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Chapter 52

Cardinal functions of measure theory

From the point of view of this book, the most important cardinals are those associated with measures and measure algebras, especially, of course, Lebesgue measure and the usual measure ν_I of $\{0,1\}^I$. In this chapter I try to cover the principal known facts about these which are theorems of ZFC. I start with a review of the theory for general measure spaces in §521, including some material which returns to the classification scheme of Chapter 21, exploring relationships between (strict) localizability, magnitude and Maharam type. §522 examines Lebesgue measure and the surprising connexions found by Bartoszyński 84 and Raisonnier & Stern 85 between the cardinals associated with the Lebesgue null ideal and the corresponding ones based on the ideal of meager subsets of $\mathbb{R}$. §523 looks at the measures ν_I for uncountable sets I, giving formulae for the additivities and cofinalities of their null ideals, and bounds for their covering numbers, uniformities and shrinking numbers. Remarkably, these cardinals are enough to tell us most of what we want to know concerning the cardinal functions of general Radon measures and semi-finite measure algebras (§524). These three sections are heavily dependent on the Galois-Tukey connections and Tukey functions of §§512-513. Precalibers do not seem to fit into this scheme, and the relatively partial information I have is in §525. The second half of the chapter deals with special topics which can be approached with the methods so far developed. In §526 I return to the ideal of subsets of $\mathbb{N}$ with asymptotic density zero, seeking to locate it in the Tukey classification. Further σ-ideals which are of interest in measure theory are the 'skew products' of §527. In §528 I examine some interesting Boolean algebras, the 'amoeba algebras' first introduced by Martin & Solovay 70, giving the results of Truss 88 on the connexions between different amoeba algebras and localization posets. Finally, in §529, I look at a handful of other structures, concentrating on results involving cardinals already described.

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521 Basic theory

In the first half of this section (down to 521L) I collect facts about the cardinal functions add, cf, non, cov, shr and shr^+ when applied to the null ideal $\mathcal{N}(\mu)$ of a measure μ, and also the π-weight of a measure. In particular I look at their relations with the constructions introduced earlier in this treatise: measure algebras and function spaces (521B), subspace measures (521F), direct sums (521G), inverse-measure-preserving functions and image measures (521H), products (521J), perfect measures (521K) and compact measures (521L). The list is long just because I have four volumes' worth of miscellaneous concepts to examine; nearly all the individual arguments are elementary.

In the second half of the section, I give a handful of easy results which may clarify some patterns from earlier volumes. In 521M-521P I look again at 'strict localizability' as considered in Chapter 21, importing the concept of 'magnitude' of a measure space from §332, hoping to throw light on the examples of §216. In 521E I consider the topological densities of measure algebras. In 521R-521S I explore possibilities for the 'countably separated' measure spaces of §§343-344, examining in particular their Maharam types. Finally, in 521T, I review some measures which arose in §464 while analyzing the L-space $\ell^\infty(I)^*$.

521A Proposition Let (X,Σ,μ) be a measure space.

(a) If $\mathcal{E}\subseteq\Sigma$ and $\#(\mathcal{E})<\operatorname{add}\mu$ then $\bigcup\mathcal{E}\in\Sigma$ and

$$\mu(\bigcup\mathcal{E})=\sup\{\mu(\bigcup\mathcal{E}_0):\mathcal{E}_0\subseteq\mathcal{E}\text{ is finite}\}.$$

(b) $\omega_1\le\operatorname{add}\mu\le\operatorname{add}\mathcal{N}(\mu)$.

(c) If μ is the measure defined by Carathéodory's method from an outer measure θ on X, then $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$.

(d) If μ is complete and locally determined, $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$.

proof (a) Induce on $\#(\mathcal{E})$. If $\mathcal{E}$ is finite, the result is trivial. For the inductive step to $\#(\mathcal{E}) = \kappa \geq \omega$, enumerate $\mathcal{E}$ as $\langle E_\xi\rangle_{\xi<\kappa}$. For each $\xi < \kappa$, set $H_\xi = E_\xi \setminus \bigcup_{\eta<\xi} E_\eta$ for each $\xi < \kappa$. Then the inductive hypothesis tells us that $H_\xi \in \Sigma$ for every ξ. Set $E = \bigcup\mathcal{E} = \bigcup_{\xi<\kappa} H_\xi$; because $\langle H_\xi\rangle_{\xi<\kappa}$ is disjoint, and $\kappa < \operatorname{add}\mu$, $E \in \Sigma$ and

$$\mu E = \sum_{\xi<\kappa} \mu H_\xi = \sup_{I\subseteq\kappa \text{ is finite}} \mu(\bigcup_{\xi\in I} H_\xi) \leq \sup_{\mathcal{E}_0\subseteq\mathcal{E} \text{ is finite}} \mu(\bigcup\mathcal{E}) \leq \mu E.$$

(b) By the definition of 'measure' (112A), μ is ω_1-additive. Suppose that $\mathcal{A} \subseteq \mathcal{N}(\mu)$ and $\#(\mathcal{A}) < \operatorname{add}\mu$. For each $A \in \mathcal{A}$, choose a measurable negligible $E_A \supseteq A$. Then (a) tells us that $E = \bigcup_{A\in\mathcal{A}} E_A$ has measure zero, so $\bigcup\mathcal{A} \subseteq E$ is negligible. As $\mathcal{A}$ is arbitrary, $\operatorname{add}\mathcal{N}(\mu) \geq \operatorname{add}\mu$.

(c) Now suppose that μ is defined by Carathéodory's method from θ. Let $\langle E_i\rangle_{i\in I}$ be a disjoint family in Σ, where $\#(I) < \operatorname{add}\mathcal{N}(\mu)$, with union E.

Let $A \subseteq X$ be any set. Then $\theta(A \cap E) = \sum_{i\in I} \theta(A \cap E_i)$. **P** Of course

$$\theta(A \cap E) \geq \sup_{J\subseteq I \text{ is finite}} \theta(A \cap \bigcup_{i\in J} E_i) = \sup_{J\subseteq I \text{ is finite}} \sum_{i\in J} \theta(A \cap E_i)$$

(induce on $\#(J)$, using the fact that $\theta B = \theta(B \cap E_i) + \theta(B \setminus E_i)$ for every $B \subseteq X$ and $i \in J$)

$$= \sum_{i\in I} \theta(A \cap E_i).$$

If $\sum_{i\in I} \theta(A\cap E_i)$ is infinite, we can stop. Otherwise, recalling that $\mathcal{N}(\mu) = \theta^{-1}[\{0\}]$, $J = \{i : A\cap E_i \notin \mathcal{N}(\mu)\}$ is countable, and $\bigcup_{i\in I\setminus J} A \cap E_i$ is negligible, because $\#(I) < \operatorname{add}\mathcal{N}(\mu)$; so

$$\theta(A \cap E) = \theta(A \cap \textstyle\bigcup_{i\in J} E_i) \leq \sum_{i\in J} \theta(A \cap E_i) = \sum_{i\in I} \theta(A \cap E_i)$$

and we have equality. **Q**

It follows that $\theta(A \cap E) + \theta(A \setminus E) \leq \theta A$. **P** For any finite $J \subseteq I$,

$$\theta(A \setminus E) + \sum_{i\in J} \theta(A \cap E_i) = \theta(A \setminus E) + \theta(A \cap \bigcup_{i\in J} E_i)$$
$$\leq \theta(A \setminus \bigcup_{i\in J} E_i) + \theta(A \cap \bigcup_{i\in J} E_i) = \theta A.$$

Taking the supremum over J, we have the result. **Q**

As A is arbitrary, $E \in \Sigma$; and setting $A = E$, we see that $\mu E = \sum_{i\in I} \mu E_i$. As $\langle E_i\rangle_{i\in I}$ is arbitrary, $\operatorname{add}\mu \geq \operatorname{add}\mathcal{N}(\mu)$ and the two additivities are equal.

(d) Now this follows immediately from (c), by 213C.

521B Proposition Let (X, Σ, μ) be a measure space and $(\mathfrak{A}, \bar\mu)$ its measure algebra.

(a) If $\mathcal{E} \subseteq \Sigma$ and $\#(\mathcal{E}) < \operatorname{add}\mu$, then $(\bigcup\mathcal{E})^\bullet = \sup_{E\in\mathcal{E}} E^\bullet$ and $(X \cap \bigcap\mathcal{E})^\bullet = \inf_{E\in\mathcal{E}} E^\bullet$ in $\mathfrak{A}$.

(b) Suppose that $A \subseteq [-\infty, \infty]^X$ is a non-empty family of Σ-measurable functions with $\#(A) < \operatorname{add}\mu$, and that $g(x) = \sup_{f\in A} f(x)$ in $[-\infty, \infty]$ for every $f \in A$. Then g is Σ-measurable.

(c) Write $\mathcal{L}^0$ for the family of μ-virtually measurable real-valued functions defined almost everywhere in X, and L^0 for the corresponding space of equivalence classes, as in §241. Suppose that $A \subseteq \mathcal{L}^0$ is such that $0 < \#(A) < \operatorname{add}\mu$ and $\{f^\bullet : f \in A\}$ is bounded above in L^0. Set $g(x) = \sup_{f\in A} f(x)$ whenever this is defined in $\mathbb{R}$; then $g \in \mathcal{L}^0$ and $g^\bullet = \sup_{f\in A} f^\bullet$ in L^0.

(d)(i) If, in (b), A consists of non-negative integrable functions and is upwards-directed, then $\int g\,d\mu = \sup_{f\in A} \int f d\mu$.

(ii) If, in (b), $f_1 \wedge f_2 = 0$ a.e. for all distinct f_1, $f_2 \in A$, then $\int g\,d\mu = \sum_{f\in A}\int f d\mu$.

proof (a) As in 521Aa, $\bigcup\mathcal{E} \in \Sigma$, and of course $(\bigcup\mathcal{E})^\bullet$ is an upper bound for $\{E^\bullet : E \in \mathcal{E}\}$. If $F \in \Sigma$ and $F^\bullet$ is an upper bound for $\{E^\bullet : E \in \mathcal{E}\}$, then, applying 521Aa to $\{E \setminus F : E \in \mathcal{E}\}$, we see that $\bigcup\mathcal{E} \setminus F$ is negligible, so $(\bigcup\mathcal{E})^\bullet \subseteq F^\bullet$. Thus $(\bigcup\mathcal{E})^\bullet$ is the least upper bound of $\{E^\bullet : E \in \mathcal{E}\}$.

Applying this to $\{X \setminus E : E \in \mathcal{E}\}$ we see that $(X \cap \bigcap\mathcal{E})^\bullet = \inf_{E\in\mathcal{E}} E^\bullet$.

(b) For any $\alpha \in \mathbb{R}$,

$$\{x : g(x) > \alpha\} = \bigcup_{f\in A}\{x : f(x) > \alpha\} \in \Sigma$$

by 521Aa.

(c) Take any $h \in \mathcal{L}^0$ such that $f^\bullet \le h^\bullet$ for every $f \in A$. For each $f \in A$, let E_f be a conegligible measurable subset of $\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h,\ f(x) \le h(x)\}$ such that $f\restriction E_f$ is measurable. Set $E = \bigcap_{f\in A} E_f$; then E is measurable and g is defined everywhere in E and $g\restriction E$ is measurable (as in (b)). Also E is conegligible, so $g \in \mathcal{L}^0$, and of course $f^\bullet \le g^\bullet$ for every $f \in A$, while $g^\bullet \le h^\bullet$. But this argument works for every h such that $h^\bullet$ is an upper bound for $\{f^\bullet : f \in A\}$, so $g^\bullet$ must be actually the supremum of $\{f^\bullet : f \in A\}$.

(d)(i) If $\sup_{f\in A}\int f d\mu$ is infinite, this is trivial. Otherwise, $\{f^\bullet : f \in A\}$ is bounded above in L^1 and therefore in L^0. By (c), $g^\bullet$ is its supremum in L^0, therefore in L^1; so

$$\int g = \int g^\bullet = \sup_{f\in A}\int f^\bullet = \sup_{f\in A}\int f,$$

as in 365Df.

(ii) Apply (i) to $A^* = \{\sup I : I \in [A]^{<\omega}\}$.

521C Just because null ideals are σ-ideals of sets, we can read off some of the elementary properties of their cardinal functions from 511J. But the presence of a measure gives us a new way to use shrinking numbers, which will be useful later.

Proposition Let (X, Σ, μ) be a measure space, and $A \subseteq X$.

(a) If $\gamma < \mu^*A$ there is a $B \subseteq A$ such that $\#(B) < \operatorname{shr}^+\mathcal{N}(\mu)$ and $\mu^*B > \gamma$.

(b) There is a $B \subseteq A$ such that $\#(B) \le \max(\omega, \operatorname{shr}\mathcal{N}(\mu))$ and $\mu^*B = \mu^*A$.

proof (a) Set $\kappa = \operatorname{shr}^+\mathcal{N}(\mu)$. Let $\mathcal{E}$ be the family of those measurable subsets of X such that there is a $B \in [A \cap E]^{<\kappa}$ with $\mu^*B = \mu E$. Then $\mathcal{E}$ is closed under finite unions (132Ed). **?** If $\mu^*B \le \gamma$ for every $B \in [A]^{<\kappa}$, then $\mu E \le \gamma$ for every $E \in \mathcal{E}$. By 215Ab, there is a non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $E \setminus \bigcup_{n\in\mathbb{N}} E_n$ is negligible for every $E \in \mathcal{E}$. Now $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \gamma < \mu^*A$ and $A' = A \setminus \bigcup_{n\in\mathbb{N}} E_n$ is not negligible. Let $B \in [A']^{<\kappa}$ be a non-negligible set. Then $\mu^*B \le \gamma$ is finite, so B has a measurable envelope F (132Ee), which belongs to $\mathcal{E}$; but $F \setminus \bigcup_{n\in\mathbb{N}} E_n \supseteq B$ is not negligible. **X** So we have a $B \in [A]^{<\kappa}$ with $\mu^*B > \gamma$, as required.

(b) If $\mu^*A = 0$ take $B = \emptyset$. Otherwise, let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a sequence in $[0, \mu^*A[$ with supremum μ^*A. For each $n \in \mathbb{N}$, (a) tells us that there is a set $B_n \subseteq A$ such that $\#(B_n) \le \operatorname{shr}(\mu)$ and $\mu^*B_n > \gamma_n$; set $B = \bigcup_{n\in\mathbb{N}} B_n$.

521D Proposition Let (X, Σ, μ) be a measure space and $(\mathfrak{A}, \bar\mu)$ its measure algebra.

(a) $\pi(\mathfrak{A}) \le \pi(\mu) \le \max(\pi(\mathfrak{A}), \operatorname{cf}\mathcal{N}(\mu))$ (definitions: 511Dc, 511Gb).

(b) If $\mu X > 0$, then $\operatorname{non}\mathcal{N}(\mu) \le \pi(\mu)$.

(c) If (X, Σ, μ) has locally determined negligible sets (definition: 213I), then $\operatorname{shr}\mathcal{N}(\mu) \le \pi(\mu)$.

(d) Suppose that there is a topology $\mathfrak{T}$ on X such that $(X, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon measure space. Then, writing $\mathfrak{A}^+$ for $\mathfrak{A} \setminus \{0\}$, the partially ordered sets $(\Sigma \setminus \mathcal{N}(\mu), \supseteq)$ and $(\mathfrak{A}^+, \supseteq)$ are Tukey equivalent and $\pi(\mu) = \pi(\mathfrak{A})$.

proof Let $\mathcal{H} \subseteq \Sigma \setminus \mathcal{N}(\mu)$ be a coinitial set with cardinal $\pi(\mu)$.

(a)(i) If $a \in \mathfrak{A}$ is non-zero, there is an $E \in \Sigma$ such that $E^\bullet = a$, and now E is not negligible, so there is an $H \in \mathcal{H}$ such that $H \subseteq E$ and $0 \ne H^\bullet \subseteq a$. Thus $\{H^\bullet : H \in \mathcal{H}\}$ is coinitial with $\mathfrak{A}^+$ and witnesses that $\pi(\mathfrak{A}) \le \#(\mathcal{H}) = \pi(\mu)$.

(ii) Let $B \subseteq \mathfrak{A}^+$ be a coinitial set with cardinal $\pi(\mathfrak{A})$, and $\mathcal{E}$ a cofinal subset of $\mathcal{N}(\mu)$ of size $\operatorname{cf}\mathcal{N}(\mu)$. For $b \in B$, let $F_b \in \Sigma$ be such that $F_b^\bullet = b$, and consider $\mathcal{G} = \{F_b \setminus E : b \in B, E \in \mathcal{E}\}$. Then $\mathcal{G} \subseteq \Sigma \setminus \mathcal{N}(\mu)$ is coinitial with $\Sigma \setminus \mathcal{N}(\mu)$. **P** If $\mu F > 0$, there is a $b \in B$ such that $b \subseteq F^\bullet$. In this case, $F_b \setminus F$ is negligible, so there is an $E \in \mathcal{E}$ such that $F_b \setminus F \subseteq E$ and $F \supseteq F_b \setminus E \in \mathcal{G}$. **Q**

It follows that $\pi(\mu) \le \#(\mathcal{G}) \le \#(B \times \mathcal{E})$ is at most the cardinal product $\pi(\mathfrak{A}) \cdot \operatorname{cf}\mathcal{N}(\mu) \le \max(\omega, \pi(\mathfrak{A}), \operatorname{cf}\mathcal{N}(\mu))$. But if $\operatorname{cf}\mathcal{N}(\mu)$ is finite it is 1, so in fact $\pi(\mu) \le \pi(\mathfrak{A}) \cdot \operatorname{cf}\mathcal{N}(\mu) = \max(\pi(\mathfrak{A}), \operatorname{cf}\mathcal{N}(\mu))$.

(b) For each $H \in \mathcal{H}$ choose $x_H \in \mathcal{H}$. Then $A = \{x_H : H \in \mathcal{H}\}$ must meet every non-negligible measurable set, so (as $\mu X > 0$) cannot itself be negligible. Thus

$$\operatorname{non}\mathcal{N}(\mu) \le \#(A) \le \#(\mathcal{H}) = \pi(\mu).$$

(c) Suppose that $B \subseteq X$ is non-negligible. Because (X, Σ, μ) has locally determined negligible sets there is an $E \in \Sigma$ such that $\mu E > 0$ and $B \cap E$ is not negligible, and now $B \cap E$ has a measurable envelope F say (132Ee again). Set $\mathcal{H}' = \{H : H \in \mathcal{H}, B \cap H \ne \emptyset\}$ and for $H \in \mathcal{H}'$ choose $x_H \in B \cap H$; set $A = \{x_H : H \in \mathcal{H}'\}$, so that $A \subseteq B$ and $\#(A) \le \pi(\mu)$. **?** If A is negligible, then $F \setminus A$ includes a non-negligible measurable set so includes a member H of $\mathcal{H}$. As $\mu H > 0$ and F is a measurable envelope of B, H meets B and belongs to $\mathcal{H}'$, and $x_H \in A \cap H$. **X** Thus A is not negligible. As B is arbitrary, $\operatorname{shr}\mathcal{N}(\mu) \le \pi(\mu)$.

(d) For $E \in \Sigma \setminus \mathcal{N}(\mu)$ let F_E be a closed non-negligible subset of E and set $\phi(E) = F_E^\bullet \in \mathfrak{A}^+$; for $a \in \mathfrak{A}^+$, let $\psi(a)$ be a self-supporting measurable set such that $\psi(a)^\bullet = a$ (414F). Then if $\phi(E) \supseteq a$, $\psi(a) \setminus F_E$ is negligible so $E \supseteq F_E \supseteq \psi(a)$. Thus (ϕ, ψ) is a Galois-Tukey connection and $(\Sigma \setminus \mathcal{N}(\mu), \supseteq, \Sigma \setminus \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathfrak{A}^+, \supseteq, \mathfrak{A}^+)$.

Moreover, if $\psi(a) \supseteq E$, then $a \supseteq \phi(E)$, so (ψ, ϕ) also is a Galois-Tukey connection and $(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) \preccurlyeq_{\mathrm{GT}} (\Sigma \setminus \mathcal{N}(\mu), \supseteq, \Sigma \setminus \mathcal{N}(\mu))$.

Thus $(\Sigma \setminus \mathcal{N}(\mu), \supseteq, \Sigma \setminus \mathcal{N}(\mu)) \equiv_{\mathrm{GT}} (\mathfrak{A}^+, \supseteq, \mathfrak{A}^+)$, that is, $(\Sigma \setminus \mathcal{N}(\mu), \supseteq) \equiv_{\mathrm{T}} (\mathfrak{A}^+, \supseteq)$. By 513E(e-i), inverted,

$$\pi(\mu) = \operatorname{ci}(\Sigma \setminus \mathcal{N}(\mu)) = \operatorname{ci}(\mathfrak{A}^+) = \pi(\mathfrak{A}).$$

521E It will be useful later in the chapter to be able to calculate the topological density of measure-algebra topologies.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra.

(a) Give $\mathfrak{A}$ its measure-algebra topology (323A).

(i) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, it is topologically dense iff it τ-generates $\mathfrak{A}$, that is, $\mathfrak{A}$ is the order-closed subalgebra of itself generated by $\mathfrak{B}$.

(ii) If $\mathfrak{A}$ is finite, then its topological density is $\#(\mathfrak{A})$; if $\mathfrak{A}$ is infinite, its topological density is equal to its Maharam type $\tau(\mathfrak{A})$.

(b) Let $\mathfrak{A}^f$ be the set of elements of $\mathfrak{A}$ with finite measure, with its strong measure-algebra topology (323Ad). Then the topological density of $\mathfrak{A}^f$ is $\#(\mathfrak{A}^f) = \#(\mathfrak{A})$ if $\mathfrak{A}$ is finite, and $\max(c(\mathfrak{A}), \tau(\mathfrak{A}))$ if $\mathfrak{A}$ is infinite.

proof (a)(i)(α) Suppose that $\mathfrak{B}$ is topologically dense. Let $\mathfrak{C}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$. If $a \in \mathfrak{A}^f$ and $c \in \mathfrak{A}$, there is a $b \in \mathfrak{C}$ such that $b \cap a = c \cap a$. **P** For each $n \in \mathbb{N}$, there is an $a_n \in \mathfrak{B}$ such that $\bar\mu(a \cap (a_n \triangle c)) \le 2^{-n}$. Set $b = \inf_{n \in \mathbb{N}} \sup_{m \ge n} a_m \in \mathfrak{C}$; then $b \cap a = c \cap a$ (apply 323F to $\langle a \cap a_n \rangle_{n \in \mathbb{N}}$). **Q**

It follows that $\mathfrak{A}^f \subseteq \mathfrak{C}$. **P** If $c \in \mathfrak{A}^f$, then whenever $c \subseteq a \in \mathfrak{A}^f$ there is a $b_a \in \mathfrak{C}$ such that $b_a \cap a = c$. Now (because $\bar\mu$ is semi-finite) $c = \inf\{b_a : c \subseteq a \in \mathfrak{A}^f\} \in \mathfrak{C}$. **Q**

Finally, again because $\bar\mu$ is semi-finite,

$$c = \sup\{a : a \in \mathfrak{A}^f, a \subseteq c\} \in \mathfrak{C}$$

for every $c \in \mathfrak{A}$, and $\mathfrak{A} = \mathfrak{C}$. Thus $\mathfrak{B}$ τ-generates $\mathfrak{A}$.

(β) Suppose that $\mathfrak{B}$ τ-generates $\mathfrak{A}$. Then the topological closure of $\mathfrak{B}$ is order-closed (323D(c-i)) and a subalgebra (323B), so must be $\mathfrak{A}$, and $\mathfrak{B}$ is topologically dense.

(ii)(α) If $\mathfrak{A}$ is finite, this is trivial, just because the measure-algebra topology is Hausdorff (323Ga). So let us henceforth suppose that $\mathfrak{A}$ is infinite, so that both $\tau(\mathfrak{A})$ and the topological density $d_{\mathfrak{T}}(\mathfrak{A})$ of $\mathfrak{A}$ are infinite.

(β) Let $A \subseteq \mathfrak{A}$ be a set with cardinal $\tau(\mathfrak{A})$ which τ-generates $\mathfrak{A}$, and let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by A. Then $\#(\mathfrak{B}) = \#(A) = \tau(\mathfrak{A})$ (331Gc), and $\mathfrak{B}$ is topologically dense in $\mathfrak{A}$, by (i); so $d_{\mathfrak{T}}(\mathfrak{A}) \le \tau(\mathfrak{A})$.

(γ) Let $A \subseteq \mathfrak{A}$ be a topologically dense set with cardinal $d_{\mathfrak{T}}(\mathfrak{A})$, and $\mathfrak{B}$ the subalgebra of $\mathfrak{A}$ generated by A. Then $\mathfrak{B}$ is topologically dense, so it τ-generates $\mathfrak{A}$, and

$$\tau(\mathfrak{A}) \le \#(\mathfrak{B}) = \#(A) = d_{\mathfrak{T}}(\mathfrak{A});$$

with (β), this means that we have equality, as claimed.

(b)(i) The case of finite $\mathfrak{A}$ is again trivial; suppose that $\mathfrak{A}$ is infinite. Let $\langle a_i \rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ consisting of non-zero elements of finite measure.

(ii) The topological density $d_{\text{top}}(\mathfrak{A}^f)$ is at most $\max(c(\mathfrak{A}), \tau(\mathfrak{A}))$. **P** For each i, the topological density of $\mathfrak{A}_{a_i}$, with its measure-algebra topology, is at most $\max(\omega, \tau(\mathfrak{A}_{a_i})) \le \tau(\mathfrak{A})$ ((a) above and 514Ed); let $B_i \subseteq \mathfrak{A}_{a_i}$ be a dense subset of this size or less. Set $B = \bigcup_{i\in I} B_i$, $D = \{\sup B' : B' \in [B]^{<\omega}\}$. Then the metric closure $\overline{D}$ of D in $\mathfrak{A}^f$ is closed under $\cup$ and includes $\mathfrak{A}_{a_i}$ for every i. If now $a \in \mathfrak{A}^f$, $a = \sup_{i\in I} a \cap a_i \in \overline{D}$. So

$$d_{\text{top}}(\mathfrak{A}^f) \le \#(D) \le \max(\omega, \#(I), \tau(\mathfrak{A})) \le \max(c(\mathfrak{A}), \tau(\mathfrak{A})). \text{ **Q**}$$

(iii) $c(\mathfrak{A}) \le d_{\text{top}}(\mathfrak{A}^f)$. **P** Let $\langle b_j \rangle_{j\in J}$ be any disjoint family in $\mathfrak{A}^+$. For each j, let $b'_j \subseteq b_j$ be a non-zero element of non-zero finite measure. Set $G_j = \{a : a \in \mathfrak{A}^f,\ \bar\mu(a \triangle b'_j) < \bar\mu b'_j\}$ for $j \in J$. Then $\langle G_j \rangle_{j\in J}$ is a disjoint family of non-empty open sets in $\mathfrak{A}^f$, so $\#(J) \le d_{\text{top}}(\mathfrak{A}^f)$ (5A4Ba). As $\langle b_j \rangle_{j\in J}$ is arbitrary, $c(\mathfrak{A}) \le d_{\text{top}}(\mathfrak{A}^f)$. **Q**

(iv) $\tau(\mathfrak{A}) \le d_{\text{top}}(\mathfrak{A}^f)$. **P** Let $A \subseteq \mathfrak{A}^f$ be a dense set with cardinal $d_{\text{top}}(\mathfrak{A}^f)$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $B = A \cup \{a_i : i \in I\}$. For any $i \in I$, set $A_i = \{a \cap a_i : a \in A\}$. Now A_i is topologically dense in $\mathfrak{A}_{a_i}$ (use 3A3Eb), so the order-closed subalgebra of $\mathfrak{A}_{a_i}$ it generates is the whole of $\mathfrak{A}_{a_i}$ (323H); by 314H, $\mathfrak{A}_{a_i} = \{b \cap a_i : b \in \mathfrak{B}\}$. As $a_i \in A \subseteq \mathfrak{B}$, $\mathfrak{B}$ includes $\mathfrak{A}_{a_i}$. As $\sup_{i\in I} a_i = 1$, $\mathfrak{B} = \mathfrak{A}$. Thus

$$\tau(\mathfrak{A}) \le \#(B) \le \max(\omega, \#(I), d_{\text{top}}(\mathfrak{A}^f)) = d_{\text{top}}(\mathfrak{A}^f)$$

(using (iii) for the last equality). **Q**

(v) Putting these together, we have the result.

