the operator of (a) above then PT is the identity operator on $L^1(\nu)$.

(d) Now consider the special case in which $Y = X$, the topology of Y is the topology generated by a codable sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(X)^f$, $\nu = \mu{\restriction}\mathcal{B}_c(Y)$ and φ is the identity function. In this case, we can identify $L^1(\nu)$ with its image in $L^1(\mu)$ under T, and P becomes a conditional expectation operator of the kind examined in 242J.

564N Product measures: Theorem Let X and Y be second-countable spaces, and μ, ν semi-finite Borel-coded measures on X, Y respectively.

(a) There is a Borel-coded measure λ on $X \times Y$ such that $\lambda(E \times F) = \mu E \cdot \nu F$ for all $E \in \mathcal{B}_c(X)$ and $F \in \mathcal{B}_c(Y)$.

(b) If ν is codably σ-finite then we can arrange that $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\lambda$ for every λ-integrable real-valued function f.

(c) If μ and ν are both codably σ-finite then λ is uniquely defined by the formula in (a).

564O Theorem Let $\langle(X_k, \rho_k)\rangle_{n\in\mathbb{N}}$ be a sequence of complete metric spaces, and suppose that we have a double sequence $\langle U_{ki}\rangle_{k,i\in\mathbb{N}}$ such that $\{U_{ki} : i \in \mathbb{N}\}$ is a base for the topology of X_k for each k. Let $\langle\mu_k\rangle_{n\in\mathbb{N}}$ be a sequence such that μ_k is a Borel-coded probability measure on X_k for each k. Set $X = \prod_{k\in\mathbb{N}} X_k$. Then X is a Polish space and there is a Borel-coded probability measure λ on X such that $\lambda(\prod_{k\in\mathbb{N}} E_k) = \prod_{k\in\mathbb{N}} \mu_k E_k$ whenever $\langle E_k\rangle_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \mathcal{B}_c(X_k)$ and $\{k : E_k \ne X_k\}$ is finite.

Version of 25.4.14

565 Lebesgue measure without choice

I come now to the construction of specific non-trivial Borel-coded measures. Primary among them is of course Lebesgue measure on $\mathbb{R}^r$; we also have Hausdorff measures (565N-565O). For Lebesgue measure I begin, as in §115, with half-open intervals. The corresponding 'outer measure' may no longer be countably subadditive, so I call it 'Lebesgue submeasure'. Carathéodory's method no longer seems quite appropriate, as it smudges the distinction between 'negligible' and 'outer measure zero', so I use 563H to show that there is a Borel-coded measure agreeing with Lebesgue submeasure on open sets (565C-565D); it is the completion of this Borel-coded measure which I will call Lebesgue measure. We have a version of Vitali's theorem for well-orderable families (in particular, for countable families) of balls (565F). From this we can prove the Fundamental Theorem of Calculus in essentially its standard form (565M).

565A Definitions Throughout this section, except when otherwise stated, $r \geq 1$ will be a fixed integer. I will say that a **half-open interval** in $\mathbb{R}^r$ is a set of the form

$$[a, b[= \{x : x \in \mathbb{R}^r,\ a(i) \leq x(i) < b(i) \text{ for } i < r\}$$

where a, $b \in \mathbb{R}^r$. For a half-open interval I, set $\lambda I = 0$ if $I = \emptyset$ and otherwise $\lambda I = \prod_{i=0}^{r-1} b(i) - a(i)$ where $I = [a, b[$. Now for $A \subseteq \mathbb{R}^r$ set

$$\theta A = \inf\{\sum_{j=0}^{\infty} \lambda I_j : \langle I_j\rangle_{j\in\mathbb{N}} \text{ is a sequence of half-open intervals covering } A\}.$$

565B Proposition In the notation of 565A,
(a) the function $\theta : \mathcal{P}\mathbb{R}^r \to [0, \infty]$ is a submeasure,
(b) $\theta I = \lambda I$ for every half-open interval $I \subseteq \mathbb{R}^r$.

Definition I will call the submeasure θ **Lebesgue submeasure** on $\mathbb{R}^r$.

565C Lemma Let $\mathcal{I}$ be the family of half-open intervals in $\mathbb{R}^r$; let θ be Lebesgue submeasure, and set

$$\Sigma = \{E : E \subseteq X,\ \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}, \qquad \nu = \theta\restriction\Sigma.$$

(a) Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathcal{I}$. Then $E = \bigcup_{n\in\mathbb{N}} I_n$ belongs to Σ and $\nu E = \sum_{n=0}^{\infty} \nu I_n$.
(b) Every open set in $\mathbb{R}^r$ belongs to Σ.
(c) If G, $H \subseteq \mathbb{R}^r$ are open, then $\nu G + \nu H = \nu(G \cap H) + \nu(G \cup H)$.
(d) If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets then $\nu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} G_n$.

565D Definition Let θ and ν be as in 565C. By 563H, there is a unique Borel-coded measure μ on $\mathbb{R}^r$ such that $\mu G = \nu G = \theta G$ for every open set $G \subseteq \mathbb{R}^r$. I will say that **Lebesgue measure** on $\mathbb{R}^r$ is the completion μ_L of μ; the sets it measures will be **Lebesgue measurable**.

565E Proposition Let $\mathcal{I}$, θ, Σ, ν, μ and μ_L be as in 565A-565D.
(a) μ is the restriction of θ to the algebra $\mathcal{B}_c(\mathbb{R}^r)$ of codable Borel sets.
(b) For every $A \subseteq \mathbb{R}^r$,

$$\theta A = \inf\{\mu_L E : E \supseteq A \text{ is Lebesgue measurable}\} = \inf\{\mu G : G \supseteq A \text{ is open}\}.$$

(c) $E \in \Sigma$ and $\mu_L E = \nu E = \theta E$ whenever E is Lebesgue measurable.
(d) μ_L is inner regular with respect to the compact sets and outer regular with respect to the open sets.

565F Vitali's Theorem Let $\mathcal{C}$ be a well-orderable family of non-singleton closed balls in $\mathbb{R}^r$. For $\mathcal{I} \subseteq \mathcal{C}$ set

$$A_{\mathcal{I}} = \bigcap_{\delta>0} \bigcup\{C : C \in \mathcal{I},\ \operatorname{diam} C \leq \delta\}.$$

Let $\mathfrak{T}$ be the family of open subsets of $\mathbb{R}^r$. Then there are functions $\Psi : \mathcal{P}\mathcal{C} \to \mathcal{P}\mathcal{C}$ and $\Theta : \mathcal{P}\mathcal{C} \times \mathbb{N} \to \mathfrak{T}$ such that $\Psi(\mathcal{I}) \subseteq \mathcal{I}$, $\Psi(\mathcal{I})$ is disjoint and countable, $\mu_L(\Theta(\mathcal{I}, k)) \leq 2^{-k}$ and $A_{\mathcal{I}} \subseteq \bigcup \Psi(\mathcal{I}) \cup \Theta(\mathcal{I}, k)$ whenever $\mathcal{I} \subseteq \mathcal{C}$ and $k \in \mathbb{N}$. In particular,

$$A_{\mathcal{I}} \setminus \bigcup \Psi(\mathcal{I}) \subseteq \bigcap_{k\in\mathbb{N}} \Theta(\mathcal{I}, k)$$

is negligible.

565G Proposition Let $A \subseteq \mathbb{R}^r$ be any set. Then its Lebesgue submeasure is

$$\theta A = \inf\{\sum_{n=0}^{\infty} \mu_L B_n : \langle B_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of closed balls covering } A\}.$$

565H Corollary Lebesgue measure is invariant under isometries.