521F Proposition Let (X, Σ, μ) be a measure space, A a subset of X and μ_A the subspace measure on A.

(a) $\mathcal{N}(\mu_A) \preccurlyeq_{\text{T}} \mathcal{N}(\mu)$, so $\operatorname{add}\mathcal{N}(\mu_A) \ge \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\mu_A) \le \operatorname{cf}\mathcal{N}(\mu)$.

(b) $(A, \in, \mathcal{N}(\mu_A)) \preccurlyeq_{\text{GT}} (X, \in, \mathcal{N}(\mu))$, so $\operatorname{non}\mathcal{N}(\mu_A) \ge \operatorname{non}\mathcal{N}(\mu)$ and $\operatorname{cov}\mathcal{N}(\mu_A) \le \operatorname{cov}\mathcal{N}(\mu)$.

(c) $\operatorname{add}\mu_A \ge \operatorname{add}\mu$.

(d) $\operatorname{shr}\mathcal{N}(\mu_A) \le \operatorname{shr}\mathcal{N}(\mu)$ and $\operatorname{shr}^+\mathcal{N}(\mu_A) \le \operatorname{shr}^+\mathcal{N}(\mu)$.

(e) If either $A \in \Sigma$ or (X, Σ, μ) has locally determined negligible sets, $\pi(\mu_A) \le \pi(\mu)$.

(f) If μ_A is semi-finite, then $\tau(\mu_A) \le \tau(\mu)$.

proof (a) Because $\mathcal{N}(\mu_A) = \mathcal{P}A \cap \mathcal{N}(\mu)$ (214Cb), the embedding $\mathcal{N}(\mu_A) \subsetneqq \mathcal{N}(\mu)$ is a Tukey function, and $\mathcal{N}(\mu_A) \preccurlyeq_{\text{T}} \mathcal{N}(\mu)$. By 513Ee, $\operatorname{add}\mathcal{N}(\mu_A) \ge \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\mu_A) \le \operatorname{cf}\mathcal{N}(\mu)$.

(b) Next, setting $\phi(x) = x$ for $x \in A$ and $\psi(F) = F \cap A$ for $F \in \mathcal{N}(\mu)$, (ϕ, ψ) witnesses that $(A, \in, \mathcal{N}(\mu_A)) \preccurlyeq_{\text{GT}} (X, \in, \mathcal{N}(\mu))$. By 512D and 512Ed,

$$\operatorname{non}\mathcal{N}(\mu_A) = \operatorname{add}(A, \in, \mathcal{N}(\mu_A)) \ge \operatorname{add}(X, \in, \mathcal{N}(\mu)) = \operatorname{non}\mathcal{N}(\mu),$$

$$\operatorname{cov}\mathcal{N}(\mu_A) = \operatorname{cov}(A, \in, \mathcal{N}(\mu_A)) \le \operatorname{cov}(X, \in, \mathcal{N}(\mu)) = \operatorname{cov}\mathcal{N}(\mu).$$

(c) If $\langle F_\xi\rangle_{\xi<\kappa}$ is a disjoint family in $\Sigma_A = \operatorname{dom}\mu_A$, where $\kappa < \operatorname{add}\mu$, then for each $\xi < \kappa$ we have an $E_\xi \in \Sigma$ such that $F_\xi = A \cap E_\xi$ and $\mu_A F_\xi = \mu E_\xi$ (214Ca). Set $E'_\xi = E_\xi \setminus \bigcup_{\eta<\xi} E_\eta$ for $\xi < \kappa$; then $E'_\xi \in \Sigma$ for each ξ, and $\langle E'_\xi\rangle_{\xi<\kappa}$ is disjoint. Set $E = \bigcup_{\xi<\kappa} E'_\xi = \bigcup_{\xi<\kappa} E_\xi$ and $F = A \cap E = \bigcup_{\xi<\kappa} F_\xi$. Then

$$\sum_{\xi<\kappa} \mu_A F_\xi \le \mu_A F \le \mu E = \sum_{\xi<\kappa} \mu E'_\xi \le \sum_{\xi<\kappa} \mu E_\xi = \sum_{\xi<\kappa} \mu_A F_\xi,$$

and we have equality. As $\langle F_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{add}\mu_A \ge \operatorname{add}\mu$.

(d) If $B \in \mathcal{P}A \setminus \mathcal{N}(\mu_A)$, there is a $C \subseteq B$ such that $C \notin \mathcal{N}(\mu)$ and $\#(C) \le \operatorname{shr}\mathcal{N}(\mu)$ (resp. $\#(C) < \operatorname{shr}^+\mathcal{N}(\mu)$); now $C \notin \mathcal{N}(\mu_A)$; as B is arbitrary, $\operatorname{shr}\mathcal{N}(\mu_A) \le \operatorname{shr}\mathcal{N}(\mu)$ (resp. $\operatorname{shr}^+\mathcal{N}(\mu_A) \le \operatorname{shr}^+\mathcal{N}(\mu)$).

(e) Let $\mathcal{H} \subseteq \Sigma\setminus\mathcal{N}(\mu)$ be a coinitial set with cardinal $\pi(\mu)$. Set $\mathcal{G} = \{H \cap A : H \in \mathcal{H}\}\setminus\mathcal{N}(\mu)$. Then $\mu_A G$ is defined and non-zero for every $G \in \mathcal{G}$. Now $\mathcal{G}$ is coinitial with $\operatorname{dom}\mu_A \setminus \mathcal{N}(\mu_A)$. **P** If $\mu_A B > 0$, there is an $E \in \Sigma$ such that $B = E \cap A$. If $A \in \Sigma$, then $B \in \Sigma$ and there is an $H \in \mathcal{H}$ such that $H \subseteq B$, while of course $H \in \mathcal{G}$. If (X, Σ, μ) has locally determined negligible sets, then, as in the proof of 521Dc, there is a non-negligible set $F \in \Sigma$ which is a measurable envelope of a subset of B. Now there is an $H \in \mathcal{H}$ included in $F \cap E$, in which case $H \cap A$ is included in B and belongs to $\mathcal{G}$. **Q** So

$$\pi(\mu_A) \le \#(\mathcal{G}) \le \#(\mathcal{H}) = \pi(\mu).$$

(f) Writing $\mathfrak{A}$, $\mathfrak{A}_A$ for the measure algebras of μ and μ_A, we have a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_A$ defined by saying that $\pi E^\bullet = (F \cap A)^\bullet$ for every $E \in \Sigma$. (The point is just that $F \cap A \in \mathcal{N}(\mu_A)$ whenever $F \in \mathcal{N}(\mu)$.) Now π is order-continuous. **P** Suppose that $C \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf C = 0$ in $\mathfrak{A}$. **?** If $b \in \mathfrak{A}_A$ is a non-zero lower bound of $\pi[C]$, then, because ν_A is semi-finite, there is a $G \in \operatorname{dom}\mu_A$ such that $0 < \mu_A G < \infty$ and $G^\bullet \subseteq b$. Let $E \in \Sigma$ be such that $G = E \cap A$ and $\mu E = \mu_A G$ (214Ca). Then $E^\bullet$ cannot be a lower bound of C; let $a \in C$ be such that $E^\bullet \setminus a \ne 0$. In this case, there is an $F \in \Sigma$ such that $F \subseteq E$ and $F^\bullet = E^\bullet \setminus a$, so that $\pi F^\bullet$ is disjoint from $\pi a \supseteq b$, and $F \cap G = (F \cap A) \cap G$ must be negligible. We know that $\mu F > 0$, so $\mu(E \setminus F) < \mu E$; but also $G \setminus (E \setminus F)$ is negligible, so

$$\mu_A G = \mu^* G \le \mu(E \setminus F) < \mu E = \mu_A G. \text{ **X**}$$

It follows that $\inf \pi[C] = 0$ in $\mathfrak{A}_A$; as C is arbitrary, π is order-continuous. **Q**

Now let $B \subseteq \mathfrak{A}$ be such that B τ-generates $\mathfrak{A}$ and $\#(B) = \tau(\mathfrak{A})$. Writing $\mathfrak{B}$ for the order-closed subalgebra of $\mathfrak{A}_A$ generated by $\pi[B]$, we see that $\pi^{-1}[\mathfrak{B}]$ is an order-closed subalgebra of $\mathfrak{A}$ including B, so must be the whole of $\mathfrak{A}$, and $\mathfrak{A}_A = \pi[\mathfrak{A}] = \mathfrak{B}$. Accordingly

$$\tau(\mu_A) = \tau(\mathfrak{A}_A) \le \#(\pi[B]) \le \#(B) = \tau(\mathfrak{A}) = \tau(\mu),$$

as claimed.

521G Proposition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a non-empty family of measure spaces with direct sum (X, Σ, μ). Then

$$\operatorname{add}\mathcal{N}(\mu) = \min_{i\in I}\operatorname{add}\mathcal{N}(\mu_i), \quad \operatorname{add}\mu = \min_{i\in I}\operatorname{add}\mu_i,$$

$$\operatorname{cov}\mathcal{N}(\mu) = \sup_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i), \quad \operatorname{non}\mathcal{N}(\mu) = \min_{i\in I}\operatorname{non}\mathcal{N}(\mu_i),$$

$$\operatorname{shr}\mathcal{N}(\mu) = \sup_{i\in I}\operatorname{shr}\mathcal{N}(\mu_i), \quad \operatorname{shr}^+\mathcal{N}(\mu) = \sup_{i\in I}\operatorname{shr}^+\mathcal{N}(\mu_i),$$

$$\tau(\mu) \le \max(\omega, \sup_{i\in I}\tau(\mu_i), \min\{\lambda : \#(I) \le 2^\lambda\})$$

and $\pi(\mu)$ is the cardinal sum $\sum_{i\in I}\pi(\mu_i)$. If I is finite, then

$$\operatorname{cf}\mathcal{N}(\mu) = \max_{i\in I}\operatorname{cf}\mathcal{N}(\mu_i).$$

proof Concerning each of $\operatorname{add}\mathcal{N}(\mu)$, $\operatorname{add}\mu$, $\operatorname{cov}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\mu)$, $\operatorname{shr}\mathcal{N}(\mu)$ and $\operatorname{shr}^+\mathcal{N}(\mu)$, 521F provides an inequality in one direction. The reverse inequalities are equally straightforward, especially if we note that $\mathcal{N}(\mu) \cong \prod_{i\in I}\mathcal{N}(\mu_i)$, so that 512Hc is relevant. For $\tau(\mu)$, 514Ef provides the formula at once if we note that the measure algebra of μ is isomorphic to the simple product of the measure algebras of the μ_i (cf. 322M).

As for $\pi(\mu)$, if for each $i \in I$ we choose a coinitial set $\mathcal{H}_i$ of $\Sigma_i \setminus \mathcal{N}(\mu_i)$ with cardinal $\pi(\mu_i)$, then

$$\mathcal{H} = \{H \times \{i\} : i \in I,\ H \in \mathcal{H}_i\}$$

is coinitial with $\Sigma\setminus\mathcal{N}(\mu)$ and witnesses that $\pi(\mu)\le\sum_{i\in I}\pi(\mu_i)$. (As in 214L, I am thinking of X as $\bigcup_{i\in I}X_i\times\{i\}$.) Conversely, if $\mathcal{H}$ is coinitial with $\Sigma\setminus\mathcal{N}(\mu)$ and for each $i\in I$ we set $\mathcal{H}_i=\{H:H\times\{i\}\in\mathcal{H}\}$, we shall have $\mathcal{H}_i$ coinitial with $\Sigma_i\setminus\mathcal{N}(\mu_i)$, so that

$$\sum_{i\in I}\pi(\mu_i)\le\sum_{i\in I}\#(\mathcal{H}_i)\le\#(\mathcal{H})=\pi(\mu)$$

and $\pi(\mu)=\sum_{i\in I}\pi(\mu_i)$.

521H Proposition Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $f:X\to Y$ an inverse-measure-preserving function.

(a)(i) $(X,\in,\mathcal{N}(\mu))\preccurlyeq_{\mathrm{GT}}(Y,\in,\mathcal{N}(\nu))$, so $\operatorname{non}\mathcal{N}(\mu)\ge\operatorname{non}\mathcal{N}(\nu)$ and $\operatorname{cov}\mathcal{N}(\mu)\le\operatorname{cov}\mathcal{N}(\nu)$.

(ii) If there is a topology on Y such that ν is a topological measure inner regular with respect to the closed sets, then $\pi(\nu)\le\pi(\mu)$.

(iii) If ν is σ-finite, then $\tau(\nu)\le\tau(\mu)$.

(b) If ν is the image measure μf^{-1}, then $\operatorname{add}\nu\ge\operatorname{add}\mu$. If, moreover, μ is complete, $\mathcal{N}(\nu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)$, so $\operatorname{add}\mathcal{N}(\mu)\le\operatorname{add}\mathcal{N}(\nu)$ and $\operatorname{cf}\mathcal{N}(\mu)\ge\operatorname{cf}\mathcal{N}(\nu)$; also $\operatorname{shr}\mathcal{N}(\mu)\ge\operatorname{shr}\mathcal{N}(\nu)$ and $\operatorname{shr}^+\mathcal{N}(\mu)\ge\operatorname{shr}^+\mathcal{N}(\nu)$.

proof (a)(i) Set $\psi(F)=f^{-1}[F]$ for $F\in\mathcal{N}(\nu)$. Then (f,ψ) is a Galois-Tukey connection from $(X,\in,\mathcal{N}(\mu))$ to $(Y,\in,\mathcal{N}(\nu))$, so

$$\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}(X,\in,\mathcal{N}(\mu))\le\operatorname{cov}(Y,\in,\mathcal{N}(\nu))=\operatorname{cov}\mathcal{N}(\nu),$$

$$\operatorname{non}\mathcal{N}(\mu)=\operatorname{add}(X,\in,\mathcal{N}(\mu))\ge\operatorname{add}(Y,\in,\mathcal{N}(\nu))=\operatorname{non}\mathcal{N}(\nu)$$

(512D, 512Ed again).

(ii) Let $\mathcal{H}$ be a coinitial subset of $\Sigma\setminus\mathcal{N}(\mu)$ with cardinal $\pi(\mu)$. Set $\mathcal{G}=\{\overline{f[H]}:H\in\mathcal{H}\}$. Because ν is a topological measure, $\mathcal{G}\subseteq\mathrm{T}$; and if $H\in\mathcal{H}$, then

$$\nu\overline{f[H]}=\mu f^{-1}[\overline{f[H]}]\ge\mu H>0,$$

so $\mathcal{G}\subseteq\mathrm{T}\setminus\mathcal{N}(\nu)$. If $F\in\mathrm{T}\setminus\mathcal{N}(\nu)$, there is a closed set $F'\subseteq F$ such that $0<\nu F'=\mu f^{-1}[F']$; there is an $H\in\mathcal{H}$ such that $H\subseteq f^{-1}[F']$; now $G=\overline{f[H]}$ belongs to $\mathcal{G}$ and is included in $F'\subseteq F$. So $\mathcal{G}$ is coinitial with $\mathrm{T}\setminus\mathcal{N}(\nu)$ and

$$\pi(\nu)\le\#(\mathcal{G})\le\#(\mathcal{H})=\pi(\mu).$$

(iii) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be the measure algebras of μ, ν respectively. Then we have a sequentially order-continuous measure-preserving Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by setting $\pi F^\bullet=f^{-1}[F]^\bullet$ for every $F\in\operatorname{dom}\nu$ (324M). If $\mathfrak{A}$ is finite then $\mathfrak{B}$ must be finite with $\#(\mathfrak{B})\le\#(\mathfrak{A})$, and consequently $\tau(\mathfrak{B})\le\tau(\mathfrak{A})$ (331Xc, or otherwise). So let us suppose that $\mathfrak{A}$ is infinite.

Writing $\mathfrak{A}^f$, $\mathfrak{B}^f$ for the respective ideals of elements of finite measure, $\pi\restriction\mathfrak{B}^f$ is a function from $\mathfrak{B}^f$ to $\mathfrak{A}^f$ which is an isometry for the measure metrics on $\mathfrak{B}^f$ and $\mathfrak{A}^f$. So the topological density $d_{\mathrm{top}}(\mathfrak{B}^f)$ is equal to $d_{\mathrm{top}}(\pi[\mathfrak{B}^f])$ and less than or equal to $d_{\mathrm{top}}(\mathfrak{A}^f)$ (5A4B(h-ii)).

Observe next that $(\mathfrak{B},\bar\nu)$ is σ-finite because ν is, and that $(\mathfrak{A},\bar\mu)$ therefore also is (324Kd). So we get

$$\tau(\nu)=\tau(\mathfrak{B})\le\max(\omega,c(\mathfrak{B}),\tau(\mathfrak{B}))=\max(\omega,d_{\mathrm{top}}(\mathfrak{B}^f))$$

(521Eb)

$$\le\max(\omega,d_{\mathrm{top}}(\mathfrak{A}^f))=\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))=\tau(\mathfrak{A})=\tau(\mu),$$

as required.

(b) If $\langle F_\xi\rangle_{\xi<\kappa}$ is a disjoint family in T, where $\kappa<\operatorname{add}\mu$, then $\langle f^{-1}[F_\xi]\rangle_{\xi<\kappa}$ is a disjoint family in Σ, so

$$\nu(\bigcup_{\xi<\kappa}F_\xi)=\mu f^{-1}[\bigcup_{\xi<\kappa}F_\xi]=\mu(\bigcup_{\xi<\kappa}f^{-1}[F_\xi]=\sum_{\xi<\kappa}\mu f^{-1}[F_\xi]=\sum_{\xi<\kappa}\nu F_\xi.$$

As $\langle F_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{add}\nu\ge\operatorname{add}\mu$.

Now suppose that μ is complete. In this case, $F \in \mathrm{T}$ whenever $F \subseteq Y$ and $f^{-1}[F] \in \mathcal{N}(\mu)$, so that $\mathcal{N}(\nu)$ is precisely $\{F : F \subseteq Y,\ f^{-1}[F] \in \mathcal{N}(\mu)\}$. It is now easy to check that $F \mapsto f^{-1}[F] : \mathcal{N}(\nu) \to \mathcal{N}(\mu)$ is a Tukey function. So $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}(\mu)$, by 513Ee again.

Take any non-negligible $A \subseteq Y$. Then $f^{-1}[A] \notin \mathcal{N}(\mu)$, so there is a set $B \subseteq f^{-1}[A]$ such that $\#(B) \le \operatorname{shr}\mathcal{N}(\mu)$ and $B \notin \mathcal{N}(\mu)$. In this case, $f[B] \subseteq A$, $f[B] \notin \mathcal{N}(\nu)$ and $\#(f[B]) \le \operatorname{shr}\mathcal{N}(\mu)$. As A is arbitrary, $\operatorname{shr}\mathcal{N}(\nu) \le \operatorname{shr}\mathcal{N}(\mu)$. The same argument, with $<$ instead of $\le$ at appropriate points, shows that $\operatorname{shr}^+\mathcal{N}(\nu) \le \operatorname{shr}^+\mathcal{N}(\mu)$.

521I Corollary Let (X, Σ, μ) be an atomless strictly localizable measure space. Then $\operatorname{non}\mathcal{N}(\mu) \ge \operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}(\mu) \le \operatorname{cov}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

proof (a) If $\mu X = 0$ this is trivial.

(b) If $0 < \mu X < \infty$, let ν be the completion of the normalized measure $\frac{1}{\mu X}\mu$. Then ν is complete and atomless, so by 343Cb there is an inverse-measure-preserving function from (X, ν) to $([0,1], \mu_1)$, where μ_1 is Lebesgue measure on $[0,1]$. Also $\mathcal{N}(\nu) = \mathcal{N}(\mu)$. By 521Ha, $\operatorname{non}\mathcal{N}(\nu) \ge \operatorname{non}\mathcal{N}(\mu_1)$ and $\operatorname{cov}\mathcal{N}(\nu) \le \operatorname{cov}\mathcal{N}(\mu_1)$. Now $([0,1], \mathcal{N}(\mu_1))$ is isomorphic to $(\mathbb{R}, \mathcal{N})$. **P** Take a bijection $h : \mathbb{R} \to [0,1]$ such that $h(x) = \frac{1}{2}(1 + \tanh x)$ for $x \in \mathbb{R} \setminus \mathbb{Q}$; then h is a suitable isomorphism. **Q** So $\operatorname{non}\mathcal{N}(\mu) = \operatorname{non}\mathcal{N}(\nu) \ge \operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}(\mu) \le \operatorname{cov}\mathcal{N}$.

(c) If X has infinite measure, let $\langle X_i \rangle_{i \in I}$ be a decomposition of X into sets of finite measure. For each $i \in I$ let μ_i be the subspace measure on X_i. Then every μ_i is atomless, so, putting (b) and 521G together,

$$\operatorname{non}\mathcal{N}(\mu) = \min_{i \in I} \operatorname{non}\mathcal{N}(\mu_i) \ge \operatorname{non}\mathcal{N},$$

$$\operatorname{cov}\mathcal{N}(\mu) = \sup_{i \in I} \operatorname{cov}\mathcal{N}(\mu_i) \le \operatorname{cov}\mathcal{N}.$$

521J For product spaces the situation is more complicated, because the product measure introduces 'new' negligible sets which are not directly definable in terms of the null ideals of the factors. In the next three sections, however, we shall find out quite a lot about the cardinal functions of Radon measures, and this information, when it comes, can be used to give results about general products of probability measures.

Proposition Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a non-empty family of probability spaces with product (X, Σ, μ).

(a)

$$\operatorname{non}\mathcal{N}(\mu) \ge \sup_{i \in I} \operatorname{non}\mathcal{N}(\mu_i), \quad \operatorname{cov}\mathcal{N}(\mu) \le \min_{i \in I} \operatorname{cov}\mathcal{N}(\mu_i),$$

$$\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu) \le \min_{i \in I} \operatorname{add}\mathcal{N}(\mu_i), \quad \operatorname{cf}\mathcal{N}(\mu) \ge \sup_{i \in I} \operatorname{cf}\mathcal{N}(\mu_i),$$

$$\operatorname{shr}\mathcal{N}(\mu) \ge \sup_{i \in I} \operatorname{shr}\mathcal{N}(\mu_i), \quad \operatorname{shr}^+\mathcal{N}(\mu) \ge \sup_{i \in I} \operatorname{shr}^+\mathcal{N}(\mu_i),$$

$$\pi(\mu) \ge \sup_{i \in I} \pi(\mu_i).$$

(b) Set $\kappa = \#(\{i : i \in I,\ \Sigma_i \ne \{\emptyset, X_i\}\})$. Then $[\kappa]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$; consequently $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$ is ω_1 if κ is uncountable, while $\operatorname{cf}\mathcal{N}(\mu)$ is at least $\operatorname{cf}[\kappa]^{\le\omega}$.

(c) Now suppose that I is countable and that we have for each $i \in I$ a probability space $(Y_i, \mathrm{T}_i, \nu_i)$ and an inverse-measure-preserving function $f_i : X_i \to Y_i$ which represents an isomorphism of the measure algebras of μ_i and ν_i. Let (Y, T, ν) be the product of $\langle (Y_i, \mathrm{T}_i, \nu_i) \rangle_{i \in I}$. Then

$$\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu) \times \prod_{i \in I} \mathcal{N}(\mu_i).$$

Consequently

$$\operatorname{add}\mathcal{N}(\mu) \ge \min(\operatorname{add}\mathcal{N}(\nu), \min_{i \in I} \operatorname{add}\mathcal{N}(\mu_i)),$$

and if I is finite

$$\operatorname{cf}\mathcal{N}(\mu) \le \max(\operatorname{cf}\mathcal{N}(\nu), \max_{i \in I} \operatorname{cf}\mathcal{N}(\mu_i)).$$

(d) If I is finite, then

$$\operatorname{non}\mathcal{N}(\mu) = \max_{i\in I}\operatorname{non}\mathcal{N}(\mu_i), \quad \operatorname{cov}\mathcal{N}(\mu) = \min_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i).$$

proof (a) Note that $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$ by 521Ad. Now with one exception the inequalities are immediate if we apply 521H to the canonical maps from X to X_i. The odd one out is the last, because we do not have a simple general result concerning the π-weight of an image measure. But in the present case we can argue as follows. Let $\mathcal{H} \subseteq \Sigma\setminus\mathcal{N}(\mu)$ be a coinitial set with cardinal $\pi(\mu)$, and take $i \in I$. Then we can identify (X,Σ,μ) with $(X',\Sigma',\mu')\times(X_i,\Sigma_i,\mu_i)$ where (X',Σ',μ') is the product of the family $\langle(X_j,\Sigma_j,\mu_j)\rangle_{j\in I,j\ne i}$ (254N). If $H \in \mathcal{H}$, $\mu(X\setminus H) = \int \mu_i^*(X_i\setminus H[\{x'\}])\mu'(dx')$ (252D) is less than 1, so there is an $x'_H \in X'$ such that $\mu_i^*(X_i\setminus H[\{x'_H\}]) < 1$, $(\mu_i)_*H[\{x'_H\}] > 0$ and there is a $G_H \in \Sigma_i\setminus\mathcal{N}(\mu_i)$ such that $G_H \subseteq H[\{x'_H\}]$. Set $\mathcal{G} = \{G_H : H \in \mathcal{H}\}$. If $\mu_i F > 0$, then $\mu(X'\times F) > 0$ and there is an $H \in \mathcal{H}$ included in $X'\times F$; in which case $G_H \subseteq H[\{x'_H\}] \subseteq F$. So $\mathcal{G}$ is coinitial with $\Sigma_i\setminus\mathcal{N}(\mu_i)$ and

$$\pi(\mu_i) \le \#(\mathcal{G}) \le \#(\mathcal{H}) \le \pi(\mu).$$

As i is arbitrary, $\sup_{i\in I}\pi(\mu_i) \le \pi(\mu)$, as claimed.

(b)(i) If $\kappa \le \omega$ then the constant function with value $\emptyset$ is a Tukey function from $[\kappa]^{\le\omega}$ to $\mathcal{N}(\mu)$. Otherwise, set $J = \{i : i \in I,\ \Sigma_i \ne \{\emptyset, X_i\}\}$ and for $i \in J$ choose a non-empty $C_i \in \Sigma_i$ such that $\mu_i C_i \le \frac{1}{2}$. Index J as $\langle i_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ and for $\xi < \kappa$ set $E_\xi = \{x : x(i_{\xi n}) \in C_{i_{\xi n}}$ for every $n \in \mathbb{N}\}$, so that $\mu E_\xi = \prod_{n\in\mathbb{N}}\mu_{i_{\xi n}}C_{i_{\xi n}} = 0$. Define $\phi : [\kappa]^{\le\omega} \to \mathcal{N}(\mu)$ by setting $\phi K = \bigcup_{\xi\in K}E_\xi$ for countable $K \subseteq \kappa$. Then ϕ is a Tukey function. **P** If $E \in \mathcal{N}(\mu)$, there is a negligible $E' \supseteq E$ which is determined by coordinates in a countable set I' (254Oc). Set $L = \{\xi : \xi < \kappa,\ i_{\xi n} \in I'$ for some $n \in \mathbb{N}\}$; then L is countable. If $\xi < \kappa$ and $E_\xi \subseteq E'$, E_ξ is determined by coordinates in $\{i_{\xi n} : n \in \mathbb{N}\}$; as neither E_ξ nor $X\setminus E'$ is empty, this must meet I', and $\xi \in L$. So $\{K : K \in [\kappa]^{\le\omega},\ \phi K \subseteq E\}$ is bounded above by $L \in [\kappa]^{\le\omega}$. As E is arbitrary, ϕ is a Tukey function. **Q** Accordingly $[\kappa]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$.

(ii) It follows that $\operatorname{add}\mathcal{N}(\mu) \le \operatorname{add}[\kappa]^{\le\omega} \le \omega_1$ if κ is uncountable, and that $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}[\kappa]^{\le\omega}$.