565I Lemma (a) Writing $C_k(\mathbb{R}^r)$ for the space of continuous real-valued functions on $\mathbb{R}^r$ with compact support, $C_k(\mathbb{R}^r) \subseteq \mathcal{L}^1(\mu_L)$.

(b) There is a countable set $D \subseteq C_k(\mathbb{R}^r)$ such that $\{g^\bullet : g \in D\}$ is norm-dense in $L^1(\mu_L)$.

565J Lemma Suppose that f is an integrable function on $\mathbb{R}^r$, and that $\int_I f \geq 0$ for every half-open interval $I \subseteq \mathbb{R}^r$. Then $f(x) \geq 0$ for almost every $x \in \mathbb{R}^r$.

565K Theorem A monotonic function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.

565L Lemma Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a bounded non-decreasing function. Then $\int F'$ is defined and is at most $\lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)$.

565M Theorem Let $F : \mathbb{R} \to \mathbb{R}$ be a function. Then the following are equiveridical:

(i) there is an integrable function f such that $F(x) = \int_{]-\infty,x[} f$ for every $x \in \mathbb{R}$,

(ii) F is of bounded variation, absolutely continuous on every bounded interval, and $\lim_{x\to-\infty} F(x) = 0$,

and in this case $F' =_{\text{a.e.}} f$.

565N Hausdorff measures Let (X, ρ) be a metric space and $s \in]0, \infty[$. As in §471, we can define **Hausdorff s-dimensional submeasure** $\theta_s : \mathcal{P}X \to [0, \infty]$ by writing

$$\theta_s A = \sup_{\delta>0} \inf\{\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^s : \langle D_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A,$$

$$\operatorname{diam} D_n \leq \delta \text{ for every } n \in \mathbb{N}\},$$

counting $\operatorname{diam}\emptyset$ as 0 and $\inf\emptyset$ as ∞. θ_s is a submeasure.

565O Theorem Let (X, ρ) be a second-countable metric space, and $s > 0$. Then there is a Borel-coded measure μ on X such that $\mu K = \theta_s K$ whenever $K \subseteq X$ is compact and $\theta_s K$ is finite.

Version of 22.8.14

566 Countable choice

With AC(ω) measure theory becomes recognisable. The definition of Lebesgue measure used in Volume 1 gives us a true countably additive Radon measure; the most important divergence from the standard theory is the possibility that every subset of $\mathbb{R}$ is Lebesgue measurable. With occasional exceptions (most notably, in the theory of infinite products) we can use the work of Volume 2. In Volume 3, we lose the two best theorems in the abstract theory of measure algebras, Maharam's theorem and the Lifting Theorem; but function spaces and ergodic theory are relatively unaffected. Even in Volume 4, a good proportion of the ideas can be applied in some form.

566D Exhaustion: Proposition [AC(ω)] (a) Let P be a partially ordered set such that $p \vee q = \sup\{p, q\}$ is defined for all p, $q \in P$, and $f : P \to \mathbb{R}$ an order-preserving function. Then there is a non-decreasing sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P such that $\lim_{n\to\infty} f(p_n) = \sup_{p\in P} f(p)$.

(b) Let (X, Σ, μ) be a measure space and $\mathcal{E} \subseteq \Sigma$ a non-empty set such that $\sup_{E\in\mathcal{E}} \mu E$ is finite and $E \cup F \in \mathcal{E}$ for every E, $F \in \mathcal{E}$. Then there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that, setting $F = \bigcup_{n\in\mathbb{N}} F_n$, $\mu F = \sup_{E\in\mathcal{E}} \mu E$ and $E \setminus F$ is negligible for every $E \in \mathcal{E}$.

(c) Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets such that

(α) $K \cup K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

(β) whenever $E \in \Sigma$ is non-negligible there is a non-negligible $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$.

Then μ is inner regular with respect to $\mathcal{K}$.

(d)(i) Let (X,Σ,μ) be a semi-finite measure space. Then μ is inner regular with respect to the family of sets of finite measure.

(ii) Let (X,Σ,μ) be a perfect measure space. Then whenever $E\in\Sigma$, $f:X\to\mathbb{R}$ is measurable and $\gamma<\mu E$, there is a compact set $K\subseteq f[E]$ such that $\mu f^{-1}[K]\ge\gamma$.

566E Proposition [AC(ω)] Let (X,Σ,μ) be a semi-finite measure space. Write $\mathcal{N}$ for the σ-ideal of μ-negligible sets.

(a) The following are equiveridical:

(i) μ is σ-finite;

(ii) either $\mu X=0$ or there is a probability measure ν on X with the same domain and the same negligible sets as μ;

(iii) there is a measurable integrable function $f:X\to\,]0,1]$;

(iv) either $\mu X=0$ or there is a measurable function $f:X\to\,]0,\infty[$ such that $\int fd\mu=1$.

(b) If μ is σ-finite, then

(i) every disjoint family in $\Sigma\setminus\mathcal{N}$ is countable;

(ii) for every $\mathcal{E}\subseteq\Sigma$ there is a countable $\mathcal{E}_0\subseteq\mathcal{E}$ such that $E\setminus\bigcup\mathcal{E}_0$ is negligible for every $E\in\mathcal{E}$.

(c) Suppose that μ is σ-finite, (Y,T,ν) is a semi-finite measure space, and $\phi:X\to Y$ is a (Σ,T)-measurable function such that $\mu\phi^{-1}[F]>0$ whenever $\nu F>0$. Then ν is σ-finite.

566F Atomless algebras: Lemma [AC(ω)] Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and μ a positive countably additive functional on $\mathfrak{A}$ such that $\mu 1=1$. Suppose that whenever $a\in\mathfrak{A}$ and $\mu a>0$ there is a $b\subseteq a$ such that $0<\mu b<\mu a$. Then there is a function $f:\mathfrak{A}\times[0,1]\to\mathfrak{A}$ such that $f(a,\alpha)\subseteq a$ and $\bar\mu f(a,\alpha)=\min(\alpha,\bar\mu a)$ for $a\in\mathfrak{A}$ and $\alpha\in[0,1]$, and $\alpha\mapsto f(a,\alpha)$ is non-decreasing for every $a\in\mathfrak{A}$.

566H Bounded additive functionals: Lemma [AC(ω)] Let $\mathfrak{A}$ be a Boolean algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ an additive functional such that $\{\nu a_n:n\in\mathbb{N}\}$ is bounded for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Then ν is bounded.

566I Infinite products: Theorem [AC(ω)] Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of perfect probability spaces such that $X=\prod_{i\in I}X_i$ is non-empty. Then there is a complete probability measure λ on X such that

(i) if $E_i\in\Sigma_i$ for every $i\in I$, and $\{i:E_i\ne X_i\}$ is countable, then $\lambda(\prod_{i\in I}E_i)$ is defined and equal to $\prod_{i\in I}\mu_iE_i$;

(ii) λ is inner regular with respect to $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

566J Theorem [AC(ω)] (a) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of metrizable Radon probability spaces such that every μ_i is strictly positive and $X=\prod_{i\in I}X_i$ is non-empty. Then the product measure on X is a quasi-Radon measure.

(b) If I is well-orderable then the product measure on $\{0,1\}^I$ is a completion regular Radon measure.

566L The Loomis-Sikorski theorem [AC(ω)] (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there are a set X, a σ-algebra Σ of subsets of X and a σ-ideal $\mathcal{I}$ of Σ such that $\mathfrak{A}\cong\Sigma/\mathcal{I}$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Then it is isomorphic to the measure algebra of a measure space.

566M Measure algebras: Proposition [AC(ω)] (a) Let $\mathfrak{A}$ be a measurable algebra.