(c)(i) Recall that any inverse-measure-preserving function f between measure spaces induces a measure-preserving Boolean homomorphism $F^\bullet \mapsto (f^{-1}[F])^\bullet$ between the measure algebras (324M). For $i \in I$ and $C \in \Sigma_i$ choose $\psi_i(C) \in \mathrm{T}_i$ such that $(f_i^{-1}[\psi_i(C)])^\bullet = C^\bullet$ in the measure algebra of μ_i, that is, $C\triangle f_i^{-1}[\psi_i(C)] \in \mathcal{N}(\mu_i)$. Next, for $E \in \mathcal{N}(\mu)$ and $n \in \mathbb{N}$, choose a family $\langle C_{Enmi}\rangle_{m\in\mathbb{N},i\in I}$ such that $C_{Enmi} \in \Sigma_i$ for every $m \in \mathbb{N}$ and $i \in I$, $E \subseteq \bigcup_{m\in\mathbb{N}}\prod_{i\in I}C_{Enmi}$, and $\sum_{m\in\mathbb{N}}\prod_{i\in I}\mu_i C_{Enmi} \le 2^{-n}$; set

$$\phi(E) = \bigl(\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\prod_{i\in I}\psi_i(C_{Enmi}), \langle\bigcup_{m,n\in\mathbb{N}}(C_{Enmi}\setminus f_i^{-1}[\psi_i(C_{Enmi})])\rangle_{i\in I}\bigr).$$

Because

$$\begin{aligned}\nu(\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\prod_{i\in I}\psi_i(C_{Enmi})) &\le \inf_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\prod_{i\in I}\nu_i\psi_i(C_{Enmi})\\ &= \inf_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\prod_{i\in I}\mu_i C_{Enmi} = 0,\end{aligned}$$

ϕ is a function from $\mathcal{N}(\nu)$ to $\mathcal{N}(\nu)\times\prod_{i\in I}\mathcal{N}(\mu_i)$.

Now ϕ is a Tukey function. **P** Suppose that $W \in \mathcal{N}(\nu)$ and that $E_i \in \mathcal{N}(\mu_i)$ for every $i \in I$. Define $f : X \to Y$ by setting $f(x) = \langle f_i(x(i))\rangle_{i\in I}$ for $x \in X$; then f is inverse-measure-preserving (254H). So $V = f^{-1}[W]\cup\bigcup_{i\in I}\{x : x(i) \in E_i\}$ is negligible. (This is where we need to know that I is countable.) Suppose that $E \in \mathcal{N}(\mu)$ is such that $\phi(E) \le (W, \langle E_i\rangle_{i\in I})$; take $x \in E$ such that $x(i) \notin E_i$ for every $i \in I$, and $n \in \mathbb{N}$. Then there is an $m \in \mathbb{N}$ such that $x \in \prod_{i\in I}C_{Enmi}$. For each $i \in I$,

$$C_{Enmi}\setminus f_i^{-1}[\psi_i(C_{Enmi})] \subseteq E_i, \quad x(i) \in C_{Enmi}\setminus E_i,$$

so $f_i(x(i)) \in \psi_i(C_{Enmi})$; thus $f(x) \in \prod_{i\in I}\psi_i(C_{Enmi})$. As n is arbitrary,

$$f(x) \in \bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\prod_{i\in I}\psi_i(C_{Enmi}) \subseteq W$$

and $x \in V$. As x is arbitrary, $E \subseteq V$. As $(W, \langle E_i\rangle_{i\in I})$ is arbitrary, ϕ is a Tukey function. **Q**

So $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu)\times\prod_{i\in I}\mathcal{N}(\mu_i)$.

(ii) Accordingly

$$\operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}(\mathcal{N}(\nu) \times \prod_{i\in I}\mathcal{N}(\mu_i)) = \min(\operatorname{add}\mathcal{N}(\nu), \min_{i\in I}\operatorname{add}\mathcal{N}(\mu_i))$$

and

$$\operatorname{cf}\mathcal{N}(\mu) \le \operatorname{cf}(\mathcal{N}(\nu) \times \prod_{i\in I}\mathcal{N}(\mu_i)) = \max(\operatorname{cf}\mathcal{N}(\nu), \max_{i\in I}\operatorname{cf}\mathcal{N}(\mu_i))$$

if I is finite.

(d)(i) For each $i \in I$ let $A_i \subseteq X_i$ be a non-negligible set with cardinal $\operatorname{non}\mathcal{N}(\mu_i)$. Then $A = \prod_{i\in I} A_i$ is not negligible (251Wm), while $\#(A) \le \max(\omega, \max_{i\in I}\operatorname{non}\mathcal{N}(\mu_i))$. If all the $\operatorname{non}\mathcal{N}(\mu_i)$ are finite, then they are all equal to 1, and A is a singleton. So we must in any case have $\#(A) = \max_{i\in I}\operatorname{non}\mathcal{N}(\mu_i)$, and $\operatorname{non}\mathcal{N}(\mu) \le \max_{i\in I}\operatorname{non}\mathcal{N}(\mu_i)$. By (a), we have equality.

(ii) Suppose that $I = \{0, 1\}$, and that $\mathcal{E}$ is a cover of $X = X_0 \times X_1$ by negligible sets. For each $E \in \mathcal{E}$, set $C_E = \{x : x \in X_0,\ E[\{x\}] \notin \mathcal{N}(\mu_1)\}$; then C_E is negligible. If $\#(\mathcal{E}) < \operatorname{cov}\mathcal{N}(\mu_0)$, then there is an $x \in X_0 \setminus \bigcup_{E\in\mathcal{E}} C_E$; in which case $\{E[\{x\}] : E \in \mathcal{E}\}$ witnesses that $\operatorname{cov}\mathcal{N}(\mu_1) \le \#(\mathcal{E})$. So $\#(\mathcal{E})$ must be at least $\min(\operatorname{cov}\mathcal{N}(\mu_0), \operatorname{cov}\mathcal{N}(\mu_1))$. As $\mathcal{E}$ is arbitrary, $\operatorname{cov}\mathcal{N}(\mu) \ge \min(\operatorname{cov}\mathcal{N}(\mu_0), \operatorname{cov}\mathcal{N}(\mu_1))$.

Now an induction on $\#(I)$ (using the associative law 254N) shows that $\operatorname{cov}\mathcal{N}(\mu) \ge \min_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i)$ whenever I is finite. Using (a) again, we have equality here also.

Remark The simplest applications of (c) here will be when the μ_i are Maharam-type-homogeneous, so that we can take the ν_i to be the usual measures on powers $\{0, 1\}^{\kappa_i}$ of $\{0, 1\}$, and ν will be isomorphic to the usual measure on $\{0, 1\}^\kappa$ where κ is the cardinal sum $\sum_{i\in I}\kappa_i$. The cardinal functions of these measures are dealt with in §523. For non-homogeneous μ_i we shall still be able to arrange for the ν_i to be completion regular Radon measures on dyadic spaces, so that the product measure ν is again a Radon measure Y (532F), and (once we have identified its measure algebra – see 334E, 334Ya) approachable by the methods of §524.

521K I turn now to 'perfect' and 'compact' measure spaces. (See §451 for the basic theory of these.)

Proposition Let (X, Σ, μ) be a perfect semi-finite measure space which is not purely atomic. Then

$$\operatorname{add}\mathcal{N}(\mu) \le \operatorname{add}\mathcal{N}, \quad \operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N},$$

$$\operatorname{shr}\mathcal{N}(\mu) \ge \operatorname{shr}\mathcal{N}, \quad \operatorname{shr}^+\mathcal{N}(\mu) \ge \operatorname{shr}^+\mathcal{N}, \quad \pi(\mu) \ge \pi(\mu_L)$$

where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$ and μ_L is Lebesgue measure on $\mathbb{R}$.

proof (a) Suppose first that μ is a complete atomless probability measure. Then there is a function $f : X \to [0, 1]$ which is inverse-measure-preserving for μ and Lebesgue measure μ_1 on $[0, 1]$ (343Cb again); and in fact μ_1 is the image measure μf^{-1}. **P** By 451O, μf^{-1} is a Radon measure; since it extends μ_1 it must actually be equal to μ_1, by 415H. **Q** So $\operatorname{add}\mathcal{N}(\mu) \le \operatorname{add}\mathcal{N}(\mu_1)$, $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N}(\mu_1)$, $\operatorname{shr}\mathcal{N}(\mu) \ge \operatorname{shr}\mathcal{N}(\mu_1)$, and $\operatorname{shr}^+\mathcal{N}(\mu) \ge \operatorname{shr}^+\mathcal{N}(\mu_1)$ and $\pi(\mu) \ge \pi(\mu_1)$, by 521H. As in the proof of 521I, $([0, 1], \mathcal{N}(\mu_1))$ is isomorphic to $(\mathbb{R}, \mathcal{N})$. Of course μ_1 is not isomorphic to μ_L. But μ_L is isomorphic to a direct sum of countably many copies of μ_1, so by 521G we know that $\pi(\mu_L)$ is the cardinal product $\omega \cdot \pi(\mu_1)$; as $\pi(\mu_1)$ is surely infinite, this is $\pi(\mu_1)$ again. So we have the result in the special case.

(b) Now suppose that (X, Σ, μ) is any semi-finite perfect measure space which is not purely atomic. Then the completion $\hat\mu$ of μ is still a semi-finite perfect measure which is not purely atomic (212Gd, 451G(c-i)), and $\mathcal{N}(\mu) = \mathcal{N}(\hat\mu)$ (212Eb). Because $\hat\mu$ is semi-finite and not purely atomic, there is a set $E \in \Sigma$ of non-zero finite measure such that the subspace measure $\hat\mu_E$ is atomless. Set $\nu = \frac{1}{\mu E}\hat\mu_E$, so that ν is an atomless complete perfect probability measure on E, while $\mathcal{N}(\nu) = \mathcal{N}(\mu_E)$. Putting (a) together with 521F, we get

$$\operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}(\hat\mu) \le \operatorname{add}\mathcal{N}(\hat\mu_E) = \operatorname{add}\mathcal{N}(\nu) \le \operatorname{add}\mathcal{N}$$

and similarly for cf, shr and shr^+.

521L Proposition (a) Let (X, Σ, μ) be a strictly localizable measure space and (Y, T, ν) a locally compact semi-finite measure space, and suppose that they have isomorphic measure algebras. Then $(X, \in, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (Y, \in, \mathcal{N}(\nu))$; consequently $\operatorname{cov}\mathcal{N}(\mu) \le \operatorname{cov}\mathcal{N}(\nu)$ and $\operatorname{non}\mathcal{N}(\nu) \le \operatorname{non}\mathcal{N}(\mu)$.

(b) Let (X, Σ, μ) be a Maharam-type-homogeneous compact probability space with Maharam type κ. Then $\operatorname{cov} \mathcal{N}(\mu) = \operatorname{cov} \mathcal{N}_\kappa$ and $\operatorname{non} \mathcal{N}(\mu) = \operatorname{non} \mathcal{N}_\kappa$, where $\mathcal{N}_\kappa$ is the null ideal of the usual measure ν_κ on $\{0,1\}^\kappa$.

(c) Let (X, Σ, μ) be a compact strictly localizable measure space with measure algebra $\mathfrak{A}$. Then

$$d(\mathfrak{A}) = \min\{\#(A) : A \subseteq X \text{ has full outer measure}\}.$$

proof (a) This follows immediately from 521Ha, because by 343B there is an inverse-measure-preserving function from X to Y.

(b) The point is that ν_κ is a compact measure (342Jd, 451Ja), so that we can apply (a) in both directions to see that $\operatorname{cov} \mathcal{N}(\mu) = \operatorname{cov} \mathcal{N}_\kappa$ and $\operatorname{non} \mathcal{N}(\mu) = \operatorname{non} \mathcal{N}_\kappa$.

(c) The case $\mu X = 0$ is trivial; suppose that $\mu X > 0$. Let $\mathcal{K}$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$.

(i) Suppose that $\langle C_\xi \rangle_{\xi < d(\mathfrak{A})}$ is a family of centered sets in $\mathfrak{A}$ covering $\mathfrak{A}^+$. For each $\xi < d(\mathfrak{A})$, set $\mathcal{K}_\xi = \{K : K \in \mathcal{K} \cap \Sigma,\ K^\bullet \in C_\xi\}$; then $\mathcal{K}_\xi$ has the finite intersection property so there is a point $x_\xi \in X \cap \bigcap \mathcal{K}_\xi$. Set $A = \{x_\xi : \xi < d(\mathfrak{A})\}$. If $K \in \mathcal{K} \cap \Sigma$ and $K \cap A = \emptyset$, then $K \notin \bigcup_{\xi < d(\mathfrak{A})} \mathcal{K}_\xi$ so $K^\bullet = 0$; it follows that every measurable subset of $X \setminus A$ is negligible and A has full outer measure, while $\#(A) \le d(\mathfrak{A})$.

(ii) Let $\hat{\mu}$ be the completion of μ, $\hat{\Sigma}$ its domain and $\theta : \mathfrak{A} \to \hat{\Sigma}$ a lifting (341K, 212Gb, 322Da). Take any $A \subseteq X$ of full outer measure for μ; then it also has full outer measure for $\hat{\mu}$ (212Eb). For $x \in A$, set $C_x = \{a : a \in \mathfrak{A},\ x \in \theta a\}$; then $\langle C_x \rangle_{x \in A}$ is a family of centered sets in $\mathfrak{A}$ with union $\mathfrak{A}^+$, so $d(\mathfrak{A}) \le \#(A)$.

521M Proposition Let (X, Σ, μ) be a complete locally determined measure space of magnitude at most $\operatorname{add} \mu$. Then it is strictly localizable.

proof Write κ for $\operatorname{add} \mu$. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of μ. Then there is a partition of unity $D \subseteq \mathfrak{A}$ consisting of elements of finite measure; as $\#(D) \le c(\mathfrak{A}) \le \kappa$, there is a family $\langle a_\xi \rangle_{\xi < \kappa}$ running over $D \cup \{0\}$. For each $\xi < \kappa$, choose $E_\xi \in \Sigma$ such that $E_\xi^\bullet = a_\xi$, and set $F_\xi = E_\xi \setminus \bigcup_{\eta < \xi} E_\eta$. Because $E_\xi \setminus F_\xi = \bigcup_{\eta < \xi} E_\xi \cap E_\eta$ is the union of fewer than $\operatorname{add} \mu$ negligible sets, it is negligible, and $F_\xi \in \Sigma$, with $F_\xi^\bullet = a_\xi$. Now $\langle F_\xi \rangle_{\xi < \kappa}$ is a disjoint family of sets of finite measure. If $E \in \Sigma$ and $\mu E > 0$, there is some $\xi < \kappa$ such that $E^\bullet \cap a_\xi \ne 0$, and now $\mu(E \cap F_\xi) > 0$. Thus $\langle F_\xi \rangle_{\xi < \kappa}$ satisfies the condition of 213Oa, and μ is strictly localizable.

521N Proposition Let (X, Σ, μ) be a complete locally determined localizable measure space of magnitude at most $\mathfrak{c}$. Then it is strictly localizable.

proof Again let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of μ, and take a partition of unity $D \subseteq \mathfrak{A}$ consisting of elements of finite measure; as $\#(D) \le c(\mathfrak{A}) \le \mathfrak{c}$, there is an injective function $h : D \to \mathcal{P}\mathbb{N}$. This time, because $\mathfrak{A}$ is Dedekind complete, we can set $b_n = \sup\{d : d \in D,\ n \in h(d)\}$ for each $n \in \mathbb{N}$. If $d \in D$, then $d = \inf_{n \in h(d)} b_n \setminus \sup_{n \in \mathbb{N} \setminus h(d)} b_n$. So if we choose $E_n \in \Sigma$ such that $E_n^\bullet = b_n$ for each n, and set $F_d = \bigcap_{n \in h(d)} E_n \setminus \bigcup_{n \in \mathbb{N} \setminus h(d)} E_n$ for $d \in D$, $\langle F_d \rangle_{d \in D}$ will be a disjoint family in Σ and $F_d^\bullet = d$ for every d. Now $\mu F_d = \bar{\mu} d$ is always finite; and if $E \in \Sigma$ is non-negligible, there is a $d \in D$ such that $0 \ne \bar{\mu}(E^\bullet \cap d) = \mu(E \cap F_d)$. Thus $\langle F_d \rangle_{d \in D}$ satisfies the condition of 213Oa, and μ is strictly localizable.

521O Proposition (a) If (X, Σ, μ) is a semi-finite measure space, its magnitude is at most $\max(\omega, 2^{\#(X)})$.

(b) If (X, Σ, μ) is a strictly localizable measure space, its magnitude is at most $\max(\omega, \#(X))$.

(c) There is an infinite semi-finite measure space (X, Σ, μ) with magnitude $2^{\#(X)}$.

(d) If $\langle A_i \rangle_{i \in I}$ is a disjoint family of subsets of X and $\#(I) > \max(\omega, \operatorname{mag}(\mu))$ then there is an $i \in I$ such that $X \setminus A_i$ has full outer measure.

proof (a)-(b) These are elementary. If (X, Σ, μ) is semi-finite, with measure algebra $\mathfrak{A}$, then

$$c(\mathfrak{A}) \le \#(\mathfrak{A}) \le \#(\Sigma) \le \#(\mathcal{P}X) = 2^{\#(X)}.$$

If μ is strictly localizable, with decomposition $\langle X_i \rangle_{i \in I}$, then $\langle X_i^\bullet \rangle_{i \in I}$ is a partition of unity in $\mathfrak{A}$ consisting of elements of finite measure, so

$$c(\mathfrak{A}) \le \max(\omega, \#(\{i : i \in I,\ \mu X_i > 0\})) \le \max(\omega, \#(X))$$

by 332E.

(c) Let $\langle X_\xi \rangle_{\xi<\omega_1}$ be a disjoint family of sets such that $\#(X_\xi) = \#(\mathcal{P}(\bigcup_{\eta<\xi} X_\eta))$ for every $\xi < \omega_1$; for each ξ, let $h_\xi : \mathcal{P}(\bigcup_{\eta<\xi} X_\eta) \to X_\xi$ be an injection. Set $X = \bigcup_{\xi<\omega_1} X_\xi$. For $A \subseteq X$ define $f_A : \omega_1 \to X$ by setting $f(\xi) = h_\xi(A \cap \bigcup_{\eta<\xi} X_\eta)$ for each ξ; let J_A be $f_A[\omega_1]$ and μ_A the countable-cocountable measure on J_A. Observe that $\#(J_A) = \omega_1$ for every $A \subseteq X$, and that if A, $B \subseteq X$ are distinct then $J_A \cap J_B$ is countable. So if we set $\mu E = \sum_{A\subseteq X} \mu_A(E \cap J_A)$ whenever $E \subseteq X$ is such that $E \cap J_A$ is countable or cocountable in J_A for every A, then μ will be a complete locally determined measure on X. Since $\mu J_A = 1$ and $\mu(J_A \cap J_B) = 0$ whenever A, $B \subseteq X$ are distinct, μ has magnitude $2^{\#(X)}$.

(d) ? Otherwise, there is for each $i \in I$ a measurable set F_i of non-zero measure such that $F_i \subseteq A_i$. Again writing $\mathfrak{A}$ for the measure algebra of μ, $\langle F_i^\bullet \rangle_{i\in I}$ is a disjoint family in $\mathfrak{A} \setminus \{0\}$ so $\#(I) \le c(\mathfrak{A}) \le \max(\omega, \operatorname{mag}(\mu))$. **X**

521P Proposition (a) If $2^\lambda < 2^\kappa$ whenever $\mathfrak{c} \le \lambda < \kappa$ and $\operatorname{cf}\lambda > \omega$, then the magnitude $\operatorname{mag}\mu$ of μ is at most $\max(\omega, \#(X))$ for every localizable measure space (X, Σ, μ).

(b) Suppose that $2^{\mathfrak{c}} = 2^{\mathfrak{c}^+}$. Then there is a localizable measure space (Y, T, ν) with $\#(Y) = \mathfrak{c}$ and $\operatorname{mag}\nu = \mathfrak{c}^+$.

Remark TEX, for once, is obscure; $2^{\mathfrak{c}^+}$ here is $\#(\mathcal{P}(\mathfrak{c}^+))$, not $(2^{\mathfrak{c}})^+$.

proof (a) If $\operatorname{mag}\mu \le \omega$ we can stop. Otherwise, set $\kappa = \operatorname{mag}\mu$. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ, so that $\kappa = c(\mathfrak{A})$ and there is a disjoint family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{A}^+$ (332F). If $\tilde\mu$ is the c.l.d. version of μ, we can identify $\mathfrak{A}$ with the measure algebra of $\tilde\mu$ (322Db).

case 1 If $\kappa \le \mathfrak{c}$, $\tilde\mu$ is strictly localizable (521N), so has a lifting θ (341K again); but now $\langle \theta a_\xi \rangle_{\xi<\kappa}$ is a disjoint family of non-empty subsets of X, so $\#(X) \ge \kappa$.

case 2 If $\kappa > \mathfrak{c}$, of course X is uncountable (521Oa). For $\xi < \kappa$, choose $E_\xi \in \Sigma$ such that $E_\xi^\bullet = a_\xi$. **?** If $\#(X) < \kappa$, there is a set $Y \subseteq X$ such that $\#(Y)$ has uncountable cofinality and $I_Y = \{\xi : \xi < \kappa,\ \mu^*(E_\xi \cap Y) > 0\}$ has cardinal greater than $\max(\mathfrak{c}, \#(X))$. **P** If $\operatorname{cf}(\#(X))$ is uncountable, take $Y = X$. Otherwise, let $\langle Y_n \rangle_{n\in\mathbb{N}}$ be an increasing sequence of subsets of X, with union X, such that $\#(Y_n)$ is an uncountable successor cardinal less than $\#(X)$ for every n. If $\xi < \kappa$, there is some n such that $E_\xi \cap Y_n$ is non-negligible, that is, $\xi \in I_{Y_n}$. So the non-decreasing sequence $\langle I_{Y_n} \rangle_{n\in\mathbb{N}}$ has union κ, and there is some $n \in \mathbb{N}$ such that $\#(I_{Y_n}) > \max(\mathfrak{c}, \#(X))$. Now we can take $Y = Y_n$. **Q**

For every $J \subseteq I_Y$, set $b_J = \sup_{\xi\in J} a_\xi$ and let $F_J \in \Sigma$ be such that $F_J^\bullet = b_J$. If J, $K \subseteq I_Y$ are distinct, there is a $\xi \in J \triangle K$, in which case $a_\xi \subseteq b_J \mathbin{\triangle} b_K$, $E_\xi \setminus (F_J \triangle F_K)$ is negligible and $Y \cap (F_J \triangle F_K)$ is non-empty. Thus $J \mapsto Y \cap F_J : \mathcal{P}I_Y \to \mathcal{P}Y$ is injective, and

$$2^{\#(I_Y)} \le 2^{\#(Y)} \le 2^{\#(X)} \le 2^{\#(I_Y)}.$$

Setting $\lambda = \max(\mathfrak{c}, \#(Y))$, $\kappa' = \#(I_Y)$ we now have $\mathfrak{c} \le \lambda < \kappa'$, $\operatorname{cf}\lambda > \omega$ and $2^\lambda = 2^{\kappa'}$, which is supposed to be impossible. **X**

So in this case also we have $\#(X) \ge \kappa$.

(b)(i) Set $I = \mathcal{P}\mathfrak{c}^+$ and $X = \{0,1\}^I \cong \{0,1\}^{2^{\mathfrak{c}}}$. Putting 5A4Be and 5A4C(a-ii) together, we see that there is a set $Y \subseteq X$, with cardinal at most $\mathfrak{c}^\omega = \mathfrak{c}$, which meets every non-empty G_δ subset of X. In particular, if $\mathcal{K} \subseteq I$ is countable and $x \in X$ there is a $y \in Y$ such that $y{\restriction}\mathcal{K} = x{\restriction}\mathcal{K}$.

(ii) Let μ be the complete locally determined localizable measure on X described in 216E, with $C = \mathfrak{c}^+$. Then Y has full outer measure in X. **P** (I follow the notation and argument of 216E.) If $\mu E > 0$, then, by the argument of part (g) of the proof of 216E, there are a $\gamma < \mathfrak{c}^+$ and a $K \in [I]^{\le\omega}$ such that $F_{\gamma K} \subseteq E$, where $F_{\gamma K} = \{x : x{\restriction}K = x_\gamma{\restriction}K\}$. But Y was chosen to meet every such set. As E is arbitrary, Y has full outer measure. **Q**

(iii) $\operatorname{mag}\mu = \mathfrak{c}^+$. **P** In the language of 216E, we have a family $\langle G_{\{\gamma\}} \rangle_{\gamma<\mathfrak{c}^+}$ of μ-atoms of measure 1, each pair with negligible intersection, and every non-negligible measurable set meets some $G_{\{\gamma\}}$ in a non-negligible set. **Q**

(iv) Now let ν be the subspace measure on Y. By 214Ie, ν is complete, locally determined and localizable. By 322I, we can identify the measure algebras of μ and ν, so $\operatorname{mag}\nu = \operatorname{mag}\mu = \mathfrak{c}^+$, while $\#(Y) = \mathfrak{c}$.

521Q Free products We have some simple calculations associated with the measure algebra free products of §325.

Proposition (a) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be semi-finite measure algebras and $(\mathfrak{C}, \bar{\lambda})$ their localizable measure algebra free product. Then

$$c(\mathfrak{C}) \le \max(\omega, c(\mathfrak{A}), c(\mathfrak{B})),$$

$$\tau(\mathfrak{C}) \le \max(\omega, \tau(\mathfrak{A}), \tau(\mathfrak{B})).$$

(b) Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{C}, \bar{\lambda})$ their probability algebra free product. Then

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \sup_{i\in I} \tau(\mathfrak{A}_i)).$$

proof (a)(i) Let $A \subseteq \mathfrak{A}$, $B \subseteq \mathfrak{B}$ be partitions of unity consisting of elements of finite measure (322Ea). Then $C = \{a \otimes b : a \in A, b \in B\}$ is a disjoint family in $\mathfrak{C}$, and

$$\sup C = \sup\{(a \otimes 1) \cap (1 \otimes b) : a \in A,\, b \in B\} = (\sup_{a\in A} a \otimes 1) \cap (\sup_{b\in B} 1 \otimes b)$$

(313Bc)

$$= (\sup A \otimes 1) \cap (1 \otimes \sup B)$$

(325Da)

$$= (1 \otimes 1) \cap (1 \otimes 1) = 1;$$

that is, C is a partition of unity. As every member of C has finite measure,

$$c(\mathfrak{C}) \le \max(\omega, \#(C)) = \max(\omega, \#(A), \#(B)) = \max(\omega, c(\mathfrak{A}), c(\mathfrak{B}))$$

by 332E.

(ii) As for Maharam types, I am just repeating the result stated and proved in 334B.

(b) This is 334D.

521R Proposition If (X, Σ, μ) is any measure space, its Maharam type is at most $2^{\#(X)}$.

proof If $\mathfrak{A}$ is the measure algebra of μ,

$$\tau(\mathfrak{A}) \le \#(\mathfrak{A}) \le \#(\Sigma) \le \#(\mathcal{P}X) = 2^{\#(X)}.$$

521S Proposition (a) A countably separated measure space has Maharam type at most $2^{\mathfrak{c}}$.

(b) There is a countably separated quasi-Radon probability space with Maharam type $2^{\mathfrak{c}}$.

(c) A countably separated semi-finite measure space has magnitude at most $2^{\mathfrak{c}}$.

(d) There is a countably separated semi-finite measure space with magnitude $2^{\mathfrak{c}}$.

proof Set $\kappa = 2^{\mathfrak{c}}$.

(a) If (X, Σ, μ) is countably separated, there is an injective function from X to $\mathbb{R}$ (343E), so $\#(X) \le \mathfrak{c}$; now use 521R.

(b) As in (b-i) of the proof of 521P, there is a set $Y \subseteq X = \{0,1\}^\kappa$, with cardinal $\mathfrak{c}$, which meets every non-empty G_δ subset of X, and therefore has full outer measure for the usual measure ν_κ of X.

In $[0,1]$ let $\langle C_y\rangle_{y\in Y}$ be a disjoint family of sets of full outer measure for Lebesgue measure μ_1 on $[0,1]$ (419I), and set $C = \{(y,t) : y \in Y,\, t \in C_y\} \subseteq Z = X \times [0,1]$. Now C has full outer measure for the product measure λ on Z. **P** Suppose that $W \subseteq Z$ and $\lambda W > 0$. Then $\int \mu_1 W[\{x\}]\nu_\kappa(dx) > 0$ (252D), so $\{x : \mu_1 W[\{x\}] > 0\}$ has non-zero measure and meets Y. Taking $y \in Y$ such that $\mu_1 W[\{y\}] > 0$, $\{y\} \times (C_y \cap W[\{y\}])$ is a non-empty subset of $C \cap W$. **Q**

The measure algebra $\mathfrak{A}$ of the subspace measure λ_C on C can therefore be identified with the measure algebra of λ (322Jb), and has Maharam type κ. Because $\langle C_y\rangle_{y\in Y}$ is disjoint, each horizontal section of C is a singleton and C is separated by the measurable sets $C\cap(X\times[0,q])$ for $q \in \mathbb{Q}\cap[0,1]$. Thus λ_C is countably separated.

If we give Z the product topology, then λ is a Radon measure (417T, or otherwise), so λ_C is quasi-Radon for the subspace topology (415B).

(c) As in (a), $\#(X) \le \mathfrak{c}$, so we can use 521Oa.

(d)(i) The first step is to build a measure space of magnitude $2^{\mathfrak{c}}$ and cardinal $\mathfrak{c}$, as follows. Let $h : \mathfrak{c} \to ([\mathfrak{c}]^{\le\omega})^2$ be a surjection; take its two components to be h_1 and h_2. For $D \subseteq \mathfrak{c}$ set $F_D = \{\xi : \xi < \mathfrak{c}, h_2(\xi) = D \cap h_1(\xi)\}$. For $I \in [\mathfrak{c}]^{\le\omega}$ set $A_I = \{\xi : \xi < \mathfrak{c}, I \not\subseteq h_1(\xi)\}$, and set $\mathcal{A} = \bigcup\{\mathcal{P}A_I : I \in [\mathfrak{c}]^{\le\omega}\}$; note that $\mathcal{A}$ is a σ-ideal of subsets of $\mathfrak{c}$.

If $D \subseteq \mathfrak{c}$, $F_D \notin \mathcal{A}$. **P** If $I \in [\mathfrak{c}]^{\le\omega}$, there is a $\xi < \mathfrak{c}$ such that $h(\xi) = (I, I\cap D)$, Now $\xi \in F_D \setminus A_I$; as I is arbitrary, $F_D \notin \mathcal{A}$. **Q** So we can define a measure ν_D on $\mathfrak{c}$ by saying that

$$\begin{aligned}\nu_D(E) &= 1 \text{ if } E \subseteq \mathfrak{c} \text{ and } F_D \setminus E \in \mathcal{A},\\ &= 0 \text{ if } E \subseteq \mathfrak{c} \text{ and } F_D \cap E \in \mathcal{A},\\ &\quad\text{undefined otherwise,}\end{aligned}$$

and $\nu_D F_D = 1$.

If D, $D' \subseteq \mathfrak{c}$ are distinct, $F_D \cap F_{D'} \in \mathcal{A}$. **P** Take $\eta \in D\triangle D'$. If $\xi \in F_D \cap F_{D'}$, then $D \cap h_1(\xi) = h_2(\xi) = D' \cap h_1(\xi)$, so $\eta \notin h_1(\xi)$ and $\xi \in A_{\{\eta\}}$. Thus $F_D \cap F_{D'} \subseteq A_{\{\eta\}} \in \mathcal{A}$. **Q**

So if we set $\nu = \sum_{D\subseteq\mathfrak{c}} \nu_D$, as defined in 234G, ν is a measure on $\mathfrak{c}$ such that $\nu F_D = 1$ and $\nu(F_D \cap F_{D'}) = 0$ for all distinct D, $D' \subseteq \mathfrak{c}$. Also ν is semi-finite, because if $\nu E > 0$ there is a $D \subseteq \mathfrak{c}$ such that $\nu_D E > 0$, in which case $\nu(E\cap F_D) = \nu_D(E\cap F_D) = 1$. So ν is a semi-finite measure on $\mathfrak{c}$ with magnitude $\#(\mathcal{P}\mathfrak{c}) = 2^{\mathfrak{c}}$. Because every ν_D is complete, so is ν (234Ha).

(ii) As in (b), let $\langle C_\xi\rangle_{\xi<\mathfrak{c}}$ be a disjoint family of subsets of $[0,1]$ all with full outer measure for Lebesgue measure μ_1. Set $Z = \mathfrak{c}\times[0,1]$ with its c.l.d. product measure $\lambda = \nu\times\mu_1$, and $C = \{(\xi,t) : \xi < \mathfrak{c}, t \in C_\xi\} \subseteq Z$. Then C has full outer measure, by the argument of (b) above. So, as in (b), the measure algebra $\mathfrak{A}$ of the subspace measure λ_C on C can be identified with the measure algebra of λ. The map $E \mapsto C \cap (E\times[0,1])$ induces a measure-preserving homomorphism from the measure algebra of ν to $\mathfrak{A}$, so $\operatorname{mag}\lambda_C = c(\mathfrak{A})$ is at least $2^{\mathfrak{c}}$; by (c), it is exactly $2^{\mathfrak{c}}$. Also as in (b), λ_C is countably separated.

521T In §464 I looked at the L-space M of bounded additive functionals on $\mathcal{P}I$ for infinite sets I, of which $I = \mathbb{N}$ is of course by far the most important, and found a band decomposition of M as $M_\tau \oplus (M_{\mathrm{m}} \cap M_\tau^\perp) \oplus M_{\mathrm{pnm}}$, where M_τ consists of the 'completely additive' functionals (and may be identified with $\ell^1(I)$), M_{m} consists of the 'measurable' functionals (that is, those integrated by the usual measure on $\mathcal{P}I$), and $M_{\mathrm{pnm}} = M_{\mathrm{m}}^\perp$ consists of the 'purely non-measurable' functionals. Any non-negative functional $\theta \in M$ can be identified with a Radon measure μ_θ on the Stone-Čech compactification βI (464P). The purely atomic measures on I correspond to members of M_τ. Among the others, the general rule is that 'simple' measures must correspond to functionals in M_{pnm}; see 464Pa and 464Xa. The next proposition, strengthening 464Qb, shows that this rule is followed by Maharam types.

Proposition Let I be a set, and suppose that a non-zero $\theta \in (M_{\mathrm{m}} \cap M_\tau^\perp)^+$, as defined in §464, corresponds to the Radon measure μ_θ on βI. Let ν be the usual measure on $\mathcal{P}I$. Then the Maharam type of μ_θ is at least $\operatorname{cov}\mathcal{N}(\nu)$.

proof Of course I has to be infinite, since not every additive functional on $\mathcal{P}I$ is completely additive; so $\operatorname{cov}\mathcal{N}(\nu)$ is not ∞. By 464Qc, we know that

$$\{(a,b) : a,\, b \subseteq I,\, \theta a = \frac{1}{2}\theta I,\, \theta(a\cap b) = \frac{1}{4}\theta I\}$$

is conegligible for the product measure $\nu\times\nu$ on $(\mathcal{P}I)^2$. Set

$$A_0=\{a:a\subseteq I,\,\theta a=\frac{1}{2},\,\{b:\theta(a\cap b)=\frac{1}{4}\theta I\}\text{ is }\nu\text{-conegligible}\};$$

then A_0 is ν-conegligible. Now take a set $A\subseteq A_0$ which is maximal subject to the requirement that $\theta(a\cap b)=\frac{1}{4}\theta I$ for all distinct a, $b\in A$. If a is any subset of I, then either $a\in A$, or $a\notin A_0$, or there is a $b\in A\setminus\{a\}$ such that $\theta(a\cap b)\ne\frac{1}{4}\theta I$; so $\mathcal{P}I$ is the union of

$$(\mathcal{P}I\setminus A_0)\cup\bigcup_{b\in A}\{a:\theta(a\cap b)\ne\frac{1}{4}\theta I\}$$

and $\operatorname{cov}\mathcal{N}(\nu)\le 1+\#(A)$. As $\operatorname{cov}\mathcal{N}(\nu)$ is surely infinite, it is in fact less than or equal to $\#(A)$.

Now consider the open-and-closed sets $\widehat{a}\subseteq\beta I$ for $a\in A$. If a, $b\in A$ are distinct,

$$\mu_\theta(\widehat{a}\triangle\widehat{b})=\mu_\theta(\widehat{a\triangle b})=\theta(a\triangle b)=\tfrac{1}{2}\theta I>0.$$

So in the measure algebra $\mathfrak{A}$ of μ_θ, $\{\widehat{a}^\bullet:a\in A\}$ is a discrete set with cardinal at least $\operatorname{cov}\mathcal{N}(\nu)$, and the topological density of $\mathfrak{A}$ is at least $\operatorname{cov}\mathcal{N}(\nu)$ (5A4B(h-ii) again). By 521E, $\tau(\mu_\theta)=\tau(\mathfrak{A})\ge\operatorname{cov}\mathcal{N}(\nu)$.

521X Basic exercises (a) Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$, and μ the restriction of Lebesgue measure to $\mathcal{B}(\mathbb{R})$. Show that $\operatorname{add}\mu=\omega_1$. (*Hint*: if $\mathfrak{c}=\omega_1$, $[\mathbb{R}]^{\omega_1}\not\subseteq\mathcal{B}(\mathbb{R})$; if $\mathfrak{c}>\omega_1$, $[\mathbb{R}]^{\omega_1}\cap\mathcal{B}(\mathbb{R})=\emptyset$; or use 423M and 423R[1].)

(b) Let (X,Σ,μ) be a semi-finite locally compact measure space. Show that $\operatorname{add}\mu$ is the least cardinal of any set $\mathcal{E}\subseteq\Sigma$ such that $\bigcup\mathcal{E}\notin\Sigma$, or ∞ if there is no such $\mathcal{E}$. (*Hint*: 451Q.)

(c) Let (X,Σ,μ) be a complete locally determined measure space, and κ a cardinal such that $\kappa<\operatorname{cov}\mathcal{N}(\mu_E)$ for every non-negligible measurable set $E\subseteq X$, writing μ_E for the subspace measure. Suppose that $A\subseteq X$ is such that both A and $X\setminus A$ are expressible as the union of at most κ members of Σ. Show that $A\in\Sigma$.

>(d)(i) Find a probability space (X,Σ,μ), with measure algebra $\mathfrak{A}$, such that $\pi(\mathfrak{A})<\pi(\mu)$. (ii) Find a probability space (X,Σ,μ), with null ideal $\mathcal{N}(\mu)$, such that $\operatorname{cf}\mathcal{N}(\mu)<\pi(\mu)$. (iii) Find a probability space (X,Σ,μ) such that $\pi(\mu)<\operatorname{cf}\mathcal{N}(\mu)$. (*Hint*: 513X(q-iii).)

(e) Let (X,Σ,μ) be a measure space and ν an indefinite-integral measure over μ. Show that $\operatorname{add}\mathcal{N}(\nu)\ge\operatorname{add}\mathcal{N}(\mu)$, $\operatorname{cf}\mathcal{N}(\nu)\le\operatorname{cf}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\nu)\ge\operatorname{non}\mathcal{N}(\mu)$, $\operatorname{cov}\mathcal{N}(\nu)\le\operatorname{cov}\mathcal{N}(\mu)$, $\operatorname{shr}\mathcal{N}(\nu)\le\operatorname{shr}\mathcal{N}(\mu)$, $\operatorname{shr}^+\mathcal{N}(\nu)\le\operatorname{shr}^+\mathcal{N}(\mu)$, $\pi(\nu)\le\pi(\mu)$, $\tau(\nu)\le\tau(\mu)$.

(f) Let (X,Σ,μ) be a semi-finite measure space which is not purely atomic. Show that $\pi(\mu)\ge\pi(\mu_L)$, where μ_L is Lebesgue measure on $\mathbb{R}$.

(g) Let (X,Σ,μ) be an atomless measure space with locally determined negligible sets (definition: 213I). Show that $\operatorname{non}\mathcal{N}(\mu)\ge\operatorname{non}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure.

(h) Let (X,Σ,μ) and (Y,T,ν) be complete locally determined measure spaces, neither of measure 0, and $\mu\times\nu$ the c.l.d. product measure on $X\times Y$. Show that

$$\operatorname{non}\mathcal{N}(\mu\times\nu)=\max(\operatorname{non}\mathcal{N}(\mu),\operatorname{non}\mathcal{N}(\nu)),$$

$$\operatorname{cov}\mathcal{N}(\mu\times\nu)\le\min(\operatorname{cov}\mathcal{N}(\mu),\operatorname{cov}\mathcal{N}(\nu))$$

with equality if either μ or ν is strictly localizable,

$$\operatorname{add}(\mu\times\nu)=\operatorname{add}\mathcal{N}(\mu\times\nu)\le\min(\operatorname{add}\mathcal{N}(\mu),\operatorname{add}\mathcal{N}(\nu)),$$

$$\operatorname{cf}\mathcal{N}(\mu\times\nu)\ge\max(\operatorname{cf}\mathcal{N}(\mu),\operatorname{cf}\mathcal{N}(\nu)),$$

$$\operatorname{shr}\mathcal{N}(\mu\times\nu)\ge\max(\operatorname{shr}\mathcal{N}(\mu),\operatorname{shr}\mathcal{N}(\nu)),$$

[1]Formerly 423Q.

$$\mathrm{shr}^+\mathcal{N}(\mu\times\nu)\ge\max(\mathrm{shr}^+\mathcal{N}(\mu),\mathrm{shr}^+\mathcal{N}(\nu)),$$

$$\pi(\mu\times\nu)\ge\max(\pi(\mu),\pi(\nu)).$$

(i) Let (X,Σ,μ) be a probability space, and $\mu^{\mathbb{N}}$ the product measure on $X^{\mathbb{N}}$. (i) Show that X has a set of full outer measure with cardinal at most $\mathrm{non}\,\mathcal{N}(\mu^{\mathbb{N}})$. (ii) Show that if $\mathcal{A}\subseteq\Sigma\setminus\mathcal{N}(\mu)$ and $\#(\mathcal{A})<\mathrm{cov}\,\mathcal{N}(\mu^{\mathbb{N}})$, then there is a countable set which meets every member of $\mathcal{A}$.

(j) Show that the direct sum of $\mathfrak{c}$ or fewer countably separated measure spaces is countably separated.

(k) Show that $2^{\mathfrak{c}}<2^{\mathfrak{c}^+}$ iff every countably separated complete locally determined localizable measure space is strictly localizable. (*Hint*: 521P, 521S, 252Yp.)

(l) Show that if (X,Σ,μ) is a purely atomic countably separated semi-finite measure space then its magnitude is at most $\max(\omega,\#(X))$ and its Maharam type is countable.

(m) Suppose that $2^\kappa\le\mathfrak{c}$ for every $\kappa<\mathfrak{c}$. Show that there is a countably separated semi-finite measure space with magnitude $2^{\mathfrak{c}}$.

(n) For a measure space (X,Σ,μ) with null ideal $\mathcal{N}(\mu)$, write $\mathrm{hcov}(\mu)$ for $\inf_{E\in\Sigma\setminus\mathcal{N}(\mu)}\mathrm{cov}(E,\mathcal{N}(\mu))$. (Count $\inf\emptyset$ as ∞, as usual.) Show that if (X,Σ,μ) and (Y,T,ν) are semi-finite measure spaces, neither having zero measure, with c.l.d. product $(X\times Y,\Lambda,\lambda)$, then $\mathrm{hcov}(\lambda)=\min(\mathrm{hcov}(\mu),\mathrm{hcov}(\nu))$.

521Y Further exercises (a) Find a probability space (X,Σ,μ), a set Y and a function $f:X\to Y$ such that, setting $\nu=\mu f^{-1}$, $\mathrm{add}\,\mathcal{N}(\mu)>\mathrm{add}\,\mathcal{N}(\nu)$, $\mathrm{cf}\,\mathcal{N}(\mu)<\mathrm{cf}\,\mathcal{N}(\nu)$, $\mathrm{shr}\,\mathcal{N}(\mu)<\mathrm{shr}\,\mathcal{N}(\nu)$ and $\pi(\mu)<\pi(\nu)$.

(b) Find a strictly localizable measure space (X,Σ,μ), a set Y, and a function $f:X\to Y$ such that, setting $\nu=\mu f^{-1}$, ν is semi-finite and $\tau(\mu)<\tau(\nu)$.

(c) Let (X,Σ,μ) and (Y,T,ν) be localizable measure spaces, and suppose that $\max(\mathrm{mag}(\nu),\tau(\nu))\le\mathfrak{c}$. Show that the c.l.d. product measure on $X\times Y$ is localizable.

(d) Show that there is a probability space (X,Σ,μ) with Maharam type greater than $\#(X)$. (*Hint*: 523Ib.)

(e) Let κ be an infinite cardinal. Let us say that a measure space (X,Σ,μ) is **κ-separated** if there is a family $\mathcal{E}\subseteq\Sigma$, with cardinal at most κ, separating the points of X. (i) Show that there is a disjoint family $\mathcal{A}$ of subsets of $\{0,1\}^\kappa$, all of full outer measure for the usual measure of $\{0,1\}^\kappa$, such that $\#(\mathcal{A})=2^\kappa$. (ii) Show that every κ-separated measure space has Maharam type at most 2^{2^κ}, and that there is a κ-separated quasi-Radon probability space with Maharam type 2^{2^κ}. (iii) Show that every semi-finite κ-separated measure space has magnitude at most 2^{2^κ}, and that there is a semi-finite κ-separated measure space with magnitude greater than 2^κ. (iv) Suppose that $\mathfrak{c}\le\kappa\le\lambda$ and $2^\kappa=2^\lambda$. Show that the usual measure on $\{0,1\}^\lambda$ is κ-separated.

521 Notes and comments The cardinal functions of an ideal can be thought of as measures of the 'complexity' of that ideal. In a measure space, it is natural to suppose that a subspace measure (at least, on a measurable subspace) will be 'simpler' than the original measure; in 521F we see that the additivity and uniformity tend to rise and the covering number, cofinality, shrinking number and π-weight tend to fall. Similarly, an image measure ought to be simpler than its parent; but here, while additivity rises and cofinality and shrinking number fall, uniformity falls and covering number rises (521H). Also there is a trap if the original measure is not complete (521Ya), and π-weight is more complicated (521H(a-ii)). There is a similar problem concerning topological π-weight, which led to the concept of network weight (5A4Ai, 5A4Bc); and just as network weight matches topological weight for compact Hausdorff spaces (5A4C(a-i)), an appropriate hypothesis on our measures can make their π-weights more coherent (521H(a-ii)).

Direct sums should not be more complex than their most complex component; 521G confirms this prejudice except in respect of cofinality. Since we are looking, in effect, at the cofinality of a product of partially ordered sets, we can expect at least as many difficulties as are to be found in pcf theory (§5A2). We should like to be able to bound the complexity of a product in terms of the complexities of the factors; here there seem to be some interesting questions, and 521J and 521Xh are, I hope, only a start.

Consider the statement

$$(\dagger)\ \text{‘}\operatorname{mag}\mu \le \#(X) \text{ for every localizable measure space } (X,\Sigma,\mu)\text{’}.$$

From 521P we see that the generalized continuum hypothesis implies (†), and also that there are simple models of set theory in which (†) is false (Kunen 80, VIII.4.7; Jech 03, 15.18). I do not know whether there is a natural combinatorial statement equiveridical with (†). If we amend (†) to

$$\text{‘}\operatorname{mag}\mu \le \#(X) \text{ for every countably separated localizable measure space } (X,\Sigma,\mu)\text{’}$$

we find ourselves with a statement equiveridical with ‘$2^{\mathfrak{c}} = 2^{\mathfrak{c}^+}$’ (cf. 521Xk).

I give space to ‘countably separated’ measures because these can be identified with the topological measures on subsets of $\mathbb{R}$, and I do not think it is immediately apparent just how complicated these can be. In fact, as shown by the proofs of parts (b) and (d) in 521S, most of the phenomena which can arise in any measure space with cardinal less than or equal to $\mathfrak{c}$ can appear in countably separated measure spaces. In 521Sb I add ‘quasi-Radon’ to show that the very strong restrictions on countably separated Radon probability measures (522Wa) depend on their perfectness, not on their τ-additivity.

The constructions in 521Oc and 521Sd both depend on almost-disjoint families of sets. Those described here are elementary. In many models of set theory, we have much more striking results, of which 521Xm is a simple example.

Some new considerations intrude rather abruptly in 521T, but the argument here is both elementary and important, quite apart from its use in helping us to understand the classification scheme in §464.

Version of 11.7.23

522 Cichoń's diagram

In this section I describe some extraordinary relationships between the cardinals associated with the ideals of meager and negligible sets in the real line. I concentrate on the strikingly symmetric pattern of Cichoń's diagram (522B); the first half of the section is taken up with proofs of the facts encapsulated in this diagram. I include a handful of results characterizing some of the most important cardinals here (522C, 522S), notes on Martin cardinals associated with the diagram (522T) and the Freese-Nation number of $\mathcal{P}\mathbb{N}$ (522U), and a brief discussion of cofinalities (522V).

522A Notation In this section, I will use the symbols $\mathcal{M}$ and $\mathcal{N}$ for the ideals of meager and negligible subsets of $\mathbb{R}$ respectively. Associated with these we have the eight cardinals $\operatorname{add}\mathcal{M}$, $\operatorname{cov}\mathcal{M}$, $\operatorname{non}\mathcal{M}$, $\operatorname{cf}\mathcal{M}$, $\operatorname{add}\mathcal{N}$, $\operatorname{cov}\mathcal{N}$, $\operatorname{non}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}$. In addition we have two cardinals associated with the partially ordered set $\mathbb{N}^{\mathbb{N}}$: the **bounding number** $\mathfrak{b} = \operatorname{add}_\omega \mathbb{N}^{\mathbb{N}}$ (see 513H for the definition of $\operatorname{add}_\omega$, and 522C for alternative descriptions of $\mathfrak{b}$) and the **dominating number** $\mathfrak{d} = \operatorname{cf}\mathbb{N}^{\mathbb{N}}$; and finally I should include $\mathfrak{c}$ itself as an eleventh cardinal in the list to be examined here. I use the notions of Galois-Tukey connection and Tukey function, and the associated relations $\preccurlyeq_{\mathrm{GT}}$, $\equiv_{\mathrm{GT}}$ and $\preccurlyeq_{\mathrm{T}}$, as described in §§512-513.

522B Cichoń's diagram The diagram itself is the following:

$$\begin{array}{ccccccccccc}
 & & \operatorname{cov}\mathcal{N} & — & \operatorname{non}\mathcal{M} & — & \operatorname{cf}\mathcal{M} & — & \operatorname{cf}\mathcal{N} & — & \mathfrak{c}\\
 & & | & & | & & | & & | & & \\
 & & | & & \mathfrak{b} & — & \mathfrak{d} & & | & & \\
 & & | & & | & & | & & | & & \\
\omega_1 & — & \operatorname{add}\mathcal{N} & — & \operatorname{add}\mathcal{M} & — & \operatorname{cov}\mathcal{M} & — & \operatorname{non}\mathcal{N} & &
\end{array}$$

The cardinals here increase from bottom left to top right; that is,

$$\omega_1 \le \operatorname{add}\mathcal{N} \le \operatorname{add}\mathcal{M} \le \mathfrak{b} \le \mathfrak{d} \le \operatorname{cf}\mathcal{M} \le \operatorname{cf}\mathcal{N} \le \mathfrak{c},$$

etc. In addition, we have two equalities:

$$\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M}), \quad \operatorname{cf}\mathcal{M} = \max(\mathfrak{d}, \operatorname{non}\mathcal{M}).$$

In the rest of this section I will prove all the inequalities declared here, seeking to demonstrate reasons for the remarkable symmetry of the diagram. I will make heavy use of the ideas of §512. Of course many of the elementary results can be proved directly without difficulty; but for the most interesting part of the argument (522K-522Q below) Tukey functions seem to be the right way to proceed.

I start with the easiest results. It will be helpful to have descriptions of $\mathfrak{b}$ and $\mathfrak{d}$ in terms of other partially ordered sets.

522C Lemma (i) On $\mathbb{N}^{\mathbb{N}}$ define a relation $\le^*$ by saying that $f \le^* g$ if if $\{n : f(n) > g(n)\}$ is finite. Then $\le^*$ is a pre-order on $\mathbb{N}^{\mathbb{N}}$; $\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \le^*) = \mathfrak{b}$ and $\operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \le^*) = \mathfrak{d}$.

(ii) On $\mathbb{N}^{\mathbb{N}}$ define a relation $\preceq$ by saying that $f \preceq g$ if either $f \le g$ or $\{n : g(n) \le f(n)\}$ is finite. Then $\preceq$ is a partial order on $\mathbb{N}^{\mathbb{N}}$, $\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preceq) = \mathfrak{b}$ and $\operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \preceq) = \mathfrak{d}$.

(iii) $(\mathbb{N}^{\mathbb{N}}, \le^*) \equiv_{\mathrm{T}} (\mathbb{N}^{\mathbb{N}}, \preceq)$.

proof (a) The checks that $\le^*$ is a pre-order and that $\preceq$ is a partial order are elementary. Write ι for the identity map from $\mathbb{N}^{\mathbb{N}}$ to itself.

(b) For $f \in \mathbb{N}^{\mathbb{N}}$ and $A \subseteq \mathbb{N}^{\mathbb{N}}$ say that $f \le' A$ if there is a $g \in A$ such that $f \le g$ (see 512F). Now $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$. **P** For $g \in \mathbb{N}^{\mathbb{N}}$, set $\psi(g) = \{h : g \le^* h \le^* g\} \in [\mathbb{N}^{\mathbb{N}}]^{\omega}$. If $f \le^* g$, then $f \le f \vee g \in \psi(g)$, so $f \le' \psi(g)$. Thus (ι, ψ) is a Galois-Tukey connection and $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$. **Q**

(c) $(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$. **P** If f, $g \in \mathbb{N}^{\mathbb{N}}$ and $f \preceq g$, then $f \le^* g$; so (ι, ι) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$ to $(\mathbb{N}^{\mathbb{N}}, \preceq \mathbb{N}^{\mathbb{N}})$, as in 512Cd. **Q**

(d) If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is countable, there is a $\psi(A) \in \mathbb{N}^{\mathbb{N}}$ such that $g \preceq \psi(A)$ for every $g \in A$. **P** If A is empty, this is trivial. Otherwise, let $\langle g_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A, and set $\psi(A)(i) = 1 + \max_{n\le i} g_n(i)$ for every $i \in \mathbb{N}$. **Q**

It follows that $(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$. **P** If $A \in [\mathbb{N}^{\mathbb{N}}]^{\le\omega}$ and $f \le' A$, then there is some $g \in A$ such that $f \le g$, so that $f \preceq \psi(A)$. Thus (ι, ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$ to $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$. **Q**

(e) Putting (b)-(d) together, we see that

$$(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) \equiv_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) \equiv_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}).$$

In particular, $(\mathbb{N}^{\mathbb{N}}, \le^*) \equiv_{\mathrm{T}} (\mathbb{N}^{\mathbb{N}}, \preceq)$ (513Ea). By 512D,

$$\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) = \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) = \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}),$$

$$\operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}).$$

But by 513Ia we have

$$\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) = \operatorname{add}_{\omega}(\mathbb{N}^{\mathbb{N}}) = \mathfrak{b},$$

so $\mathfrak{b} = \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \le^*) = \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preceq)$. In the other direction,

$$\mathfrak{d} = \operatorname{cf}\mathbb{N}^{\mathbb{N}} = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \le \max(\omega, \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}))$$

(512Gf)

$$\le \max(\omega, \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}))$$

(512Gc)

$$= \max(\omega, \mathfrak{d}) = \mathfrak{d}.$$

So

$$\mathfrak{d} = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}) = \operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \leq^*) = \operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \preccurlyeq),$$

and the proof is complete.

522D Proposition $\mathfrak{b} \leq \mathfrak{d}$.

proof Use 511He and 522C.

522E Proposition $\operatorname{add}\mathcal{N} \leq \operatorname{cov}\mathcal{N}$, $\operatorname{add}\mathcal{M} \leq \operatorname{cov}\mathcal{M}$, $\operatorname{non}\mathcal{M} \leq \operatorname{cf}\mathcal{M}$ and $\operatorname{non}\mathcal{N} \leq \operatorname{cf}\mathcal{N}$.

proof We need only observe that both $\mathcal{M}$ and $\mathcal{N}$ are proper ideals of $\mathcal{P}\mathbb{R}$ with union $\mathbb{R}$, and use 511Jc.

522F Proposition $\omega_1 \leq \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N} \leq \mathfrak{c}$.

proof Of course $\omega_1 \leq \operatorname{add}\mathcal{N}$ because $\mathcal{N}$ is a σ-ideal of sets. As for $\operatorname{cf}\mathcal{N}$, we know that the family of negligible Borel sets is cofinal with $\mathcal{N}$ (134Fb) and has at most $\mathfrak{c}$ members (Fa), so $\operatorname{cf}\mathcal{N} \leq \mathfrak{c}$.