(i) For any $A\subseteq\mathfrak{A}$ there is a countable $B\subseteq A$ with the same upper bounds as A.

(ii) $\mathfrak{A}$ is Dedekind complete.

(iii) If $D\subseteq\mathfrak{A}$ is order-dense and $c\in D$ whenever $c\subseteq d\in D$, there is a partition of unity included in D.

(b) Let $(\mathfrak{A},\bar\mu)$ be a σ-finite measure algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ such that $(\mathfrak{B},\bar\mu{\restriction}\mathfrak{B})$ is a semi-finite measure algebra. Then $(\mathfrak{B},\bar\mu{\restriction}\mathfrak{B})$ is a σ-finite measure algebra.

566N Characterizing the usual measure on $\{0,1\}^{\mathbb{N}}$: Theorem [AC(ω)] (a) Let (X,Σ,μ) be an atomless, perfect, complete, countably separated probability space. Then it is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual measure.

(b) Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra of countable Maharam type. Then it is isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$.

(c) An atomless measurable algebra of countable Maharam type is homogeneous.

(d) For any infinite set I, the measure algebra of the usual measure on $\{0,1\}^I$ is homogeneous.

566O Boolean values: Proposition [AC(ω)] (a) Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$, and $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ the Borel σ-algebra. If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi:\mathfrak{B}\to\mathfrak{A}$ is a Boolean homomorphism, π has a unique extension to a sequentially order-continuous Boolean homomorphism from $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ to $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there is a bijection between $L^0=L^0(\mathfrak{A})$ and the set Φ of sequentially order-continuous Boolean homomorphisms from the algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ to $\mathfrak{A}$, defined by saying that $u\in L^0$ corresponds to $\phi\in\Phi$ iff $[\![u>\alpha]\!]=\phi(]\alpha,\infty[)$ for every $\alpha\in\mathbb{R}$.

(c) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Write Σ_{um} for the algebra of universally measurable subsets of $\mathbb{R}$. Then for any $u\in L^0(\mathfrak{A})$, we have a sequentially order-continuous Boolean homomorphism $E\mapsto[\![u\in E]\!]:\Sigma_{\mathrm{um}}\to\mathfrak{A}$ such that

$$\begin{aligned}[\![u\in E]\!]&=\sup\{[\![u\in F]\!]:F\subseteq E\text{ is Borel}\}=\sup\{[\![u\in K]\!]:K\subseteq E\text{ is compact}\}\\&=\inf\{[\![u\in F]\!]:F\supseteq E\text{ is Borel}\}=\inf\{[\![u\in G]\!]:G\supseteq E\text{ is open}\}\end{aligned}$$

for every $E\in\Sigma_{\mathrm{um}}$, while

$$[\![u\in\,]\alpha,\infty[\,]\!]=[\![u>\alpha]\!]$$

for every $\alpha\in\mathbb{R}$.

566P Weak compactness: Theorem [AC(ω)] Let U be a Hilbert space. Then bounded sets in U are relatively weakly compact.

566Q Theorem [AC(ω)] Let U be an L-space. Then a subset of U is weakly relatively compact iff it is uniformly integrable.

566R Automorphisms of measurable algebras: Theorem [AC(ω)] Let $\mathfrak{A}$ be a measurable algebra.

(a) Every automorphism of $\mathfrak{A}$ has a separator.

(b) Every $\pi\in\operatorname{Aut}\mathfrak{A}$ is a product of at most three exchanging involutions belonging to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

566T Proposition [AC(ω)] Let I be any set, and X a separable metrizable space. Then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X^I)$ of X^I is equal to the σ-algebra $\widehat{\bigotimes}_I\mathcal{B}(X)$ generated by sets of the form $\{x:x(i)\in E\}$ for $i\in I$ and Borel sets $E\subseteq X$.

566Z Problem Is it relatively consistent with ZF + AC(ω) to suppose that there is a non-zero atomless rigid measurable algebra?

Version of 31.10.14

567 Determinacy

So far, this chapter has been looking at set theories which are weaker than the standard theory ZFC, and checking which of the principal results of measure theory can still be proved. I now turn to an axiom which directly contradicts the axiom of choice, and leads to a very different world. This is AD, the 'axiom of determinacy', defined in terms of strategies for infinite games (567A-567C). The first step is to confirm that we automatically have a weak version of countable choice which is enough to make Lebesgue measure well-behaved (567D-567E). Next, in separable metrizable spaces all subsets are universally measurable and have the Baire property (567G). Consequently (at least when we can use AC(ω)) linear operators between Banach spaces are bounded (567H), additive functionals on σ-complete Boolean algebras are countably additive (567J), and many L-spaces are reflexive (567K). In a different direction, we find that ω_1 is two-valued-measurable (567L) and that there are many surjections from $\mathbb{R}$ onto ordinals (567M).

At the end of the section I include two celebrated results in ZFC (567N, 567O) which depend on some of the same ideas.

567A Infinite games (a) Let X be a non-empty set and A a subset of $X^{\mathbb{N}}$. In the corresponding infinite game Game(X, A), players I and II choose members of X alternately, so that I chooses $x(0), x(2), \ldots$ and II chooses $x(1), x(3), \ldots$; a **play** of the game is an element of $X^{\mathbb{N}}$; player I wins the play x if $x \in A$, otherwise II wins. A **strategy** for I is a function $\sigma : \bigcup_{n\in\mathbb{N}} X^n \to X$; a play $x \in X^{\mathbb{N}}$ is **consistent** with σ if $x(2n) = \sigma(\langle x(2i+1)\rangle_{i<n})$ for every n; σ is a **winning strategy** if every play consistent with σ belongs to A. Similarly, a strategy for II is a function $\tau : \bigcup_{n\ge1} X^n \to X$; a play x is consistent with τ if $x(2n+1) = \tau(\langle x(2i)\rangle_{i\le n})$ for every n; and τ is a winning strategy for II if $x \notin A$ whenever $x \in X^{\mathbb{N}}$ and x is consistent with τ.

(b) A set $A \subseteq X^{\mathbb{N}}$ is **determined** if either I or II has a winning strategy in Game(X, A).

(c) It will sometimes be convenient to describe games with 'rules', so that the players are required to choose points in subsets of X (determined by the moves so far) at each move. Such a description can be regarded as specifying A in the form $(A' \cup G) \setminus H$, where G is the set of plays in which II is the first to break a rule, H is the set of plays in which I is the first to break a rule, and A' is the set of plays in which both obey the rules and I wins.

(d) Not infrequently the 'rules' will specify different sets for the moves of the two players, so that I always chooses a point in X_1 and II always chooses a point in X_2; setting $X = X_1 \cup X_2$ we can reduce this to the formalization above.

567B Theorem Let X be a non-empty well-orderable set. Give X its discrete topology and $X^{\mathbb{N}}$ the product topology. If $F \subseteq X^{\mathbb{N}}$ is closed then Game(X, F) is determined.

567C The axiom of determinacy (a) The standard 'axiom of determinacy' is the statement

(AD) Every subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

Evidently it will follow that every subset of $X^{\mathbb{N}}$ is determined for any countable set X.

(b) At the same time, it will be useful to consider a weak form of the axiom of countable choice: for any set X, write AC$(X;\omega)$ for the statement

$\prod_{n\in\mathbb{N}} A_n \ne \emptyset$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty subsets of X.

567D Theorem AD implies AC$(\mathbb{R};\omega)$.

567E Consequences of AC($\mathbb{R};\omega$) Suppose that AC$(\mathbb{R};\omega)$ is true.

(a) If a set X is the image of a subset Y of $\mathbb{R}$ under a function f, then AC$(X;\omega)$ is true.