522G Proposition (Rothberger 1938a) $\operatorname{cov}\mathcal{N} \leq \operatorname{non}\mathcal{M}$ and $\operatorname{cov}\mathcal{M} \leq \operatorname{non}\mathcal{N}$.

proof The point is just that there is a comeager negligible set $E \subseteq \mathbb{R}$. **P** Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$, and set

$$E = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n}]q_n - 2^{-n}, q_n + 2^{-n}[. \ \mathbf{Q}$$

Because $x \mapsto a + x$ and $x \mapsto a - x$ are measure-preserving homeomorphisms, $a + E$ is negligible and $a - E$ is comeager for every $a \in \mathbb{R}$. Let $A \subseteq \mathbb{R}$ be a non-meager set with cardinal $\operatorname{non}\mathcal{M}$. Then $A \cap (a - E) \neq \emptyset$ for every $a \in \mathbb{R}$, that is, $\{x + E : x \in A\}$ covers $\mathbb{R}$; so $\operatorname{cov}\mathcal{N} \leq \#(A) = \operatorname{non}\mathcal{M}$.

For the other inequality, note that $F = \mathbb{R} \setminus E$ is conegligible and meager; so the same argument shows that $\operatorname{cov}\mathcal{M} \leq \operatorname{non} N$.

522H Proposition $\operatorname{add}\mathcal{M} \leq \mathfrak{b}$ and $\mathfrak{d} \leq \operatorname{cf}\mathcal{M}$.

proof (a) Start by choosing a countable base $\mathcal{U}$ for the topology of $\mathbb{R}$, not containing $\emptyset$, and enumerate it as $\langle U_k\rangle_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$ let $\langle V_{kl}\rangle_{l\in\mathbb{N}}$ be a disjoint sequence of non-empty open subsets of U_k; finally, enumerate $V_{kl} \cap \mathbb{Q}$ as $\langle x_{kli}\rangle_{i\in\mathbb{N}}$ for each k, $l \in \mathbb{N}$.

(b) Fix $f \in \mathbb{N}^{\mathbb{N}}$ for the moment. Set $E_k(f) = \{x_{kli} : l \in \mathbb{N}, i \leq f(l)\} \subseteq U_k$ for each $k \in \mathbb{N}$. This is nowhere dense because if G is a non-empty open set, either $G \cap \bigcup_{l\in\mathbb{N}} V_{kl} = \emptyset$ and $G \cap E_k(f) = \emptyset$, or there is an l such that $G \cap V_{kl}$ is non-empty, in which case $G \cap V_{kl} \cap E_k(f)$ is finite and $G \setminus \overline{E_k(f)} \supseteq G \cap V_{kl} \setminus E_k(f)$ is non-empty.

Now choose $\langle k_n\rangle_{n\in\mathbb{N}}$, $\langle l_n\rangle_{n\in\mathbb{N}}$ inductively as follows. Given $\langle k_i\rangle_{i<n}$, $\bigcup_{i<n} E_{k_i}(f)$ is nowhere dense, so there is an $l_n \in \mathbb{N}$ such that $\overline{U}_{l_n} \subseteq U_n \setminus \bigcup_{i<n} E_{k_i}(f)$. Now if $U_n \subseteq \overline{\bigcup_{i\leq n} U_{l_i}}$, set $k_n = 0$; otherwise, take k_n such that $U_{k_n} \subseteq U_n \setminus \overline{\bigcup_{i\leq n} U_{l_i}}$, and continue. At the end of the induction, set $\phi(f) = \overline{\bigcup_{n\in\mathbb{N}} E_{k_n}}$.

The construction ensures that $\overline{U}_{l_n} \cap E_{k_m} = \emptyset$ for all m and n, so that U_{l_n} is always a non-empty open subset of $U_n \setminus \phi(f)$; accordingly $\phi(f)$ is nowhere dense. If $G \subseteq \mathbb{R}$ is a non-empty open set meeting $\phi(f)$, there is a $k \in \mathbb{N}$ such that $E_k(f) \subseteq G \cap \phi(f)$. **P** Let $n \in \mathbb{N}$ be such that $U_n \subseteq G$ and $U_n \cap \phi(f) \neq \emptyset$. Then there is an $i \in \mathbb{N}$ such that $U_n \cap E_{k_i}(f) \neq \emptyset$; as $E_{k_i}(f) \cap \overline{U}_{l_j}$ is empty for every j, $U_n \not\subseteq \overline{\bigcup_{j\leq n} U_{l_j}}$ and

$$E_{k_n}(f) \subseteq U_{k_n} \subseteq U_n \subseteq G,$$

so we can take $k = k_n$. **Q**

(c) In the other direction, given $M \in \mathcal{M}$, choose a sequence $\langle F_n(M)\rangle_{n\in\mathbb{N}}$ of closed nowhere dense closed sets covering M. For k, l, $n \in \mathbb{N}$ set $g_{Mnk}(l) = \min\{j : x_{klj} \notin F_n(M)\}$. Since $\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preccurlyeq) = \operatorname{add}_\omega(\mathbb{N}^{\mathbb{N}}, \leq) \geq \omega_1$ (522C(ii) with 513Ib, or use the construction in part (d) of the proof of 522C), $\{g_{Mnk} : n, k \in \mathbb{N}\}$ is bounded above in $(\mathbb{N}^{\mathbb{N}}, \preccurlyeq)$; take $\psi(M) \in \mathbb{N}^{\mathbb{N}}$ such that $g_{Mnk} \leq \psi(M)$ for all n and k.

(d) Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \preccurlyeq, \mathbb{N}^{\mathbb{N}})$ to $(\mathcal{M}, \subseteq, \mathcal{M})$. **P** Suppose that $f \in \mathbb{N}^{\mathbb{N}}$ and $M \in \mathcal{M}$ are such that $\phi(f) \subseteq M$. Because $\phi(f)$ is closed and not empty and included in $\bigcup_{n\in\mathbb{N}} F_n(M)$, Baire's theorem (3A3G or 4A2Ma) tells us that there are $n \in \mathbb{N}$ and an open set G such that $\emptyset \neq G \cap \phi(f) \subseteq F_n(M)$. By the last remark in (b), there is a $k \in \mathbb{N}$ such that $E_k(f) \subseteq G \cap \phi(f)$. But this means that, for any $l \in \mathbb{N}$,

$x_{kli} \in G \cap \phi(f)$ for every $i \le f(l)$, while if $j = g_{Mnk}(l)$ then $x_{klj} \notin G \cap \phi(f)$. So $f(l) \le g_{Mnk}(l)$ for every l, and $f \preceq g_{Mnk} \preceq \psi(M)$. **Q**

(e) So in fact ϕ is a Tukey function from $(\mathbb{N}^{\mathbb{N}}, \preceq)$ to $(\mathcal{M}, \subseteq)$ (513Ea). It follows at once that

$$\operatorname{add} \mathcal{M} \le \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preceq), \quad \operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \preceq) \le \operatorname{cf} \mathcal{M}$$

(513Ee), that is, $\operatorname{add} \mathcal{M} \le \mathfrak{b}$ and $\mathfrak{d} \le \operatorname{cf} \mathcal{M}$, by 522C(ii).

522I Proposition $\mathfrak{b} \le \operatorname{non} \mathcal{M}$ and $\operatorname{cov} \mathcal{M} \le \mathfrak{d}$.

proof Again Let $\preceq$ be the partial order on $\mathbb{N}^{\mathbb{N}}$ described in 522C(ii). Then $(\mathbb{R} \setminus \mathbb{Q}, \in, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$. **P** Let $\phi : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{N}^{\mathbb{N}}$ be a homeomorphism (4A2Ub[2]). For $f \in \mathbb{N}^{\mathbb{N}}$, set $K_f = \{g : g \le f\}$; then K_f is compact, so $\phi^{-1}[K_f]$ is compact. Because $\phi^{-1}[K_f]$ is disjoint from $\mathbb{Q}$, it is nowhere dense. Set

$$\psi(f) = \bigcup \{\phi^{-1}[K_g] : g \in \mathbb{N}^{\mathbb{N}}, \ \{n : g(n) \ne f(n)\} \text{ is finite}\}.$$

Because there are only countably many functions eventually equal to f, $\psi(f) \in \mathcal{M}$.

Suppose that $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f \in \mathbb{N}^{\mathbb{N}}$ are such that $\phi(x) \preceq f$. Set $g = \phi(x) \vee f$; then $g(n) = f(n)$ for all but finitely many n, and $\phi(x) \le g$, so $x \in \phi^{-1}[K_g] \subseteq \psi(f)$. This shows that (ϕ, ψ) is a Galois-Tukey connection from $(\mathbb{R} \setminus \mathbb{Q}, \in, \mathcal{M})$ to $(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$, so that $(\mathbb{R} \setminus \mathbb{Q}, \in, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$. **Q**

It follows (using 522C(ii) and 512D) that

$$\begin{aligned} \mathfrak{b} &= \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}) \le \operatorname{add}(\mathbb{R} \setminus \mathbb{Q}, \in, \mathcal{M}) \\ &= \min\{\#(A) : A \subseteq \mathbb{R} \setminus \mathbb{Q}, \ A \notin \mathcal{M}\} \le \operatorname{non} \mathcal{M}, \end{aligned}$$

$$\begin{aligned} \mathfrak{d} &= \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}) \ge \operatorname{cov}(\mathbb{R} \setminus \mathbb{Q}, \in, \mathcal{M}) = \min\{\#(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{M}, \ \mathbb{R} \setminus \mathbb{Q} \subseteq \bigcup \mathcal{A}\} \\ &= \min\{\#(\mathcal{A} \cup \{\mathbb{Q}\}) : \mathcal{A} \subseteq \mathcal{M}, \ \mathbb{R} = \bigcup (\mathcal{A} \cup \{\mathbb{Q}\})\} \ge \operatorname{cov} \mathcal{M}. \end{aligned}$$

522J Theorem (see Truss 77 and Miller 81) $\operatorname{add} \mathcal{M} = \min(\mathfrak{b}, \operatorname{cov} \mathcal{M})$ and $\operatorname{cf} \mathcal{M} = \max(\mathfrak{d}, \operatorname{non} \mathcal{M})$.

proof My aim this time is to prove that

$$(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes (\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le \omega}),$$

defining $\le'$ as in the proof of 522H and $\ltimes$ as in 512I.

(a) Let $\langle q_n \rangle_{n \in \mathbb{N}}$ be a sequence running over $\mathbb{Q}$ with cofinal repetitions. For $f \in \mathbb{N}^{\mathbb{N}}$, set

$$E_f = \mathbb{R} \setminus \bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} \left]q_m - 2^{-f(m)}, q_m + 2^{-f(m)}\right[,$$

so that E_f is a meager set disjoint from $\mathbb{Q}$. Observe that if $\langle H_n \rangle_{n \in \mathbb{N}}$ is any sequence of closed sets disjoint from $\mathbb{Q}$, then there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $\bigcup_{n \in \mathbb{N}} H_n \subseteq E_f$. **P** For each $n \in \mathbb{N}$, let $f(n)$ be such that $\left]q_n - 2^{-f(n)}, q_n + 2^{-f(n)}\right[$ does not meet $\bigcup_{m \le n} H_m$. **Q**

For $M \in \mathcal{M}$, choose a sequence $\langle F_n(M) \rangle_{n \in \mathbb{N}}$ of nowhere dense closed sets covering M. For $x \in \mathbb{R}$, if $\mathbb{Q} \cap (\bigcup_{n \in \mathbb{N}} F_n(M) - x)$ is not empty, set $p_M(x)(n) = 0$ for every $n \in \mathbb{N}$; otherwise, take $p_M(x) = f$ for some $f \in \mathbb{N}^{\mathbb{N}}$ such that $E_f \supseteq \bigcup_{n \in \mathbb{N}} F_n(M) - x$. Now set $\phi(M) = (\bigcup_{n \in \mathbb{N}} F_n(M) + \mathbb{Q}, p_M)$. This defines $\phi : \mathcal{M} \to \mathcal{M} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{R}}$.

(b) In the other direction, define $\psi : \mathbb{R} \times [\mathbb{N}^{\mathbb{N}}]^{\le \omega} \to \mathcal{M}$ by setting $\psi(x, B) = \bigcup_{f \in B}(x + E_f)$ for $x \in \mathbb{R}$ and $B \in [\mathbb{N}^{\mathbb{N}}]^{\le \omega}$. Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{M}, \subseteq, \mathcal{M})$ to $(\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes (\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le \omega})$. **P** $(\mathbb{R}, \in, \mathcal{M})^{\perp} = (\mathcal{M}, \not\ni, \mathbb{R})$, so

$$(\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes (\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le \omega}) = (\mathcal{M} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{R}}, T, \mathbb{R} \times [\mathbb{N}^{\mathbb{N}}]^{\le \omega}),$$

where $((M, p), (x, B)) \in T$ iff $x \notin M$ and $p(x) \le g$ for some $g \in B$. Now suppose that $M \in \mathcal{M}$ and $(x, B) \in \mathbb{R} \times [\mathbb{N}^{\mathbb{N}}]^{\le \omega}$ are such that $(\phi(M), (x, B)) \in T$. Then $x \notin \bigcup_{n \in \mathbb{N}} F_n(M) + \mathbb{Q}$, so $\mathbb{Q} \cap (\bigcup_{n \in \mathbb{N}} F_n(M) - x) = \emptyset$, while $p_M(x) \le g$ for some $g \in B$. But this means that

[2] Later editions only.

$$E_g \supseteq E_{p_M(x)} \supseteq \bigcup_{n\in\mathbb{N}} F_n(M) - x \supseteq M - x, \quad M \subseteq E_g + x \subseteq \psi(x,B).$$

As M and (x,B) are arbitrary, (ϕ,ψ) is a Galois-Tukey connection, as claimed. **Q**

(c) It follows that

$$\operatorname{cf}\mathcal{M} = \operatorname{cov}(\mathcal{M},\subseteq,\mathcal{M})$$

(512Ea, as before)

$$\le \operatorname{cov}((\mathbb{R},\in,\mathcal{M})^\perp \ltimes (\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega}))$$
$$= \max(\operatorname{cov}(\mathbb{R},\in,\mathcal{M})^\perp, \operatorname{cov}(\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega}))$$

(512Jb)

$$= \max(\operatorname{add}(\mathbb{R},\in,\mathcal{M}),\mathfrak{d}) = \max(\operatorname{non}\mathcal{M},\mathfrak{d})$$

by 512Ed and the calculation in part (e) of the proof of 522H. On the other hand

$$\min(\operatorname{cov}\mathcal{M},\mathfrak{b}) = \min(\operatorname{cov}\mathcal{M},\operatorname{add}(\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega}))$$
$$= \min(\operatorname{cov}(\mathbb{R},\in,\mathcal{M}),\operatorname{add}(\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega}))$$
$$= \min(\operatorname{add}(\mathbb{R},\in,\mathcal{M})^\perp,\operatorname{add}(\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega}))$$
$$= \operatorname{add}((\mathbb{R},\in,\mathcal{M})^\perp \ltimes (\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega}))$$

(512Jc)

$$\le \operatorname{add}(\mathcal{M},\subseteq,\mathcal{M})$$

((b) above and 512Db)

$$= \operatorname{add}\mathcal{M}$$

(512Ea, as ever). Since we already know from 522E and 522H that $\operatorname{add}\mathcal{M} \le \min(\mathfrak{b},\operatorname{cov}\mathcal{M})$ and that $\max(\mathfrak{d},\operatorname{non}\mathcal{M}) \le \operatorname{cf}\mathcal{M}$, we have the result.

522K Localization The last step in proving the facts announced in 522B depends on the following construction. Let I be any set. Write $\mathcal{S}_I$ for the family of sets $S \subseteq \mathbb{N}\times I$ such that each vertical section $S[\{n\}]$ has at most 2^n members. For $f \in I^\mathbb{N}$ and $S \subseteq \mathbb{N}\times I$ say that $f \subseteq^* S$ if $\{n : n \in \mathbb{N},\, (n,f(n)) \notin S\}$ is finite; that is, $f \setminus S$ is finite, if we identify f with its graph. I will say that the supported relation $(I^\mathbb{N},\subseteq^*,\mathcal{S}_I)$ is the **I-localization relation**. By far the most important case (and the only one needed in this section) is when I is countably infinite; when $I=\mathbb{N}$ I will generally write $\mathcal{S}$ rather than $\mathcal{S}_\mathbb{N}$.

Members of $\mathcal{S}_I$, or similar sets, are sometimes called **slaloms**. The particular formula '$\#(S[\{n\}]) \le 2^n$' is convenient for the results of this section, but it is worth knowing that all functions diverging to ∞ give rise to equivalent partially ordered sets.

***522L Lemma** Let I be an infinite set. For any $\alpha \in \mathbb{N}^\mathbb{N}$ write

$$\mathcal{S}_I^{(\alpha)} = \{S : S \subseteq \mathbb{N}\times I,\, \#(S[\{n\}]) \le \alpha(n) \text{ for every } n \in \mathbb{N}\}.$$

Then $(I^\mathbb{N},\subseteq^*,\mathcal{S}_I^{(\alpha)}) \equiv_{\mathrm{GT}} (I^\mathbb{N},\subseteq^*,\mathcal{S}_I^{(\beta)})$ whenever α, $\beta \in \mathbb{N}^\mathbb{N}$ and $\lim_{n\to\infty}\alpha(n) = \lim_{n\to\infty}\beta(n) = \infty$.

proof Let $g \in \mathbb{N}^\mathbb{N}$ be a strictly increasing sequence such that $\beta(n) \le \alpha(i)$ whenever $n \in \mathbb{N}$ and $i \ge g(n)$, and let $h_n : I \to I^{g(n+1)\setminus g(n)}$ be a bijection for each n. Define $\phi : I^\mathbb{N} \to I^\mathbb{N}$ by setting $\phi(f)(n) = h_n^{-1}(f\restriction g(n+1)\setminus g(n))$ for $f \in I^\mathbb{N}$ and $n \in \mathbb{N}$. Define $\psi : \mathcal{S}_I^{(\beta)} \to \mathcal{P}(\mathbb{N}\times I)$ by setting $\psi(S) = \bigcup_{(n,i)\in S} h_n(i)$, identifying each $h_n(i) \in I^{g(n+1)\setminus g(n)} \subseteq (g(n+1)\setminus g(n))\times I$ with a subset of $\mathbb{N}\times I$. Now for $g(n) \le j < g(n+1)$, $\psi(S)[\{j\}] = \{h_n(i)(j) : i \in S[\{n\}]\}$ has at most $\beta(n) \le \alpha(j)$ members, while $\psi(S)[\{j\}] = \emptyset$ for $j < g(0)$, so $\psi(S) \in \mathcal{S}_I^{(\alpha)}$ for every $S \in \mathcal{S}_I^{(\beta)}$.

If $f \in I^\mathbb{N}$ and $S \in \mathcal{S}_I^{(\beta)}$ and $\phi(f) \subseteq^* S$, then there is an $n_0 \in \mathbb{N}$ such that $\phi(f)(n) \in S[\{n\}]$ for every $n \ge n_0$. So

$$f{\restriction}g(n+1)\setminus g(n)=h_n(\phi(f)(n))\subseteq\psi(S)$$

for every $n\ge n_0$, $(m,f(m))\in\psi(S)$ for every $m\ge g(n_0)$ and $f\subseteq^*\psi(S)$. This means that (ϕ,ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S}_I^{(\alpha)})$ to $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S}_I^{(\beta)})$. Similarly, $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S}_I^{(\beta)})\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S}_I^{(\alpha)})$ and the two supported relations are equivalent.

522M Proposition Let $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ be the $\mathbb{N}$-localization relation. Then $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})\equiv_{\mathrm{GT}}(\mathcal{N},\subseteq,\mathcal{N})$.

proof (a) Let $\langle G_{ij}\rangle_{i,j\in\mathbb{N}}$ be a stochastically independent family of open subsets of $[0,1]$ such that the Lebesgue measure μG_{ij} of G_{ij} is 2^{-i} for all i, $j\in\mathbb{N}$. For $f\in\mathbb{N}^{\mathbb{N}}$, set $\phi(f)=\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}G_{m,f(m)}$. Then $\phi(f)$ is negligible.

For each $E\in\mathcal{N}$, choose a non-empty compact self-supporting set $K_E\subseteq[0,1]\setminus E$ (416Dc). Let $\langle W_{En}\rangle_{n\in\mathbb{N}}$ enumerate a base for the relative topology on K_E not containing $\emptyset$; because K_E is self-supporting, no W_{En} is negligible. Set

$$I_{Eni}=\{j:j\in\mathbb{N},\,W_{En}\cap G_{ij}=\emptyset\}$$

for n, $i\in\mathbb{N}$. Then

$$\textstyle\sum_{i=0}^\infty 2^{-i}\#(I_{Eni})=\sum\{\mu G_{ij}:i,\,j\in\mathbb{N},\,G_{ij}\cap W_{En}=\emptyset\}$$

is finite, by the Borel-Cantelli lemma (273K). For each n, let $k(E,n)\in\mathbb{N}$ be such that $2^{-i}\#(I_{Eni})\le 2^{-n-1}$ for $i\ge k(E,n)$, and set

$$\psi(E)=\textstyle\bigcup_{n\in\mathbb{N}}\{(i,j):i,\,j\in\mathbb{N},\,i\ge k(E,n),\,j\in I_{Eni}\}.$$

Then

$$\#(\{j:(i,j)\in\psi(E)\})\le\sum_{n\in\mathbb{N},k(E,n)\le i}\#(I_{Eni})\le\sum_{n\in\mathbb{N},k(E,n)\le i}2^{-n-1}2^i\le 2^i$$

for every $i\in\mathbb{N}$, so $\psi(E)\in\mathcal{S}$.

Now (ϕ,ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ to $(\mathcal{N},\subseteq,\mathcal{N})$. **P** Suppose that $f\in\mathbb{N}^{\mathbb{N}}$ and $E\in\mathcal{N}$ are such that $\phi(f)\subseteq E$. Then $K_E\cap\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}G_{m,f(m)}=\emptyset$. By Baire's theorem, there is some $m\in\mathbb{N}$ such that $\bigcup_{i\ge m}G_{i,f(i)}\cap K_E$ is not dense in K_E, that is, there is an $n\in\mathbb{N}$ such that $W_{En}\cap\bigcup_{i\ge m}G_{i,f(i)}=\emptyset$ so $f(i)\in I_{Eni}$ for every $i\ge m$. But this means that $(i,f(i))\in\psi(E)$ for every $i\ge\max(m,k(E,n))$, so that $f\subseteq^*\psi(E)$. As f and E are arbitrary, we have the result. **Q**

Thus $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})\preccurlyeq_{\mathrm{GT}}(\mathcal{N},\subseteq,\mathcal{N})$.

(b) Let $\mathcal{H}$ be the family of finite unions of bounded open intervals in $\mathbb{R}$ with rational endpoints. Then $\mathcal{H}$ is countable. For each $n\in\mathbb{N}$, let $\langle H_{ni}\rangle_{i\in\mathbb{N}}$ be an enumeration of $\{H:H\in\mathcal{H},\,\mu H\le 4^{-n}\}$. Now for each $E\in\mathcal{N}$ there is an $f\in\mathbb{N}^{\mathbb{N}}$ such that $E\subseteq\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}H_{m,f(m)}$. **P** For each $n\in\mathbb{N}$, let $\langle J_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence of open intervals with rational endpoints such that $E\subseteq\bigcup_{i\in\mathbb{N}}J_{ni}$ and $\sum_{i=0}^\infty\mu J_{ni}\le 2^{-n-1}$. Re-enumerating $\langle J_{ni}\rangle_{n\in\mathbb{N},i\in\mathbb{N}}$ as $\langle J_i\rangle_{i\in\mathbb{N}}$, we have a sequence of open intervals with rational endpoints such that $\sum_{i=0}^\infty\mu J_i\le 1$ and $E\subseteq\bigcup_{i\ge n}J_i$ for every n. Let $\langle k(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $k(0)=0$ and $\sum_{i=k(n)}^\infty\mu J_i\le 4^{-n}$ for every $n\in\mathbb{N}$. Then $V_n=\bigcup_{k(n)\le i<k(n+1)}J_i$ belongs to $\mathcal{H}$ and has measure at most 4^{-n} for each n, so we can define $f\in\mathbb{N}^{\mathbb{N}}$ by saying that $H_{n,f(n)}=V_n$ for each n, and we shall have an appropriate function. **Q**

We can therefore find a function $\phi:\mathcal{N}\to\mathbb{N}^{\mathbb{N}}$ such that $E\subseteq\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}H_{m,\phi(E)(m)}$ for every $E\in\mathcal{N}$. In the reverse direction, define

$$\psi(S)=\textstyle\bigcap_{n\in\mathbb{N}}\bigcup\{H_{mi}:m\ge n,\,(m,i)\in S\}$$

for $S\in\mathcal{S}$; because

$$\textstyle\sum_{(m,i)\in S}\mu H_{mi}\le\sum_{m=0}^\infty 2^m4^{-m}<\infty,$$

$\psi(S)\in\mathcal{N}$.

Now (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{N},\subseteq,\mathcal{N})$ to $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$. **P** If $E\in\mathcal{N}$ and $S\in\mathcal{S}$ are such that $\phi(E)\subseteq^*S$, then

$$E \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} H_{m,\phi(E)(m)} \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n,(m,i)\in S} H_{mi} = \psi(S).\ \mathbf{Q}$$

So $(\mathcal{N}, \subseteq, \mathcal{N}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ and the proof is complete.

522N Lemma Let X be a topological space with a countable π-base. Then there is for each $n \in \mathbb{N}$ a countable family $\mathcal{U}_n$ of open subsets of X such that every dense open subset of X includes some member of $\mathcal{U}_n$ and $\bigcap \mathcal{V} \ne \emptyset$ for every $\mathcal{V} \in [\mathcal{U}_n]^{\le n}$.

proof Induce on n. Start by taking $\mathcal{U}$ to be a countable π-base for the topology of X which is closed under finite unions. Set $\mathcal{U}_0 = \{\emptyset\}$. For the inductive step to $n+1$, let $\langle H_i\rangle_{i\in\mathbb{N}}$ be a sequence running over $\mathcal{U}_n$, and set

$$\mathcal{J}_i = \{J : J \subseteq i, \bigcap_{j\in J} H_j \ne \emptyset\}$$

for $i \in \mathbb{N}$,

$$\mathcal{U}_{n+1} = \{U \cup H_i : i \in \mathbb{N}, U \in \mathcal{U}, U \cap \bigcap_{j\in J} H_j \ne \emptyset \text{ whenever } J \in \mathcal{J}_i\}.$$

Then $\mathcal{U}_{n+1}$ is a countable family of open sets. If $G \subseteq X$ is a dense open set, let $i \in \mathbb{N}$ be such that $H_i \subseteq G$. Then $\mathcal{J}_i$ is finite, so we can find a $U \in \mathcal{U}$ such that $U \subseteq G$ and $U \cap \bigcap_{j\in J} U_j \ne \emptyset$ for every $J \in \mathcal{J}_i$; then $U \cup H_i$ belongs to $\mathcal{U}_{n+1}$ and is included in G. If $\mathcal{V} \subseteq \mathcal{U}_{n+1}$ and $\#(\mathcal{V}) \le n+1$, then if $\mathcal{V}$ is empty we certainly have $\bigcap \mathcal{V} \ne \emptyset$. Otherwise, express $\mathcal{V}$ as $\{U_k \cup H_{i(k)} : k \le n\}$ where $U_k \cap \bigcap_{j\in J} H_j \ne \emptyset$ whenever $J \in \mathcal{J}_{i(k)}$; do this in such a way that $i(k) \le i(n)$ for every $k < n$. By the inductive hypothesis, $\bigcap_{k<n} H_{i(k)} \ne \emptyset$; if $i(k) = i(n)$ for some $k < n$, then of course $\bigcap_{k\le n} H_{i(k)} \ne \emptyset$; otherwise, $U_n \cap \bigcap_{k<n} H_{i(k)} \ne \emptyset$. In either case, $\bigcap \mathcal{V}$ is non-empty. So $\mathcal{U}_{n+1}$ has the required properties and the induction continues.

522O Proposition Let $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ be the $\mathbb{N}$-localization relation. Then $(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$.

proof Let $\langle U_n\rangle_{n\in\mathbb{N}}$ enumerate a π-base for the topology of $\mathbb{R}$ not containing $\emptyset$. By 522N, there is for each $n \in \mathbb{N}$ a countable family $\mathcal{V}_n$ of open subsets of U_n such that $\bigcap \mathcal{V} \ne \emptyset$ for every $\mathcal{V} \in [\mathcal{V}_n]^{\le 2^n}$ and every dense open subset of U_n includes some member of $\mathcal{V}_n$. Enumerate $\mathcal{V}_n$ as $\langle V_{nm}\rangle_{m\in\mathbb{N}}$.

For each $M \in \mathcal{M}$, let $\langle F_n(M)\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of nowhere dense sets covering M, and let $\phi(M) \in \mathbb{N}^{\mathbb{N}}$ be such that $F_n(M) \cap V_{n,\phi(M)(n)} = \emptyset$ for every n. In the other direction, for $S \in \mathcal{S}$ set

$$\psi(S) = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} (U_m \cap \bigcap_{i\in S[\{m\}]} V_{mi});$$

then because $\bigcap_{i\in S[\{m\}]} V_{mi}$ is non-empty for every n, $\bigcup_{m\ge n} (U_m \cap \bigcap_{i\in S[\{m\}]} V_{mi})$ is a dense open set for every n, and $\psi(S)$ is meager.

Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{M}, \subseteq, \mathcal{M})$ to $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$. $\mathbf{P}$ Suppose that $M \in \mathcal{M}$ and $S \in \mathcal{S}$ are such that $\phi(M) \subseteq^* S$. Let $n \in \mathbb{N}$ be such that $\phi(M)(k) \in S[\{k\}]$ for every $k \ge n$. Then

$$F_m(M) \cap \bigcap_{i\in S[\{k\}]} V_{ki} \subseteq F_k(M) \cap V_{k,\phi(M)(k)} = \emptyset$$

whenever $k \ge \max(m, n)$, so

$$F_m(M) \subseteq \mathbb{R} \setminus \bigcup_{k\ge \max(m,n)} \bigcap_{i\in S[\{k\}]} V_{ki} \subseteq \psi(S)$$

for every m, and $M \subseteq \psi(S)$. $\mathbf{Q}$

So we have the result.

522P Corollary $\mathcal{M} \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

proof Putting 522M and 522O and 512Cb together, we see that $(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$, that is, $\mathcal{M} \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

522Q Theorem (Bartoszyński 84, Raisonnier & Stern 85) $\mathrm{add}\,\mathcal{N} \le \mathrm{add}\,\mathcal{M}$ and $\mathrm{cf}\,\mathcal{M} \le \mathrm{cf}\,\mathcal{N}$.

proof 522P, 513Ee.

522R The exactness of Cichoń's diagram The list of inequalities displayed in Cichoń's diagram is complete in the following sense: it is known that all assignments of the values ω_1, ω_2 to the eleven cardinals of the diagram which are allowed by the diagram together with the two equalities $\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$, $\operatorname{cf}\mathcal{M} = \max(\mathfrak{d}, \operatorname{non}\mathcal{M})$ are relatively consistent with the axioms of ZFC. So, for instance, it is possible to have

$$\omega_1 = \operatorname{add}\mathcal{N} = \operatorname{cov}\mathcal{N} = \operatorname{add}\mathcal{M} = \mathfrak{b} = \operatorname{non}\mathcal{M},$$

$$\operatorname{cov}\mathcal{M} = \mathfrak{d} = \operatorname{cf}\mathcal{M} = \operatorname{non}\mathcal{N} = \operatorname{cf}\mathcal{N} = \mathfrak{c} = \omega_2.$$

In §§552 and 554 below I will describe forcing constructions exhibiting a few of these combinations; for the rest, I refer you to Bartoszyński & Judah 95, §§5.2, 7.5 and 7.6. I remark also that not all the forcing methods used are effective beyond ω_2, so that if we allow $\mathfrak{c} = \omega_3$ then some puzzles remain.

522S The cardinals non $\mathcal{M}$, cov $\mathcal{M}$ All the cardinals in Cichoń's diagram appear in many different ways in set-theoretic real analysis. But $\operatorname{add}\mathcal{N}$, the additivity of Lebesgue measure, the bounding number $\mathfrak{b}$, and $\operatorname{cov}\mathcal{M}$, the Novák number of $\mathbb{R}$, seem to be particularly important. The additivity of measure will play a large role in the next section. Here I will give two striking characterizations of $\operatorname{cov}\mathcal{M}$ and a dual characterization of $\operatorname{non}\mathcal{M}$.

Theorem (a) $n(\mathbb{R}) = \operatorname{cov}\mathcal{M} = \mathfrak{m}_{\text{countable}}$.

(b) (Bartoszyński 87) $\operatorname{cov}\mathcal{M}$ is the least cardinal of any set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $f \in A$ such that $f(n) \neq g(n)$ for every $n \in \mathbb{N}$.

(c) (Bartoszyński 87) $\operatorname{non}\mathcal{M}$ is the least cardinal of any set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $f \in A$ such that $\{n : f(n) = g(n)\}$ is infinite.

proof (a) Because $\mathbb{R}$ is a Baire space, the Novák number $n(\mathbb{R})$ is equal to $\operatorname{cov}\mathcal{M}$ (512Eb). By 517P(d-ii) or 517P(d-iii), $n(\mathbb{R}) = \mathfrak{m}_{\text{countable}}$.

(b) Let κ be the smallest cardinal of any $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $f \in A$ such that $f \cap g = \emptyset$, identifying the functions f and g with their graphs in $\mathbb{N} \times \mathbb{N}$.

(i) $\operatorname{cov}\mathcal{M} \leq \kappa$. **P** Suppose that $A \subseteq \mathbb{N}^{\mathbb{N}}$ and that $\#(A) < \operatorname{cov}\mathcal{M}$. Set $P = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, ordered by extension of functions. Then P is a non-empty countable partially ordered set. For each $f \in A$ set $Q_f = \{p : p \in P, p \cap f \neq \emptyset\}$; then Q_f is cofinal with P. Set $\mathcal{Q} = \{Q_f : f \in A\}$. Then

$$\#(\mathcal{Q}) \leq \#(A) < \operatorname{cov}\mathcal{M} = \mathfrak{m}_{\text{countable}} \leq \mathfrak{m}^{\uparrow}(P),$$

so there is an upwards-linked $R \subseteq P$ meeting every member of $\mathcal{Q}$. Now $g_0 = \bigcup R \subseteq \mathbb{N} \times \mathbb{N}$ is a function; taking $g \in \mathbb{N}^{\mathbb{N}}$ to be any extension of g_0 to the whole of $\mathbb{N}$, $g \cap f \neq \emptyset$ for every $f \in A$.

As A is arbitrary, this shows that $\kappa \geq \operatorname{cov}\mathcal{M}$. **Q**

In particular, $\kappa \geq \omega_1$, as can also be seen by elementary arguments.

(ii) Let $\langle K_n \rangle_{n\in\mathbb{N}}$ be any sequence of non-empty countable sets, and write F for the set of all functions f such that $\operatorname{dom} f$ is an infinite subset of $\mathbb{N}$ and $f(n) \in K_n$ for every $n \in \operatorname{dom} f$. Then if $A \in [F]^{<\kappa}$ there is a $g \in \prod_{n\in\mathbb{N}} K_n$ such that $f \cap g \neq \emptyset$ for every $f \in A$. **P** For each $n \in \mathbb{N}$, let F_n be $\bigcup\{\prod_{i\in I} K_i : I \in [\mathbb{N}]^{n+1}\}$. For $f \in F$ and $n \in \mathbb{N}$ take any $(n+1)$-element subset of f and call it $p_f(n)$, so that $p_f(n) \in F_n$. Now each F_n is countably infinite, and

$$A' = \{p_f : f \in A\} \subseteq \prod_{n\in\mathbb{N}} F_n \cong \mathbb{N}^{\mathbb{N}}$$

has cardinal less than κ, so there is a $\phi \in \prod_{n\in\mathbb{N}} F_n$ such that $\phi \cap p_f \neq \emptyset$ for every $f \in A$.

Now choose $\langle i_k \rangle_{k\in\mathbb{N}}$ inductively so that $i_k \in \operatorname{dom}\phi(k) \setminus \{i_j : j < k\}$ for each $k \in \mathbb{N}$, and take $g \in \prod_{n\in\mathbb{N}} K_n$ such that $g(i_k) = \phi(k)(i_k)$ for every k. Then for any $f \in A$ there is a $k \in \mathbb{N}$ such that $\phi(k) = p_f(k) \subseteq f$, so that $g(i_k) = f(i_k)$ and $f \cap g \neq \emptyset$, as required. **Q**

(iii) If $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $f_0 \in \mathbb{N}^{\mathbb{N}}$ and $\#(A) < \kappa$, then there is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that $g(0) > 0$, $g(n+1) \geq f_0(g(n))$ for every n and $\{n : f(g(n)) \leq g(n+1)\}$ is infinite for every $f \in A$. **P** For $f \in A$ set

$$f^*(0) = 0, \quad \tilde{f}(n) = \max_{i\leq n} f(i), \quad f^*(n+1) = n + \tilde{f}(\tilde{f}(f^*(n)))$$

for each n, so that $f \leq \tilde{f}$, $\tilde{f}$ and f^* are non-decreasing, and f^* is unbounded. Consider $B = \{f^* \restriction \mathbb{N} \setminus n : f \in A, n \in \mathbb{N}\}$; then $\#(B) \leq \max(\#(A), \omega) < \kappa$, so by (ii) (or otherwise) there is an $h \in \mathbb{N}^{\mathbb{N}}$ meeting every member of B. Now $h \cap f^*$ is infinite for every $f \in A$. Set

$$g(0)=1+h(0),\quad g(n+1)=1+\max_{i\le n+1}h(i)+\max_{i\le n}f_0(g(i))$$

for $n\in\mathbb{N}$, so that $h(n)<g(n)$ and $f_0(g(n))\le g(n+1)$ for every n, and g is non-decreasing.

? Suppose, if possible, that $f\in A$ is such that $\{n:f(g(n))\le g(n+1)\}$ is finite. Let $n_0\in\mathbb{N}$ be such that $f(g(n))\ge g(n+1)$ for every $n\ge n_0$. If $i\ge n_0$ then

$$\tilde f(g(i))\ge f(g(i))\ge g(i+1),$$

so if $i\ge n_0$ and $j\in\mathbb{N}$ are such that $f^*(j)\ge g(i)$, then

$$f^*(j+1)\ge\tilde f(\tilde f(f^*(j)))\ge\tilde f(\tilde f(g(i)))\ge\tilde f(g(i+1))\ge g(i+2)$$

because $\tilde f$ is non-decreasing. But f^* is unbounded; taking k such that $f^*(k)\ge g(n_0)$, we have $f^*(k+i)\ge g(n_0+2i)$ for every $i\in\mathbb{N}$; because both f^* and g are non-decreasing, this means that $f^*(n)\ge g(n)$ whenever $n\ge\max(k,2k-n_0)$. But there must be such an n with $f^*(n)=h(n)<g(n)$, so this is impossible. **X**

Thus g has the required property. **Q**

(iv) Now suppose that P is a countable partially ordered set, $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q})<\kappa$, and $p_0\in P$. Let $\langle p_i\rangle_{i\ge1}$ be such that $\langle p_i\rangle_{i\in\mathbb{N}}$ runs over P with cofinal repetitions. Let $f\in\mathbb{N}^{\mathbb{N}}$ be a strictly increasing function such that whenever $n\in\mathbb{N}$ and $i<n$ then there is a $j\in f(n)\setminus n$ such that $p_i\le p_j$. For each $Q\in\mathcal{Q}$ let $f_Q\in\mathbb{N}^{\mathbb{N}}$ be a strictly increasing function such that whenever $n\in\mathbb{N}$ and $i<n$ there is a $j\in f_Q(n)\setminus n$ such that $p_i\le p_j\in Q$. By (iii), we can find $g\in\mathbb{N}^{\mathbb{N}}$ such that $g(0)>0$, $g(n+1)\ge f(g(n))$ for every n and $I_Q=\{n:g(n+1)\ge f_Q(g(n))\}$ is infinite for every $Q\in\mathcal{Q}$.

For each $n\in\mathbb{N}$, set $J_n=g(n+1)\setminus g(n)$, and let Φ_n be the set of functions $h:g(n)\to J_n$ such that $p_i\le p_{h(i)}$ for every $i<g(n)$; because $g(n+1)\ge f(g(n))$ this is non-empty. For $Q\in\mathcal{Q}$ and $n\in I_Q$ let $\phi_Q(n)\in\Phi_n$ be such that $p_{\phi_Q(n)(i)}\in Q$ for every $i<g(n)$; such a function exists because $g(n+1)\ge f_Q(g(n))$. Now all the Φ_n are countable (indeed finite), so (ii) tells us that there is a $\phi\in\prod_{n\in\mathbb{N}}\Phi_n$ such that $\phi\cap\phi_Q$ is non-empty for every $Q\in\mathcal{Q}$.

Define $\langle i_n\rangle_{n\in\mathbb{N}}$ by setting $i_0=0$ and $i_{n+1}=\phi(n)(i_n)$ for $n\in\mathbb{N}$; because $\operatorname{dom}\phi(n)=g(n)>0$ and $\phi(n)(i)<g(n+1)$ whenever $i<g(n)$, i_n is well-defined for each n. Because $\phi(n)\in\Phi_n$ for each n, $p_{i_n}\le p_{i_{n+1}}$ for each n. If $Q\in\mathcal{Q}$ there is some n such that $\phi(n)=\phi_Q(n)$, so that

$$p_{i_{n+1}}=p_{\phi(n)(i_n)}=p_{\phi_Q(n)(i_n)}\in Q.$$

But this means that $R=\{p_{i_k}:k\in\mathbb{N}\}$ is an upwards-linked (indeed, totally ordered) subset of P meeting every member of $\mathcal{Q}$ and containing p_0. As p_0 and $\mathcal{Q}$ are arbitrary, $\mathfrak{m}^{\uparrow}(P)\ge\kappa$. As P is arbitrary, $\mathfrak{m}_{\text{countable}}\ge\kappa$ and $\kappa=\mathfrak{m}_{\text{countable}}=\operatorname{cov}\mathcal{M}$, as claimed.

(c) Write λ for the least cardinal of any set $A\subseteq\mathbb{N}^{\mathbb{N}}$ such that for every $g\in\mathbb{N}^{\mathbb{N}}$ there is an $f\in A$ such that $f\cap g$ is infinite. Of course λ is infinite.

(i) Let $A_0\subseteq\mathbb{R}$ be a non-meager set with cardinal $\operatorname{non}\mathcal{M}$. Then $A_0\setminus\mathbb{Q}$ is not a meager subset of $\mathbb{R}$ and therefore is not a meager subset of $\mathbb{R}\setminus\mathbb{Q}$, since any subset of $\mathbb{R}\setminus\mathbb{Q}$ which is nowhere dense in $\mathbb{R}\setminus\mathbb{Q}$ is also nowhere dense in $\mathbb{R}$. So $\mathbb{R}\setminus\mathbb{Q}$ has a non-meager subset with cardinal $\operatorname{non}\mathcal{M}$. But $\mathbb{R}\setminus\mathbb{Q}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ (4A2Ub again), so there is a non-meager set $A\subseteq\mathbb{N}^{\mathbb{N}}$ with cardinal $\operatorname{non}\mathcal{M}$.

Now take any $g\in\mathbb{N}^{\mathbb{N}}$. Then $\bigcup_{i\ge n}\{f:f\in\mathbb{N}^{\mathbb{N}},\ f(i)=g(i)\}$ is a dense open subset of $\mathbb{N}^{\mathbb{N}}$ for every $n\in\mathbb{N}$, so

$$\{f:f\in\mathbb{N}^{\mathbb{N}},\ f\cap g\text{ is infinite}\}=\bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n}\{f:f(i)=g(i)\}$$

is a dense G_δ set and meets A; that is, there is an $f\in A$ such that $f\cap g$ is infinite. Accordingly $\lambda\le\operatorname{non}\mathcal{M}$.

(ii) Let $A\subseteq\mathbb{N}^{\mathbb{N}}$ be a set with cardinal λ such that for every $g\in\mathbb{N}^{\mathbb{N}}$ there is an $f\in A$ such that $f\cap g$ is infinite. Write S_2 for $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$.

(α) There is a set $A_1\subseteq\mathbb{N}^{\mathbb{N}}$, with cardinal at most λ, such that whenever $g\in\mathbb{N}^{\mathbb{N}}$ and $D\subseteq\mathbb{N}$ is infinite, there is an $f\in A_1$ such that $f\cap g\restriction D$ is infinite. **P** For $n\in\mathbb{N}$ set $F_n=\bigcup_{I\in[\mathbb{N}]^{n+1}}\mathbb{N}^I$. Because each F_n is countably infinite, there is a family $H\subseteq\prod_{n\in\mathbb{N}}F_n$, with cardinal λ, such that for every $\psi\in\prod_{n\in\mathbb{N}}F_n$ there is a $\phi\in H$ such that $\phi\cap\psi$ is infinite. For $\phi\in H$ define $k_\phi\in\mathbb{N}^{\mathbb{N}}$ by setting $k_\phi(n)=\min(\operatorname{dom}\phi(n)\setminus\{k_\phi(i):i<n\})$ for each n, so that $k_\phi\in\mathbb{N}^{\mathbb{N}}$ is injective, and choose $f_\phi\in\mathbb{N}^{\mathbb{N}}$ such that $f_\phi(k_\phi(n))=\phi(n)(k_\phi(n))$ for every n. Now suppose that $g\in\mathbb{N}^{\mathbb{N}}$ and $D\subseteq\mathbb{N}^{\mathbb{N}}$ is infinite. Define $\psi\in\prod_{n\in\mathbb{N}}F_n$ by taking $\psi(n)(i)=g(i)$ whenever

$i \in D$ and $\#(D \cap i) \le n$. We know that there is a $\phi \in H$ such that $K = \{n : \phi(n) = \psi(n)\}$ is infinite, and that for every $n \in K$

$$k_\phi(n) \in \operatorname{dom}\phi(n) = \operatorname{dom}\psi(n) \subseteq D,$$

$$f_\phi(k_\phi(n)) = \phi(n)(k_\phi(n)) = \psi(n)(k_\phi(n)) = g(k_\phi(n)).$$

As $k_\phi[K]$ is an infinite subset of D, $f_\phi \cap g\restriction D$ is infinite. So we can take $A_1 = \{f_\phi : \phi \in H\}$. **Q**

(β) Because S_2 is countably infinite, we can copy A onto a set $A_2 \subseteq S_2^{\mathbb{N}}$, with cardinal λ, such that for every $\psi \in S_2^{\mathbb{N}}$ there is a $\phi \in A_2$ such that $\phi \cap \psi$ is infinite. For each $\phi \in A_2$ let $h_\phi \in \mathbb{N}^{\mathbb{N}}$ be a strictly increasing function such that $h_\phi(k) + \#(\phi(n)) \le h_\phi(k+2)$ whenever $k \in \mathbb{N}$ and $h_\phi(k) \le n < h_\phi(k+1)$.

(γ) For $\phi \in A_2$ and $f \in A_1$ define $x_{\phi f}$, $y_{\phi f} \in \{0,1\}^{\mathbb{N}}$ by saying (in the notation of 5A1C) that

$$x_{\phi f} = \sigma_0^\frown\sigma_1^\frown\ldots, \quad y_{\phi f} = \tau_0^\frown\tau_1^\frown\ldots,$$

where

$$\begin{aligned} \sigma_k &= \phi(f(k)) \text{ if } h_\phi(2k) \le f(k) < h_\phi(2k+1), \\ &= {<}0{>} \text{ otherwise}, \\ \tau_k &= \phi(f(k)) \text{ if } h_\phi(2k+1) \le f(k) < h_\phi(2k+2), \\ &= {<}0{>} \text{ otherwise}. \end{aligned}$$

Note that

$$\#(\sigma_k) \le h_\phi(2k+2) - h_\phi(2k), \quad \#(\tau_k) \le h_\phi(2k+3) - h_\phi(2k+1)$$

for every k. Write

$$A_3 = \{x_{\phi f} : \phi \in A_2,\, f \in A_1\} \cup \{y_{\phi f} : \phi \in A_2,\, f \in A_1\}$$

so that $A_3 \subseteq \{0,1\}^{\mathbb{N}}$ has cardinal at most λ.

(δ) A_3 is a non-meager subset of $\{0,1\}^{\mathbb{N}}$. **P** Let H be a dense G_δ subset of $\{0,1\}^{\mathbb{N}}$. Then we can express H as $\bigcap_{n\in\mathbb{N}} G_n$ where $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of dense open subsets of $\{0,1\}^{\mathbb{N}}$. For each n,

$$G'_n = \bigcap_{m\le n}\bigcap_{\tau\in\{0,1\}^m}\{x : x \in \{0,1\}^{\mathbb{N}},\, \tau^\frown x \in G_n\}$$

is a dense open subset of $\{0,1\}^{\mathbb{N}}$ so there is an $\upsilon_n \in S_2$ such that $\{x : \upsilon_n \subseteq x \in \{0,1\}^{\mathbb{N}}\} \subseteq G'_n$. Let $\phi \in A_2$ be such that $C = \{n : n \in \mathbb{N},\, \phi(n) = \upsilon_n\}$ is infinite. Set $D_0 = \{k : C \cap h_\phi(2k+1) \setminus h_\phi(2k) \ne \emptyset\}$, $D_1 = \{k : C \cap h_\phi(2k+2) \setminus h_\phi(2k+1) \ne \emptyset\}$; then at least one of D_0, D_1 is infinite.

Suppose that D_0 is infinite. For $k \in D_0$ set $n_k = \min(C \cap h_\phi(2k+1) \setminus h_\phi(2k))$. Then there is an $f \in A_1$ such that

$$E_0 = \{k : k \in D_0,\, f(k) = n_k\}$$

is infinite. In this case, for $k \in E_0$, $h_\phi(2k) \le f(k) < h_\phi(2k+1)$ so $\#(\phi(f(k))) \le h_\phi(2k+2) - h_\phi(2k)$ and $\sigma_k = \phi(f(k)) = \phi(n_k) = \upsilon_{n_k}$. Accordingly $\sigma_k^\frown\sigma_{k+1}^\frown\ldots \in G'_{n_k}$.

At the same time, writing τ for $\sigma_0^\frown\ldots^\frown\sigma_{k-1}$,

$$\#(\tau) \le \sum_{i=0}^{k-1} h_\phi(2i+2) - h_\phi(2i) \le h_\phi(2k) \le n_k$$

and

$$x_{\phi f} = \tau^\frown\upsilon_{n_k}^\frown\sigma_{k+1}^\frown\cdots$$

belongs to G_{n_k}. Since $\langle n_k\rangle_{k\in\mathbb{N}}$ is strictly increasing, $x_{\phi f}$ belongs to G_n for infinitely many n and therefore belongs to H.

Similarly, if D_1 is infinite, we can set $m_k = \min(C \cap h_\phi(2k+2) \setminus h_\phi(2k+1))$ for $k \in D_1$, take $f \in A_1$ such that $E_1 = \{k : k \in D_1,\, f(k) = m_k\}$ is infinite, and for $k \in E_1$ see that $\#(\tau_0^\frown\ldots^\frown\tau_{k-1}) \le m_k$ and $\tau_k = \upsilon_{m_k}$ so that $y_{\phi f} \in G_{m_k}$. Thus in either case we have a member of A_3 belonging to H; as H is arbitrary, A_3 is non-meager. **Q**

(ϵ) Finally, let $\theta : \{0,1\}^{\mathbb{N}} \to [0,1]$ be the standard surjection defined by setting $\theta(x) = \sum_{i=0}^{\infty} 2^{-i-1}x(i)$ for $x \in \{0,1\}^{\mathbb{N}}$. Then θ is continuous and irreducible, so the inverse image of a dense open subset of $[0,1]$ is a dense open subset of $\{0,1\}^{\mathbb{N}}$, and $\theta[A_3]$ is non-meager in $[0,1]$. As the interior of $[0,1]$ in $\mathbb{R}$ is dense in $[0,1]$, $\theta[A_3]$ is non-meager in $\mathbb{R}$ and

$$\operatorname{non}\mathcal{M} \le \#(\theta[A_3]) \le \#(A_3) \le \lambda.$$

Together with (i) above, this shows that $\operatorname{non}\mathcal{M} = \lambda$, as claimed.

522T Martin numbers Following the identification of $\operatorname{cov}\mathcal{M}$ with $\mathfrak{m}_{\text{countable}}$, we can amalgamate the diagrams in 522B and 517Ob, as follows:

$$\begin{tikzcd}
 & & \operatorname{cov}\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \operatorname{non}\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \mathfrak{c} \\
 & & & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & & \\
 & & \operatorname{add}\mathcal{N} \arrow[r, dash] \arrow[d, dash] & \operatorname{add}\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \operatorname{cov}\mathcal{M} \arrow[r, dash] & \operatorname{non}\mathcal{N} & \\
 & & \mathfrak{m}_{\sigma\text{-linked}} \arrow[r, dash] \arrow[d, dash] & \mathfrak{p} \arrow[d, dash] & & & \\
\omega_1 \arrow[r, dash] & \mathfrak{m} \arrow[r, dash] & \mathfrak{m}_{\text{K}} \arrow[r, dash] & \mathfrak{m}_{\text{pc}\omega_1} & & &
\end{tikzcd}$$

proof The two new inequalities to be proved are $\mathfrak{m}_{\sigma\text{-linked}} \le \operatorname{add}\mathcal{N}$ and $\mathfrak{p} \le \operatorname{add}\mathcal{M}$.

(a) Let $\mathcal{S}^{\infty}$ be the '$(\mathbb{N},\infty)$-localization poset'

$$\{p : p \subseteq \mathbb{N}\times\mathbb{N},\ \#(p[\{n\}]) \le 2^n \text{ for every } n,\ \sup_{n\in\mathbb{N}} \#(p[\{n\}]) < \infty\},$$

ordered by $\subseteq$. For $p \in \mathcal{S}^{\infty}$ set $\|p\| = \max_{n\in\mathbb{N}} \#(p[\{n\}])$. Then $\mathcal{S}^{\infty}$ is σ-linked upwards. **P** If p, $q \in \mathcal{S}^{\infty}$, $\|p\| \le n$, $\|q\| \le n$ and $p[\{i\}] = q[\{i\}]$ for every $i \le n$, then $p \cup q \in \mathcal{S}^{\infty}$. So for any $n \in \mathbb{N}$ and $\langle J_i\rangle_{i\le n} \in \prod_{i\le n}[\mathbb{N}]^{\le 2^i}$ we have an upwards-linked set

$$\{p : p \in \mathcal{S}^{\infty},\ \|p\| \le n,\ p[\{i\}] = J_i \text{ for every } i \le n\};$$

as there are only countably many such families $\langle J_i\rangle_{i\le n}$, $\mathcal{S}^{\infty}$ is σ-linked upwards. **Q**

Accordingly $\mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{m}^{\uparrow}(\mathcal{S}^{\infty})$. Next, $\mathfrak{m}^{\uparrow}(\mathcal{S}^{\infty}) \le \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$, where $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ is the $\mathbb{N}$-localization relation. **P** Suppose that $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $\#(A) < \mathfrak{m}^{\uparrow}(\mathcal{S}^{\infty})$. For each $f \in A$, set $Q_f = \{p : p \in \mathcal{S}^{\infty}, f \subseteq^* p\}$. If $p \in \mathcal{S}^{\infty}$ and $\|p\| = n$, then $p \subseteq p \cup \{(i, f(i)) : i \ge n\} \in Q_f$; so Q_f is cofinal with $\mathcal{S}^{\infty}$. As $\#(\{Q_f : f \in A\}) < \mathfrak{m}^{\uparrow}(\mathcal{S}^{\infty})$, there is an upwards-directed $R \subseteq \mathcal{S}^{\infty}$ meeting Q_f for every $f \in A$. Set $S = \bigcup R$. For each $n \in \mathbb{N}$, $\{p[\{n\}] : p \in R\}$ is an upwards-directed family of subsets of $\mathbb{N}$, all of size at most 2^n, with union $S[\{n\}]$. So $\#(S[\{n\}]) \le 2^n$; as n is arbitrary, $S \in \mathcal{S}$. If $f \in A$, there is a $p \in R \cap Q_f$, and now $f \subseteq^* p \subseteq S$. As A is arbitrary, we have the result. **Q**

Now

$$\mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{m}^{\uparrow}(\mathcal{S}^{\infty}) \le \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}) = \operatorname{add}(\mathcal{N}, \subseteq, \mathcal{N})$$

(522M, 512Db)

$$= \operatorname{add}\mathcal{N},$$

as required.