(b) In particular, taking $S^* = \bigcup_{n\ge1} \mathbb{N}^n$, AC$(\mathcal{P}S^*;\omega)$ is true. It follows that (in any second-countable space X) every sequence of codable Borel sets is codable and the family of codable Borel sets is a σ-algebra, coinciding with the Borel σ-algebra $\mathcal{B}(X)$ on its ordinary definition. Moreover, since $\mathcal{B}(X)$ is an image of $\mathcal{P}S^*$, we have AC$(\mathcal{B}(X);\omega)$. Similarly, the family of codable Borel functions becomes the ordinary family of Borel-measurable functions, and we have countable choice for sets of Borel real-valued functions on X.

(c) Consequently the results of §562-565 give us large parts of the elementary theory of Borel measures on second-countable spaces. At the same time, if X is second-countable, the union of a sequence of meager subsets of X is meager, so the Baire-property algebra of X is a σ-algebra.

(d) We also find that the supremum of a sequence of countable ordinals is again countable.

567F Lemma [AC($\mathbb{R};\omega$)] Suppose that $A \subseteq \{0,1\}^{\mathbb{N}}$ is a continuous image of a subset B of $\{0,1\}^{\mathbb{N}}$ such that $(h^{-1}[B] \cap F) \cup H \subseteq \mathbb{N}^{\mathbb{N}}$ is determined whenever $h : \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is continuous, $F \subseteq \mathbb{N}^{\mathbb{N}}$ is closed and $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open.

(a) A is universally measurable.

(b) A has the Baire property in $\{0,1\}^{\mathbb{N}}$.

567G Theorem [AD] In any Hausdorff second-countable space, every subset is universally measurable and has the Baire property.

567H Theorem (a) [AD] Let X be a Polish group and Y a topological group which is either separable or Lindelöf. Then every group homomorphism from X to Y is continuous.

(b) [AD+AC(ω)] Let X be an abelian topological group which is complete under a metric defining its topology, and Y a topological group which is either separable or Lindelöf. Then every group homomorphism from X to Y is continuous.

(c) [AD+AC(ω)] Let X be a complete metrizable linear topological space, Y a linear topological space and $T : X \to Y$ a linear operator. Then T is continuous. In particular, every linear operator between Banach spaces is a bounded operator.

567I Proposition [AC($\mathbb{R};\omega$)] Let $\widehat{\mathcal{B}}$ be the Baire-property algebra of $\mathcal{P}\mathbb{N}$. Then every $\widehat{\mathcal{B}}$-measurable real-valued additive functional on $\mathcal{P}\mathbb{N}$ is of the form $a \mapsto \sum_{n\in a} \gamma_n$ for some $\langle \gamma_n \rangle_{n\in\mathbb{N}} \in \ell^1$.

567J Proposition [AD] A finitely additive functional on a Dedekind σ-complete Boolean algebra is countably additive.

567K Theorem [AD+AC(ω)] If U is an L-space with a weak order unit, it is reflexive.

567L Theorem [AD] ω_1 is two-valued-measurable.

Remark In the present context I will use the formulation 'an initial ordinal κ is two-valued-measurable if there is a proper κ-additive 2-saturated ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ containing singletons', where here 'κ-additive' means that $\bigcup_{\eta<\xi} J_\eta \in \mathcal{I}$ whenever $\xi < \kappa$ and $\langle J_\eta \rangle_{\eta<\xi}$ is a family in $\mathcal{I}$.

567M Theorem [AD] Let α be an ordinal such that there is a surjection from $\mathcal{P}\mathbb{N}$ onto α. Then there is a surjection from $\mathcal{P}\mathbb{N}$ onto $\mathcal{P}\alpha$.

567N Theorem [AC] Assume that there is a two-valued-measurable cardinal. Then every coanalytic subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

567O Corollary [AC] If there is a two-valued-measurable cardinal, then every PCA subset of any Polish space is universally measurable.