(b)(i) Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{R}$, not containing $\emptyset$. Consider the set P of pairs (σ, F) where $\sigma \in \bigcup_{n\in\mathbb{N}} \mathcal{U}^n$ and $F \subseteq \mathbb{R}$ is nowhere dense, together with the relation $\le$ where $(\sigma, F) \le (\sigma', F')$ if σ' extends σ, $F' \supseteq F$ and $F \cap \sigma'(i) = \emptyset$ whenever $i \in \operatorname{dom}\sigma' \setminus \operatorname{dom}\sigma$. Then $\le$ is a partial order on P. **P** If $(\sigma, F) \le (\sigma', F') \le (\sigma'', F'')$ then we surely have $\sigma \subseteq \sigma' \subseteq \sigma''$ and $F \subseteq F' \subseteq F''$. If $i \in \operatorname{dom}\sigma'' \setminus \operatorname{dom}\sigma$, then either $i \in \operatorname{dom}\sigma' \setminus \operatorname{dom}\sigma$ and $\sigma''(i) = \sigma'(i)$ must be disjoint from F, or $i \in \operatorname{dom}\sigma'' \setminus \operatorname{dom}\sigma'$ and $\sigma''(i)$ must be disjoint from $F' \supseteq F$. Thus in either case $F \cap \sigma''(i) = \emptyset$; as i is arbitrary, $(\sigma, F) \le (\sigma'', F'')$. Thus $\le$ is transitive. Evidently it is also reflexive and anti-symmetric, so it is a partial order. **Q**

(ii) $(P, \le)$ is σ-centered upwards. **P** If $(\sigma, F_0), \dots, (\sigma, F_k)$ are members of P with a common first member, then they have a common upper bound $(\sigma, \bigcup_{i\le k} F_i)$ in P. So for any $n \in \mathbb{N}$ and $\sigma \in \mathcal{U}^n$ the set $\{(\sigma, F) : F \subseteq \mathbb{R}$ is nowhere dense$\}$ is upwards-centered in P; as $\bigcup_{n\in\mathbb{N}} \mathcal{U}^n$ is countable, P is σ-centered upwards. **Q**

(iii) For each $V \in \mathcal{U}$ and $n \in \mathbb{N}$ set

$$Q_{nV} = \{(\sigma, F) : (\sigma, F) \in P,\ V \cap \textstyle\bigcup_{n \le i < \operatorname{dom}\sigma} \sigma(i) \ne \emptyset\}.$$

Then Q_{nV} is cofinal with P. **P** If $(\sigma, F) \in P$, set $m = \max(n, \operatorname{dom}\sigma) + 1$, and take $U \in \mathcal{U}$ such that $U \subseteq V \setminus F$. Setting

$$\begin{aligned} \sigma'(i) &= \sigma(i) \text{ for } i < \operatorname{dom}\sigma, \\ &= U \text{ for } \operatorname{dom}\sigma \le i < m, \end{aligned}$$

we find that $(\sigma, F) \le (\sigma', F) \in Q_{nV}$. **Q**

For each nowhere dense set $H \subseteq \mathbb{R}$,

$$Q'_H = \{(\sigma, F) : (\sigma, F) \in P,\ H \subseteq F\}$$

is cofinal with P. **P** For any $(\sigma, F) \in P$, we have $(\sigma, F) \le (\sigma, F \cup H) \in Q'_H$. **Q**

(iv) Now suppose that $\mathcal{A} \subseteq \mathcal{M}$ and $\#(\mathcal{A}) < \mathfrak{p}$. Then each member of $\mathcal{A}$ is covered by a sequence of nowhere dense sets, so there is a family $\mathcal{H}$ of nowhere dense sets with the same union as $\mathcal{A}$ and with $\#(\mathcal{H}) \le \max(\omega, \#(\mathcal{A}))$. In this case

$$\mathcal{Q} = \{Q_{nV} : n \in \mathbb{N},\ V \in \mathcal{U}\} \cup \{Q'_H : H \in \mathcal{H}\}$$

is a family of cofinal subsets of P and

$$\#(\mathcal{Q}) \le \max(\omega, \#(\mathcal{A})) < \mathfrak{p} \le \mathfrak{m}^{\uparrow}(P).$$

There is therefore an upwards-directed $R \subseteq P$ meeting every member of $\mathcal{Q}$. If (σ, F) and (σ', F') belong to R, they must be upwards-compatible in P, and in particular σ and σ' have a common extension; we therefore have a function $\phi = \bigcup_{(\sigma,F)\in R} \sigma$ from a subset of $\mathbb{N}$ to $\mathcal{U}$. If $n \in \mathbb{N}$ and $V \in \mathcal{U}$, then there is a $(\sigma, F) \in R \cap Q_{nV}$, so that there is some $i \ge n$ such that $\phi(i) = \sigma(i)$ meets V. As V is arbitrary, the open set $W_n = \bigcup_{i\in\operatorname{dom}\phi, i\ge n} \phi(i)$ is dense; as n is arbitrary, $M = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} W_n$ is meager. Now $H \subseteq M$ for every $H \in \mathcal{H}$. **P** There is a $(\sigma, F) \in R \cap Q'_H$. Set $n = \operatorname{dom}\sigma$. If $i \in \operatorname{dom}\phi \setminus n$, there is a $(\sigma', F') \in R$ such that $i \in \operatorname{dom}\sigma'$; because R is upwards-directed, we may suppose that $(\sigma, F) \le (\sigma', F')$. But in this case $\phi(i) = \sigma'(i)$ must be disjoint from F and therefore from H. As i is arbitrary, $H \cap W_n = \emptyset$ and $H \subseteq M$. **Q**

As H is arbitrary, $\bigcup \mathcal{A} = \bigcup \mathcal{H} \subseteq M$. As $\mathcal{A}$ is arbitrary, $\operatorname{add}\mathcal{M} \ge \mathfrak{p}$, as claimed.

Remark In fact $\mathfrak{m}^{\uparrow}(\mathcal{S}^{\infty})$ is exactly equal to $\operatorname{add}\mathcal{N}$; see 528N.

***522U FN($\mathcal{P}\mathbb{N}$)** For any cardinal which is known to lie between ω_1 and $\mathfrak{c}$, it is natural, and often profitable, to try to locate it on Cichoń's diagram. For the Freese-Nation number of $\mathcal{P}\mathbb{N}$, which appeared more than once in §518, we have the following results.

Proposition (Fuchino Koppelberg & Shelah 96, Fuchino Geschke & Soukup 01) (a) $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \mathfrak{b}$.

(b) $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \operatorname{cov}\mathcal{N}$.

(c) If $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ then $\operatorname{shr}\mathcal{M} = \omega_1$, so

$$\mathfrak{m} = \mathfrak{m}_{\mathrm{K}} = \mathfrak{m}_{\mathrm{pc}\omega_1} = \mathfrak{m}_{\sigma\text{-linked}} = \mathfrak{p} = \operatorname{add}\mathcal{N} = \operatorname{add}\mathcal{M} = \mathfrak{b} = \operatorname{cov}\mathcal{N} = \operatorname{non}\mathcal{M} = \omega_1.$$

(d) If $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and $\kappa \ge \mathfrak{m}_{\mathrm{countable}}$ is such that $\operatorname{cf}[\kappa]^{\le\omega} \le \kappa \le \mathfrak{c}$, then $\kappa = \mathfrak{c}$. So if $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and $\mathfrak{m}_{\mathrm{countable}} < \omega_\omega$, then

$$\mathfrak{m}_{\mathrm{countable}} = \operatorname{non}\mathcal{N} = \mathfrak{d} = \operatorname{cf}\mathcal{M} = \operatorname{cf}\mathcal{N} = \mathfrak{c}.$$

(e) There is a set $A \subseteq \mathbb{R}$ with cardinal $\mathfrak{m}_{\text{countable}}$ such that every meager set meets A in a set with cardinal less than $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

proof (a) Let $\le^*$ and $\preceq$ be the pre-order and partial order on $\mathbb{N}^{\mathbb{N}}$ described in 522C, so that $\mathfrak{b} = \mathrm{add}(\mathbb{N}^{\mathbb{N}}, \preceq)$. Write κ for $\mathrm{FN}(\mathcal{P}\mathbb{N})$; by 518D, $\kappa = \mathrm{FN}(\mathbb{N}^{\mathbb{N}}, \le)$ and we have a Freese-Nation function $\phi : \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}^{\mathbb{N}}]^{<\kappa}$ for $\le$. For $f \in \mathbb{N}^{\mathbb{N}}$, set $\psi(f) = \bigcup\{\phi(g) : g \le^* f \le^* g\}$; then $\#(\psi(f)) \le \kappa$.

? Suppose, if possible, that $\kappa < \mathfrak{b}$. Choose a family $\langle f_\xi\rangle_{\xi\le\kappa}$ in $\mathbb{N}^{\mathbb{N}}$ inductively, as follows. Given $\langle f_\eta\rangle_{\eta<\xi}$ where $\xi \le \kappa$, $\bigcup_{\eta<\xi}\psi(f_\eta)$ has cardinal at most $\kappa < \mathfrak{b}$, so has a $\preceq$-upper bound f'_ξ; now set $f_\xi(i) = f'_\xi(i) + 1$ for every i, and continue.

Next choose $\langle h_\xi\rangle_{\xi<\kappa}$ in $\phi(f_\kappa)$ as follows. For each $\xi < \kappa$, $f_\xi \in \phi(f_\xi)$ (511Hh) so $f_\xi \in \psi(f_\xi)$, $f_\xi \preceq f'_\kappa$ and $f_\xi \preceq f_\kappa$. So if we set $g_\xi = f_\xi \wedge f_\kappa$ then $g_\xi \le^* f_\xi \le^* g_\xi$ while $g_\xi \le f_\kappa$. There is therefore an $h_\xi \in \phi(g_\xi)\cap\phi(f_\kappa)$ such that $g_\xi \le h_\xi \le f_\kappa$. Now if $\eta < \xi < \kappa$, $h_\eta \in \phi(g_\eta) \subseteq \psi(f_\eta)$ so $h_\eta \preceq f'_\xi$. Accordingly

$$\begin{aligned}\{i : h_\xi(i) \le h_\eta(i)\} &\subseteq \{i : h_\xi(i) \le f'_\xi(i)\} \cup \{i : f'_\xi(i) < h_\eta(i)\}\\ &\subseteq \{i : h_\xi(i) < f_\xi(i)\} \cup \{i : f'_\xi(i) < h_\eta(i)\}\\ &\subseteq \{i : g_\xi(i) < f_\xi(i)\} \cup \{i : f'_\xi(i) < h_\eta(i)\}\end{aligned}$$

is finite and $h_\xi \ne h_\eta$. But this means that $\{h_\xi : \xi < \kappa\}$ has cardinal κ and $\#(\phi(f_\kappa)) = \kappa$, contrary to the choice of ϕ. **X**

Thus $\mathfrak{b} \le \kappa = \mathrm{FN}(\mathcal{P}\mathbb{N})$, as claimed. In particular, $\mathrm{FN}(\mathcal{P}\mathbb{N})$ is uncountable.

(b)(i) We need to know the following fact: if $\mathcal{E}$ is a family of non-negligible Lebesgue measurable subsets of $\mathbb{R}$, and $\#(\mathcal{E}) < \mathrm{cov}\,\mathcal{N}$, there is a countable set meeting every member of $\mathcal{E}$. **P** For each $E \in \mathcal{E}$, $\mathbb{R}\setminus(\mathbb{Q}+E)$ is negligible (439Eb), so there is an $x \in \mathbb{R}\cap\bigcap_{E\in\mathcal{E}}\mathbb{Q}+E$; now $\mathbb{Q}+x$ is countable and meets every member of $\mathcal{E}$. **Q**

(ii) Set $\kappa = \mathrm{FN}(\mathcal{P}\mathbb{N})$. If $\mathcal{C}$ is the family of closed sets in $\mathbb{R}$, then $(\mathcal{C}, \subseteq) \cong (\mathfrak{T}, \supseteq)$, so $\mathrm{FN}(\mathcal{C}) = \mathrm{FN}(\mathfrak{T}) = \kappa$ (518D). Let $f : \mathcal{C} \to [\mathcal{C}]^{<\kappa}$ be a Freese-Nation function.

(iii) ? If $\kappa < \mathrm{cov}\,\mathcal{N}$, write K for the set of infinite successor cardinals $\lambda \le \kappa$, and for $\lambda \in K$ set $D_\lambda = \{x : x \in \mathbb{R},\ \#(f(\{x\})) < \lambda\}$. As $\mathbb{R} = \bigcup_{\lambda\in K} D_\lambda$, there must be some $\lambda \in K$ such that D_λ cannot be covered by κ negligible sets. Choose $\langle M_\xi\rangle_{\xi\le\lambda}$ and $\langle H_{\xi n}\rangle_{\xi<\lambda,n\in\mathbb{N}}$ inductively, as follows. $M_\xi = \emptyset$. Given that $M_\xi \subseteq \mathcal{C}$ and $\#(M_\xi) \le \kappa$, (i) tells us that there is a countable set $A_\xi \subseteq \mathbb{R}$ meeting every non-negligible member of M_ξ; let $\langle H_{\xi n}\rangle_{n\in\mathbb{N}}$ be a sequence of closed subsets of $\mathbb{R}\setminus A_\xi$ such that $\bigcup_{n\in\mathbb{N}} H_{\xi n}$ is conegligible. Now set

$$M_{\xi+1} = M_\xi \cup \{H_{\xi n} : n \in \mathbb{N}\} \cup \bigcup_{F\in M_\xi} f(F) \in [\mathcal{C}]^{\le\kappa}.$$

At non-zero limit ordinals $\xi \le \lambda$, set $M_\xi = \bigcup_{\eta<\xi} M_\eta$.

By the choice of λ, there is an $x \in D_\lambda$ which does not belong to any negligible set belonging to M_λ, nor to any of the sets $\mathbb{R}\setminus\bigcup_{n\in\mathbb{N}} H_{\xi n}$ for $\xi < \lambda$. Now $\#(M_\lambda \cap f(\{x\})) < \lambda$; because λ is regular, there is a $\xi < \lambda$ such that $M_\lambda \cap f(\{x\}) \subseteq M_\xi$. Let $n \in \mathbb{N}$ be such that $x \in H_{\xi n}$. Then there must be an $F \in f(\{x\}) \cap f(H_{\xi n})$ such that $x \in F \subseteq H_{\xi n}$. In this case, $H_{\xi n} \in M_{\xi+1}$ and $F \in M_{\xi+2} \subseteq M_\lambda$, so in fact $F \in M_\xi$. Because $x \in F$, F cannot be negligible, so $A_\xi \cap F \ne \emptyset$; but $H_{\xi n}$ was chosen to be disjoint from A_ξ. **X**

(iv) Thus $\kappa \ge \mathrm{cov}\,\mathcal{N}$, as claimed.

(c) Let $A \subseteq \mathbb{R}$ be a non-meager set.

(i) By 518D, $\mathrm{FN}(\mathfrak{T}) = \omega_1$, where $\mathfrak{T}$ is the topology of $\mathbb{R}$. Let $f : \mathfrak{T} \to [\mathfrak{T}]^{\le\omega}$ be a Freese-Nation function. There is a set M such that

(α) whenever $G \in M \cap \mathfrak{T}$ then $f(G) \subseteq M$;

(β) whenever $t \in M \cap \mathbb{R}$ then $\mathbb{R}\setminus\{t\} \in M$;

(γ) whenever $\mathcal{G} \subseteq M$ is a countable family of dense open subsets of $\mathbb{R}$, $M \cap A \cap \bigcap\mathcal{G}$ is non-empty;

(δ) $\#(M) \le \omega_1$.

P Build a non-decreasing family $\langle M_\xi\rangle_{\xi<\omega_1}$ of countable sets as follows. $M_0 = \emptyset$. Given that M_ξ is countable, let $M_{\xi+1}$ be a countable set including M_ξ such that

(α) whenever $G \in M_\xi \cap \mathfrak{T}$ then $f(G) \subseteq M_{\xi+1}$;

(β) whenever $t \in M_\xi \cap \mathbb{R}$ then $\mathbb{R} \setminus \{t\} \in M_{\xi+1}$;

(γ) $M_{\xi+1} \cap A \cap \bigcap\{G : G \in M_\xi$ is a dense open subset of $\mathbb{R}\}$ is not empty.

For countable limit ordinals $\xi > 0$, set $M_\xi = \bigcup_{\eta<\xi} M_\eta$. At the end of the construction, set $M = \bigcup_{\xi<\omega_1} M_\xi$. **Q**

(ii) If $H \subseteq \mathbb{R}$ is an open set, there is a countable family $\mathcal{G} \subseteq M \cap \mathfrak{T}$ such that $M \cap \mathbb{R} \cap \bigcap \mathcal{G} \subseteq H \subseteq \bigcap \mathcal{G}$. **P** Set $\mathcal{G} = \{G : G \in f(H) \cap M,\ H \subseteq G\}$; then certainly $H \subseteq \bigcap \mathcal{G}$ and $\mathcal{G}$ is countable. If $t \in M \cap \mathbb{R} \setminus H$, then $H \subseteq \mathbb{R} \setminus \{t\}$ so there is a $G \in f(H) \cap f(\mathbb{R} \setminus \{t\})$ such that $H \subseteq G \subseteq \mathbb{R} \setminus \{t\}$; since $\mathbb{R} \setminus \{t\} \in M$, $G \in M$; and $t \notin G$. As t is arbitrary, $M \cap \mathbb{R} \cap \bigcap \mathcal{G} \subseteq H$. **Q**

(iii) Now consider $B = A \cap M$. Then $\#(B) \le \omega_1$. **?** If B is meager, let $\langle H_n \rangle_{n\in\mathbb{N}}$ be a sequence of dense open sets such that $B \cap \bigcap_{n\in\mathbb{N}} H_n = \emptyset$. For each $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a countable family of dense open sets belonging to M such that $M \cap \mathbb{R} \cap \bigcap \mathcal{G}_n \subseteq H_n$ (using (ii)). Set $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$; then $\mathcal{G} \subseteq M$ is a countable family of dense open sets, so there is a $t \in M \cap A \cap \bigcap \mathcal{G}$, by condition ($\gamma$) in the specification of M. But now $t \in M \cap A \cap \bigcap \mathcal{G}_n \subseteq H_n$ for each n, so $t \in B \cap \bigcap_{n\in\mathbb{N}} H_n$, which is impossible. **X**

Thus A has a non-meager subset with cardinal at most ω_1; as A is arbitrary, $\operatorname{shr} \mathcal{M} = \omega_1$.

(d)(i) Again let $\mathfrak{T}$ be the topology of $\mathbb{R}$ and $f : \mathfrak{T} \to [\mathfrak{T}]^{\le\omega}$ a Freese-Nation function. This time, we can find a set M such that

($\dagger$) for every $g \in \mathbb{N}^\mathbb{N}$ there is an $h \in M \cap \mathbb{N}^\mathbb{N}$ such that $g(n) \ne h(n)$ for every $n \in \mathbb{N}$;

(α) whenever $G \in M \cap \mathfrak{T}$ then $f(G) \subseteq M$;

(β) $M \cap [M]^{\le\omega}$ is cofinal with $[M]^{\le\omega}$;

(γ) whenever $D \in M$ is countable, then there is a double sequence $\langle G_{ij} \rangle_{i,j\in\mathbb{N}}$ belonging to M such that every G_{ij} belongs to $\mathfrak{T}$, $\langle G_{ij} \rangle_{j\in\mathbb{N}}$ is disjoint for each $i \in \mathbb{N}$ and whenever $G \in D$ is an open subset of $\mathbb{R}$ with infinite complement, there is an $i \in \mathbb{N}$ such that $G_{ij} \setminus G$ is non-empty for every $j \in \mathbb{N}$;

(δ) whenever $\langle G_{ij} \rangle_{i,j\in\mathbb{N}} \in M$ is a double sequence of open subsets of $\mathbb{R}$, and $h \in M \cap \mathbb{N}^\mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} G_{i,h(i)} \in M$;

(ϵ) $\#(M) = \kappa$.

P Build a non-decreasing family $\langle M_\xi \rangle_{\xi<\omega_1}$ of sets with cardinal κ as follows. Start with a set $M_0 \subseteq \mathbb{N}^\mathbb{N}$ such that $\#(M_0) = \kappa$ and for every $g \in \mathbb{N}^\mathbb{N}$ there is an $h \in M_0$ such that $g(n) \ne h(n)$ for every $n \in \mathbb{N}$ (using 522Sb). Given that $\#(M_\xi) = \kappa$, then $\operatorname{cf}[M_\xi]^{\le\omega} = \kappa$. Let $M_{\xi+1} \supseteq M_\xi$ be such that

(α) whenever $G \in M_\xi \cap \mathfrak{T}$ then $f(G) \subseteq M_{\xi+1}$;

(β) $M_{\xi+1} \cap [M_\xi]^{\le\omega}$ is cofinal with $[M_\xi]^{\le\omega}$;

(γ) whenever $D \in M_\xi$ is countable, then there is a double sequence $\langle G_{ij} \rangle_{i,j\in\mathbb{N}} \in M_{\xi+1}$ such that every G_{ij} is an open set in $\mathbb{R}$, $\langle G_{ij} \rangle_{j\in\mathbb{N}}$ is disjoint for each $i \in \mathbb{N}$ and whenever $G \in D$ is an open subset of $\mathbb{R}$ with infinite complement, there is an $i \in \mathbb{N}$ such that $G_{ij} \setminus G$ is non-empty for every $j \in \mathbb{N}$;

(δ) whenever $\langle G_{ij} \rangle_{i,j\in\mathbb{N}} \in M_\xi$ is a double sequence of open subsets of $\mathbb{R}$, and $h \in M_\xi \cap \mathbb{N}^\mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} G_{i,h(i)} \in M_{\xi+1}$;

(ϵ) $\#(M_{\xi+1}) = \kappa$.

For limit ordinals $\xi > 0$, set $M_\xi = \bigcup_{\eta<\xi} M_\eta$. At the end of the construction, set $M = \bigcup_{\xi<\omega_1} M_\xi$. Then

$$M \cap [M]^{\le\omega} = \bigcup_{\xi<\omega_1} M_{\xi+1} \cap [M_\xi]^{\le\omega}$$

is cofinal with $\bigcup_{\xi<\omega_1} [M_\xi]^{\le\omega} = [M]^{\le\omega}$, and it is easy to see that the other conditions are satisfied. **Q**

(ii) **?** Now suppose, if possible, that there is a $t \in \mathbb{R}$ such that $\mathbb{R} \setminus I \notin M$ for any finite set I containing t. Set

$$\mathcal{G} = \{G : G \in M \cap f(\mathbb{R} \setminus \{t\}),\ t \notin G\}.$$

Then $\mathcal{G}$ is a countable subset of M and $\mathbb{R} \setminus G$ is infinite for every $G \in \mathcal{G}$. Let $D \in M$ be a countable set including $\mathcal{G}$. Then we have a double sequence $\langle G_{ij} \rangle_{i,j\in\mathbb{N}} \in M$ such that $\langle G_{ij} \rangle_{j\in\mathbb{N}}$ is a disjoint sequence of open sets for each $i \in \mathbb{N}$ and whenever $G \in D$ is an open subset of $\mathbb{R}$ with infinite complement, there is an $i \in \mathbb{N}$ such that $G_{ij} \setminus G$ is non-empty for every $j \in \mathbb{N}$. In particular, this last clause is true for every $G \in \mathcal{G}$. For each $i \in \mathbb{N}$ choose $g(i) \in \mathbb{N}$ such that $t \notin G_{ij}$ for any $j \ne g(i)$; let $h \in M \cap \mathbb{N}^\mathbb{N}$ be such that $h(i) \ne g(i)$

for every i, and set $H=\bigcup_{i\in\mathbb{N}}G_{i,h(i)}\in M$; note that $t\notin H$. Now there is a $G\in f(H)\cap f(\mathbb{R}\setminus\{t\})$ such that $H\subseteq G$ and $t\notin G$. As $f(H)\subseteq M$, $G\in M$, so $G\in\mathcal{G}$. But this means that $G_{i,h(i)}\subseteq G$ for every $i\in\mathbb{N}$; and we chose $\langle G_{ij}\rangle_{i,j\in\mathbb{N}}$ so that this could not be so. **X**

Thus $\mathcal{I}=\{I:I\in[\mathbb{R}]^{<\omega},\ \mathbb{R}\setminus I\in M\}$ covers $\mathbb{R}$. As $\#(\mathcal{I})\le\#(M)\le\kappa$, $\#(\mathbb{R})\le\kappa$ and $\kappa=\mathfrak{c}$, as claimed.

(iii) Finally, if $\mathfrak{m}_{\text{countable}}<\omega_\omega$, then we can take $\kappa=\mathfrak{m}_{\text{countable}}$, by 5A1F(e-iv), and get $\mathfrak{m}_{\text{countable}}=\ldots=\mathfrak{c}$.

(e) Because $\mathrm{FN}(\mathfrak{T})=\mathrm{FN}(\mathcal{P}\mathbb{N})$, 518E tells us that there is a set $A\subseteq\mathbb{R}$, with cardinal $n(\mathbb{R})=\mathfrak{m}_{\text{countable}}$, such that $\#(A\cap F)<\mathrm{FN}^*(\mathfrak{T})=\mathrm{FN}^*(\mathcal{P}\mathbb{N})$ for every nowhere dense set $F\subseteq\mathbb{R}$. As $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$ certainly has uncountable cofinality, A meets every meager set in a set with cardinal less than $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

522V Cofinalities For any cardinal associated with a mathematical structure, we can ask whether there are any limitations on what that cardinal can be. The commonest form of such limitations, when they appear, is a restriction on the possible cofinalities of the cardinal. I run through the known results concerning the cardinals of Cichoń's diagram. Most are elementary, but part (f) requires a substantial argument.

Proposition (a) $\mathrm{cf}\,\mathfrak{c}\ge\mathfrak{p}$.

(b) $\mathrm{add}\,\mathcal{N}$, $\mathrm{add}\,\mathcal{M}$ and $\mathfrak{b}$ are regular.

(c) $\mathrm{cf}(\mathrm{cf}\,\mathcal{N})\ge\mathrm{add}\,\mathcal{N}$, $\mathrm{cf}(\mathrm{cf}\,\mathcal{M})\ge\mathrm{add}\,\mathcal{M}$ and $\mathrm{cf}\,\mathfrak{d}\ge\mathfrak{b}$.

(d) $\mathrm{cf}(\mathrm{non}\,\mathcal{N})\ge\mathrm{add}\,\mathcal{N}$, $\mathrm{cf}(\mathrm{non}\,\mathcal{M})\ge\mathrm{add}\,\mathcal{M}$.

(e) If $\mathrm{cf}\,\mathcal{M}=\mathfrak{m}_{\text{countable}}$ then $\mathrm{cf}(\mathrm{cf}\,\mathcal{M})\ge\mathrm{non}\,\mathcal{M}$; if $\mathrm{cf}\,\mathcal{N}=\mathrm{cov}\,\mathcal{N}$, then $\mathrm{cf}(\mathrm{cf}\,\mathcal{N})\ge\mathrm{non}\,\mathcal{N}$.

(f) (Bartoszyński & Judah 89) $\mathrm{cf}(\mathfrak{m}_{\text{countable}})\ge\mathrm{add}\,\mathcal{N}$.

proof (a) If $\omega\le\kappa<\mathfrak{p}$ then $2^\kappa=\mathfrak{c}$, by 517Rb, so $\mathrm{cf}\,\mathfrak{c}>\kappa$ by 5A1Fd.

(b) Use 513C(a-i); to see that $\mathfrak{b}$ is regular, use its characterization as the additivity of a partially ordered set in 522C(ii).

(c) Use 513C(a-ii); this time, we need to know that $\mathfrak{d}$ is the cofinality of a partially ordered set for which $\mathfrak{b}$ is the additivity.

(d)-(e) 513Cb with 522Sa,

(f)(i) Write $\mathcal{M}_1$ for the ideal of meager subsets of $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}^\mathbb{N}$ is given its usual topology. Let $(\mathbb{N}^\mathbb{N},\subseteq^*,\mathcal{S})$ be the $\mathbb{N}$-localization relation (522K), and set $\mathcal{S}^{(0)}=\{S:S\in\mathcal{S},\ \lim_{n\to\infty}2^{-n}\#(S[\{n\}])=0\}$. I will write `finint` and `disj` for the relations $\{(A,B):A\cap B\text{ is finite}\}$, $\{(A,B):A\cap B=\emptyset\}$. Following the same mild abuse of notation as in 512Aa and elsewhere, I will write $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^\mathbb{N})$ and $(\mathbb{N}^\mathbb{N},\mathtt{disj},\mathbb{N}^\mathbb{N})$ for the supported relations $(\mathcal{S}^{(0)},R_1,\mathbb{N}^\mathbb{N})$ and $(\mathbb{N}^\mathbb{N},R_2,\mathbb{N}^\mathbb{N})$, where

$$R_1=\{(S,f):S\in\mathcal{S}^{(0)},\ f\in\mathbb{N}^\mathbb{N},\ \{n:(n,f(n))\in S\}\text{ is finite}\},$$

$$R_2=\{(f,g):f,\ g\in\mathbb{N}^\mathbb{N},\ f(n)\ne g(n)\text{ for every }n\}.$$

(ii)(α) $(\mathbb{N}^\mathbb{N},\in,\mathcal{M}_1)\preccurlyeq_{\mathrm{GT}}(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^\mathbb{N})$. **P** For $f\in\mathbb{N}^\mathbb{N}$, set $\phi(f)=f$ (identifying f with its graph, as usual); for $g\in\mathbb{N}^\mathbb{N}$, set $\psi(g)=\{h:h\in\mathbb{N}^\mathbb{N},\ h\cap g\text{ is finite}\}$. Then $\phi(f)\in\mathcal{S}^{(0)}$ for every $f\in\mathbb{N}^\mathbb{N}$, and $\psi(g)\in\mathcal{M}_1$ for every $g\in\mathbb{N}^\mathbb{N}$, because all the sets $\{h:h\cap g\subseteq n\}$ are nowhere dense. If f, $g\in\mathbb{N}^\mathbb{N}$ and $(\phi(f),g)\in\mathtt{finint}$, then $f\cap g$ is finite so $f\in\psi(g)$; thus (ϕ,ψ) is a Galois-Tukey connection from $(\mathbb{N}^\mathbb{N},\in,\mathcal{M}_1)$ to $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^\mathbb{N})$. **Q**

(β) $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^\mathbb{N})\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^\mathbb{N},\mathtt{disj},\mathbb{N}^\mathbb{N})$. **P** Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ such that $\#(I_n)=2^n$ for each n. For $n\in\mathbb{N}$, let $\theta_n:\mathbb{N}^{I_n}\to\mathbb{N}$ be a bijection. For $S\in\mathcal{S}^{(0)}$, choose $\phi(S)\in\mathbb{N}^\mathbb{N}$ such that whenever $(n,i)\in S$ then $\phi(S)\cap\theta_n^{-1}(i)\ne\emptyset$, where once again both the function $\phi(S)$ and the function $\theta_n^{-1}(i)$ are identified with their graphs; this is possible because on each set I_n there are at most 2^n functions with domain I_n that $\phi(S)$ has to meet. For $g\in\mathbb{N}^\mathbb{N}$, define $\psi(g)\in\mathbb{N}^\mathbb{N}$ by saying that $\psi(g)(n)=\theta_n(g\restriction I_n)$ for every n.

Now suppose that $S\in\mathcal{S}^{(0)}$ and $g\in\mathbb{N}^\mathbb{N}$ are such that $S\cap\psi(g)$ is infinite. Then there is certainly an n such that $(n,\psi(g)(n))\in S$. In this case,

$$\emptyset\ne\phi(S)\cap\theta_n^{-1}(\psi(g)(n))=\phi(S)\cap g\restriction I_n,$$

so $\phi(S)\cap g$ is non-empty. Turning this round, if $(\phi(S),g)\in\mathtt{disj}$ then $(S,\psi(g))\in\mathtt{finint}$; that is, (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$ to $(\mathbb{N}^{\mathbb{N}},\mathtt{disj},\mathbb{N}^{\mathbb{N}})$. **Q**

(γ) $\operatorname{cov}(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})=\mathfrak{m}_{\text{countable}}$. **P**

$$\mathfrak{m}_{\text{countable}}=n(\mathbb{N}^{\mathbb{N}})=\operatorname{cov}\mathcal{M}_1$$

(517Pd)

$$=\operatorname{cov}(\mathbb{N}^{\mathbb{N}},\in,\mathcal{M}_1)\le\operatorname{cov}(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$$

(512Da and (α) above)

$$\le\operatorname{cov}(\mathbb{N}^{\mathbb{N}},\mathtt{disj},\mathbb{N}^{\mathbb{N}})$$

((β) above)

$$=\mathfrak{m}_{\text{countable}}$$

(522Sb, with 522Sa). **Q**

(iii) Suppose that $\kappa<\operatorname{add}\mathcal{N}$ and that $\langle S_\xi\rangle_{\xi<\kappa}$ is any family in $\mathcal{S}^{(0)}$. Then there is an $S^*\in\mathcal{S}^{(0)}$ such that $S_\xi\setminus S^*$ is finite for every $\xi<\kappa$. **P** For $\xi<\kappa$, $n\in\mathbb{N}$ let $f_\xi(n)\in\mathbb{N}$ be such that $\#(S_\xi[\{i\}])\le 2^{i-2n}$ for every $i\ge f_\xi(n)$. Because $\kappa<\operatorname{add}\mathcal{N}\le\mathfrak{b}$, there is an $f\in\mathbb{N}^{\mathbb{N}}$ such that $\{n:f_\xi(n)>f(n)\}$ is finite for every $\xi<\kappa$ (522C(ii)); of course we may suppose that $f(0)=0$ and that f is strictly increasing and that $f(n)\ge 2n$ for every n. Set $J_n=f(n+1)\setminus f(n)$ for each n. For each $\xi<\kappa$, let m_ξ be such that $f_\xi(n)\le f(n)$ for every $n\ge m_\xi$; set $S'_\xi=\{(i,j):(i,j)\in S_\xi,\ i\ge f(m_\xi)\}$. Then $S_\xi\setminus S'_\xi$ is finite and $\#(S'_\xi[\{i\}])\le 2^{i-2n}$ whenever $i\in J_n$.

For each $n\in\mathbb{N}$, let $\mathcal{K}_n$ be the family of those sets $K\subseteq J_n\times\mathbb{N}$ such that $\#(K[\{i\}])\le 2^{i-2n}$ for every $i\in J_n$, and $\theta_n:\mathcal{K}_n\to\mathbb{N}$ a bijection; set $h_\xi(n)=\theta_n(S'_\xi\cap(J_n\times\mathbb{N}))$ for each $\xi<\kappa$.

By 522M, $\operatorname{add}(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})=\operatorname{add}\mathcal{N}$ is greater than κ, so there is an $S\in\mathcal{S}$ such that $h_\xi\subseteq^* S$ for every $\xi<\kappa$. Set $S^*=\bigcup_{(n,j)\in S}\theta_n^{-1}(j)$. For any $n\in\mathbb{N}$ and $i\in J_n$, $\#(\theta_n^{-1}(j)[\{i\}])\le 2^{i-2n}$ for every $j\in\mathbb{N}$, so that $S^*[\{i\}]=\bigcup_{(n,j)\in S}\theta_n^{-1}(j)[\{i\}]$ has cardinal at most 2^{i-n}. This means that $S^*\in\mathcal{S}^{(0)}$.

Take any $\xi<\kappa$. As $h_\xi\subseteq^* S$, there is some $m\in\mathbb{N}$ such that $(n,h_\xi(n))\in S$, that is, $(n,\theta_n(S'_\xi\cap(J_n\times\mathbb{N})))\in S$, for every $n\ge m$. But this means that $S'_\xi\cap(J_n\times\mathbb{N})\subseteq S^*$ for every $n\ge m$, so $S'_\xi\setminus S^*$ is finite; it follows at once that $S_\xi\setminus S^*$ is finite. Thus we have a suitable S^*. **Q**

(iv) ? Now suppose, if possible, that $\operatorname{cf}(\mathfrak{m}_{\text{countable}})=\kappa<\operatorname{add}\mathcal{N}$. By (ii-$\gamma$), there is a set $A\subseteq\mathbb{N}^{\mathbb{N}}$ of size $\mathfrak{m}_{\text{countable}}$ such that for every $S\in\mathcal{S}^{(0)}$ there is an $f\in A$ such that $S\cap f$ is finite. Express A as $\bigcup_{\xi<\kappa}A_\xi$ where $\#(A_\xi)<\mathfrak{m}_{\text{countable}}$ for every $\xi<\kappa$. By (ii-γ) again, we can find for each $\xi<\kappa$ an $S_\xi\in\mathcal{S}^{(0)}$ such that $S_\xi\cap f$ is infinite for every $f\in A_\xi$. By (iii), there is an $S^*\in\mathcal{S}^{(0)}$ such that $S_\xi\setminus S^*$ is finite for every $\xi<\kappa$. But this means that $S^*\cap f$ must be infinite for every $f\in A_\xi$ and every $\xi<\kappa$; which contradicts the choice of A. **X**

So we are forced to conclude that $\operatorname{cf}(\mathfrak{m}_{\text{countable}})\ge\operatorname{add}\mathcal{N}$, as stated.

522W Other spaces All the theorems above refer to the specific σ-ideals $\mathcal{M}$ and $\mathcal{N}$ of subsets of $\mathbb{R}$ or the specific partially ordered set $\mathbb{N}^{\mathbb{N}}$. Of course the structures involved appear in many other guises. In particular, we have the following results.

(a)(i) Let (X,Σ,μ) be an atomless countably separated (definition: 343D) σ-finite perfect (definition: 342K) measure space of non-zero measure, and $\mathcal{N}(\mu)$ the null ideal of μ. Then $(X,\mathcal{N}(\mu))$ is isomorphic to $(\mathbb{R},\mathcal{N})$; in particular, $\operatorname{add}\mathcal{N}(\mu)=\operatorname{add}\mathcal{N}$, $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}$, $\operatorname{non}\mathcal{N}(\mu)=\operatorname{non}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\mu)=\operatorname{cf}\mathcal{N}$. **P** The first thing to note is that because μ is σ-finite there is a probability measure ν on X with the same measurable sets and the same negligible sets as μ (215B(vii)); and of course ν is still atomless, countably separated and perfect. Next, the completion $\hat\nu$ of ν is again atomless, countably separated and perfect (212Gd, 343H(vi), 451G(c-i)) and has the same negligible sets as ν (212Eb). In the same way, starting from Lebesgue measure instead of μ, we have a complete atomless countably separated perfect probability measure λ on $\mathbb{R}$ with the same negligible sets as Lebesgue measure. But now $(X,\hat\nu)$ and $(\mathbb{R},\lambda)$ are isomorphic (344I), so that $(X,\mathcal{N}(\mu))$ and $(\mathbb{R},\mathcal{N})$ are isomorphic. **Q**

(ii) The most important examples of spaces satisfying the conditions of (i) are Lebesgue measure on the unit interval and the usual measure on $\{0,1\}^{\mathbb{N}}$. But the ideas go much farther. On a Hausdorff space with a countable network (e.g., any separable metrizable space, or any analytic Hausdorff space), any topological measure is countably separated (433B). So any non-zero atomless Radon measure on such a space will have a null ideal isomorphic to $\mathcal{N}$. (The measure will be σ-finite because it is a locally finite measure on a Lindelöf space, and perfect by 416Wa.)

(iii) As we shall see in §523, there are many more measure spaces (X,μ) for which $\mathcal{N}(\mu)$ is close enough to $\mathcal{N}$ to have the same additivity and cofinality, and even uniformity and covering number match in a number of interesting cases.

(b)(i) Similarly, the structure $(\mathbb{R},\mathcal{M})$ is duplicated in any non-empty Polish space X without isolated points, in the sense that $(X,\mathcal{B}(X),\mathcal{M}(X)) \cong (\mathbb{R},\mathcal{B},\mathcal{M})$, where $\mathcal{B}$ and $\mathcal{B}(X)$ are the Borel σ-algebras of $\mathbb{R}$ and X respectively, and $\mathcal{M}(X)$ is the ideal of meager subsets of X. **P** Note first that $\mathbb{N}^{\mathbb{N}}$, with its usual topology, has an uncountable nowhere dense closed set; e.g., $\{f : f(2n) = 0 \text{ for every } n\}$. Now we know that X has a dense G_δ set X_1 homeomorphic to $\mathbb{N}^{\mathbb{N}}$ (5A4Le), and X_1 must also have an uncountable nowhere dense closed set F_1; since $X_1 \setminus F_1$ is again a non-empty Polish space without isolated points (4A2Qd), it too has a dense G_δ set X_2 homeomorphic to $\mathbb{N}^{\mathbb{N}}$, and X_2 is a dense G_δ set in X with uncountable complement. Similarly, $\mathbb{R}$ has a dense G_δ subset H which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and has uncountable complement.

Let $\mathcal{M}(X_2)$, $\mathcal{M}(H)$ be the ideals of meager subsets of X_2 and H when they are given their subspace topologies. Because X_2 is dense, a closed subset of X is nowhere dense in X iff its intersection with X_2 is nowhere dense in X_2; accordingly $\mathcal{M}(X_2)$ is precisely $\{M \cap X_2 : M \in \mathcal{M}(X)\}$. Similarly, $\mathcal{M}(H) = \{M \cap H : M \in \mathcal{M}\}$.

Consider the complements $X \setminus X_2$, $\mathbb{R} \setminus H$. These are uncountable Borel subsets of Polish spaces. They are therefore Borel isomorphic (424G, 424Cb); let $\phi : X \setminus X_2 \to \mathbb{R} \setminus H$ be a Borel isomorphism. Next, X_2 and H are homeomorphic to $\mathbb{N}^{\mathbb{N}}$, therefore to each other; let $\psi : X_2 \to H$ be a homeomorphism. Finally, set $\theta = \psi \cup \phi$, so that $\theta : X \to \mathbb{R}$ is a Borel isomorphism. For $M \subseteq X$,

$$M \in \mathcal{M}(X) \iff M \cap X_2 \in \mathcal{M}(X)$$

(because $X \setminus X_2$ is meager)

$$\iff M \cap X_2 \in \mathcal{M}(X_2) \iff \psi[M \cap X_2] \in \mathcal{M}(H)$$
$$\iff \theta[M] \cap H \in \mathcal{M}(H) \iff \theta[M] \in \mathcal{M}.$$

So θ is an isomorphism between the structures $(X,\mathcal{B}(X),\mathcal{M}(X))$ and $(\mathbb{R},\mathcal{B},\mathcal{M})$. **Q**

(ii) Again, the most important special cases here are $X = [0,1]$, $X = \{0,1\}^{\mathbb{N}}$ and $X = \mathbb{N}^{\mathbb{N}}$.

522X Basic exercises >(a) Let $\mathcal{K}$ be the σ-ideal of subsets of $\mathbb{N}^{\mathbb{N}}$ generated by the compact sets. Show that $(\mathcal{K},\subseteq)$ is Tukey equivalent to the pre-ordered sets of 522C, so that $\operatorname{add}\mathcal{K} = \mathfrak{b}$ and $\operatorname{cf}\mathcal{K} = \mathfrak{d}$.

(b) (O.Kalenda) Set $(P,\sqsubseteq) = (\mathbb{N},\le) \times (\mathbb{N}^{\mathbb{N}},\preceq)$ where $\preceq$ is the partial ordering of 522C(ii). Show that $(\mathbb{N},\le) \preccurlyeq_{\mathrm{T}} (P,\sqsubseteq) \preccurlyeq_{\mathrm{T}} (\mathbb{N}^{\mathbb{N}},\le)$, $(P,\sqsubseteq) \not\preccurlyeq_{\mathrm{T}} (\mathbb{N},\le)$ and $(\mathbb{N}^{\mathbb{N}},\le) \not\preccurlyeq_{\mathrm{T}} (P,\sqsubseteq)$.

(c) Let (X,Σ,μ) be an atomless semi-finite measure space with $\mu X > 0$. Show that $\#(X) \ge \operatorname{non}\mathcal{N}$. (*Hint*: 343Cc.)

>(e) Show that there are just 23 assignments of values to the cardinals of Cichoń's diagram which are allowed by the results in 522D-522Q and have $\mathfrak{c} = \omega_2$.

(g) Show that if $\operatorname{cov}\mathcal{N} > \omega_1$ then every $\mathbf{\Delta}^1_2$ (= PCA-&-CPCA) set in a Polish space is universally measurable. (*Hint*: 423Tb[3], 521Xc.)

[3]Formerly 423Rb.

(h) Let Z be the Stone space of the measure algebra $\mathfrak{A}$ of Lebesgue measure. Show that the Novák number $n(Z)$ of Z and the Martin number $\mathfrak{m}(\mathfrak{A})$ of $\mathfrak{A}$ are both equal to $\operatorname{cov}\mathcal{N}$. (*Hint*: 341Q, 416V, 517K.)

522Y Further exercises (a) Show that if $\operatorname{add}\mathcal{N} = \operatorname{cf}\mathcal{N}$ then $(\mathbb{R}, \mathcal{M})$ and $(\mathbb{R}, \mathcal{N})$ are isomorphic, in the sense that there is a permutation $f : \mathbb{R} \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ is meager iff $f[A]$ is Lebesgue negligible.

(b) Show that if $\operatorname{cov}\mathcal{N} > \omega_1$ then $\operatorname{cov}\mathcal{N} \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$.

(c) Let P and Q be partially ordered sets such that Q has no greatest member, $\sim$ an equivalence relation on P, and $\pi : P \to Q$ a surjective function such that, for p_0, $p_1 \in P$, $\pi(p_0) \le \pi(p_1)$ iff there is a $p \sim p_0$ such that $p \le p_1$. Suppose that κ is a cardinal such that no $\sim$-equivalence class has cardinal greater than κ. Show that $\operatorname{add}(Q) \le \max(\mathrm{FN}(P), \kappa)$.

(d) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that whenever $A \subseteq \mathbb{R}$ is non-meager there is a set $B \in [A]^{\omega_1}$ such that every uncountable subset of B is non-meager[4].

(e) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \mathfrak{p}$ and that $\kappa \ge \mathfrak{m}_{\mathrm{countable}}$ is such that $\operatorname{cf}[\kappa]^{<\mathfrak{p}} \le \kappa$. Show that $\kappa \ge \mathfrak{c}$.

(f) (S.Geschke) Show that if $\mathrm{FN}^*(\mathcal{P}\mathbb{N}) \le \mathfrak{m}_{\mathrm{countable}}$ then $\operatorname{non}\mathcal{M} \le \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(g) Let $\mathcal{S}^{(0)}$ be the family described in the proof of 522Vf. For any sets A, B say that $A \subseteq^* B$ if $A \setminus B$ is finite, and define $\le^*$ as in 522C. Show that $(\mathcal{S}^{(0)}, \subseteq^*, \mathcal{S}^{(0)}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}^{(0)})$.

(h) Suppose that we have supported relations (A, R, B) and (A, S, A) such that $R \circ S \subseteq R$, that is, $(a, b) \in R$ whenever $(a, a') \in S$ and $(a', b) \in R$. Show that if $\omega \le \operatorname{cov}(A, R, B) < \infty$ then $\operatorname{cf}(\operatorname{cov}(A, R, B)) \ge \operatorname{add}(A, S, A)$.

(i) Let X be any topological space with countable π-weight and write $\mathcal{M}(X)$ for the family of meager subsets of X. Show that $\mathcal{M}(X) \preccurlyeq_{\mathrm{T}} \mathcal{M}$.

522Z Problem Is it the case that $(\mathbb{R}, \in, \mathcal{M}) \equiv_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \mathtt{finint}, \mathbb{N}^{\mathbb{N}})$? (See 522S and the proof of 522V.)

522 Notes and comments All the significant ideas of this section may be found in Bartoszyński & Judah 95, with a good deal more.

For many years it appeared that 'measure' and 'category' on the real line, or at least the structures $(\mathbb{R}, \mathcal{B}, \mathcal{N})$ and $(\mathbb{R}, \mathcal{B}, \mathcal{M})$ where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$, were in a symmetric duality. It was perfectly well understood that the algebras $\mathfrak{A} = \mathcal{B}/\mathcal{B} \cap \mathcal{N}$ and $\mathfrak{G} = \mathcal{B}/\mathcal{B} \cap \mathcal{M}$ – what in this book I call the 'Lebesgue measure algebra' and the 'category algebra of $\mathbb{R}$' – are very different, but their complexities seemed to be balanced, and such results as 522G encouraged us to suppose that anything provable in ZFC relating measure to category ought to respect the symmetry. It therefore came as a surprise to most of us when Bartoszyński and Raisonnier & Stern (independently, but both drawing inspiration from ideas of Shelah 84, themselves responding to a difficulty noted in Solovay 70) showed that $\operatorname{add}\mathcal{N} \le \operatorname{add}\mathcal{M}$ in all models of set theory. (It was already known that $\operatorname{add}\mathcal{N}$ could be strictly less than $\operatorname{add}\mathcal{M}$.)

The diagram in its present form emphasizes a new dual symmetry, corresponding to the duality of Galois-Tukey connections (512Ab). No doubt this also is only part of the true picture. It gives no hint, for instance, of a striking difference between $\operatorname{cov}\mathcal{M}$ and $\operatorname{cov}\mathcal{N}$. While $\operatorname{cov}\mathcal{M} = \mathfrak{m}_{\mathrm{countable}}$ must have uncountable cofinality (522Vf), $\operatorname{cov}\mathcal{N}$ can be ω_ω (Shelah 00). In 522H-522I and 522Sb-522Sc there are hints of a different symmetry which I have not been able to formalize convincingly (see 522Z).

I have hardly mentioned shrinking numbers here. This is because while $\operatorname{shr}\mathcal{M}$ and $\operatorname{shr}\mathcal{N}$ can be located in Cichoń's diagram (we have $\operatorname{non}\mathcal{M} \le \operatorname{shr}\mathcal{M} \le \operatorname{cf}\mathcal{M}$ and $\operatorname{non}\mathcal{N} \le \operatorname{shr}\mathcal{N} \le \operatorname{cf}\mathcal{N}$, by 511Jc), they are not known to be connected organically with the rest of the diagram. I will return to them in a more general context in 523M. I have also not said where the π-weight of Lebesgue measure (see 511Gb) fits in; this is in fact equal to $\operatorname{cf}\mathcal{N}$, as will appear in 524P.

[4] In the language of 554C, every non-meager subset of the real line includes a Lusin set.

In 522T I give two classic 'Martin's axiom' arguments. They are typical in that the structure of the proof is to establish that there is a suitable partially ordered set for which a 'generic' upwards-directed subset will provide an object to witness the truth of some assertion. 'Generic', in this context, means 'meeting sufficiently many cofinal sets'. If there were any more definite method of finding the object sought, we would use it; these constructions are always even more ethereal than those which depend on unscrupulous use of the axiom of choice. 'Really' they are names of propositions in a suitable forcing language, since (as a rule) we can lift Martin numbers above ω_1 only by entering a universe created by forcing. But in this chapter, at least, I will try to avoid such considerations, and use arguments which are expressible in the ordinary language of ZFC, even though their non-trivial applications depend on assumptions beyond ZFC.

Of the partially ordered sets $\mathcal{S}^\infty$ and P in the proof of 522T, the former comes readily to hand as soon as we cast the problem in terms of the supported relation $(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S})$; we need only realize that we can express members of $\mathcal{S}$ as limits of upwards-directed subsets of a subfamily in which there is some room to manoeuvre, so that we have enough cofinal sets. The latter is more interesting. It belongs to one of the standard types in that the partially ordered set is made up of pairs (σ, F) in which σ is the 'working part', from which the desired meager set

$$M = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in R, i\ge n} \sigma(i)$$

will be constructed, and F is a 'side condition', designed to ensure that the partial order of P interacts correctly with the problem. In such cases, there is generally a not-quite-trivial step to be made in proving that the ordering is transitive ((b-i) of the proof of 522T). Note that we have two classes of cofinal set to declare in (b-iii) of the proof here; the Q_{nV} are there to ensure that M is meager, and the Q'_H to ensure that it includes every member of $\mathcal{H}$. And a final element which must appear in every proof of this kind, is the check that the partial order found is of the correct type, σ-linked in (a) and σ-centered in (b).

In 522U I suggest that it is natural to try to locate any newly defined cardinal among those displayed in Cichoń's diagram. Of course there is no presumption that it will be possible to do this tidily, or that we can expect any final structure to be low-dimensional; the picture in 522T is already neater than we are entitled to expect, and the complications in 522U (and 522Yd-522Yf) are a warning that our luck may be running out. However, we can surprisingly often find relationships like the ones between $\mathrm{FN}(\mathcal{P}\mathbb{N})$, $\mathfrak{b}$, $\operatorname{shr}\mathcal{M}$ and $\mathfrak{m}_{\text{countable}}$ here, which is one of my reasons for using this approach. It is very remarkable that under fairly weak assumptions on cardinal arithmetic (the hypothesis '$\mathfrak{m}_{\text{countable}} < \omega_\omega$' in 522Ud is much stronger than is necessary, since in 'ordinary' models of set theory we have $\operatorname{cf}[\kappa]^{\le\omega} = \kappa$ whenever $\operatorname{cf}\kappa > \omega$ – see 5A6Bc and 5A6C), the axiom '$\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$' splits Cichoń's diagram neatly into two halves. For an explanation of why it was worth looking for such a split, see Fuchino Geschke & Soukup 01.

For the sake of exactness and simplicity, I have maintained rigorously the convention that $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meager and negligible sets in $\mathbb{R}$ with Lebesgue measure. But from the point of view of the diagram, they are 'really' representatives of classes of ideals defined on non-empty Polish spaces without isolated points, on the one hand, and on atomless countably separated σ-finite perfect measure spaces of non-zero measure on the other (522W). The most natural expression of the duality between the supported relations $(\mathbb{R}, \in, \mathcal{M})$ and $(\mathbb{R}, \in, \mathcal{N})$ (522G) depends, of course, on the fact that both structures are invariant under translation; but even this is duplicated in $\mathbb{R}^r$ and in infinite compact metrizable groups like $\{0,1\}^\mathbb{N}$.

At some stage I ought to mention a point concerning the language of this chapter. It is natural to think of such expressions as $\operatorname{add}\mathcal{N}$ as names for objects which exist in some ideal universe. Starting from such a position, the sentence 'it is possible that $\operatorname{add}\mathcal{N} < \operatorname{add}\mathcal{M}$' has to be interpreted as 'there is a possible mathematical universe in which $\operatorname{add}\mathcal{N} < \operatorname{add}\mathcal{M}$'. But this can make sense only if '$\operatorname{add}\mathcal{N}$' can refer to different objects in different universes, and has a meaning independent of any particular incarnation. I think that in fact we have to start again, and say that the expression $\operatorname{add}\mathcal{N}$ is not a name for an object, but an abbreviation of a definition. We can then speak of the interpretations of that definition in different worlds. In fact we have to go much farther back than the names for cardinals in this section. $\mathcal{P}\mathbb{N}$ and $\mathbb{R}$ also have to be considered primarily as definitions. The set $\mathbb{N}$ itself has a relatively privileged position; but even here it is perhaps safest to regard the symbol $\mathbb{N}$ as a name for a formula in the language of set theory rather than anything else. Fortunately, one can do mathematics without aiming at perfect consistency or logical purity, and I will make no attempt to disinfect my own language beyond what seems to be demanded by the ideas I am trying to express at each moment; but you should be aware that there are possibilities for confusion here, and that at some point you will need to find your own way of balancing among them. My own practice, when the path does not seem clear, is to re-read KUNEN 80.

Version of 24.8.24

523 The measure of $\{0,1\}^I$

In §522 I tried to give an account of current knowledge concerning the most important cardinals associated with Lebesgue measure. The next step is to investigate the usual measure ν_I on $\{0,1\}^I$ for an arbitrary set I. Here I discuss the cardinals associated with these measures. Obviously they depend only on $\#(I)$, and are trivial if I is finite. I start with the basic diagram relating the cardinal functions of ν_κ and ν_λ for different cardinals κ and λ (523B). I take the opportunity to mention some simple facts about the measures ν_I (523C-523D). Then I look at additivities (523E), covering numbers (523F-523G), uniformities (523H-523L), shrinking numbers (523M) and cofinalities (523N). I end with a description of these cardinals under the generalized continuum hypothesis (523P).

523A Notation For any set I, I will write ν_I for the usual measure on $\{0,1\}^I$ and $\mathcal{N}_I$ for its null ideal. Recall that $(\{0,1\}^\omega, \mathcal{N}_\omega)$ is isomorphic to $(\mathbb{R}, \mathcal{N})$, where $\mathcal{N}$ is the Lebesgue null ideal (522Wa).

523B The basic diagram Suppose that κ and λ are infinite cardinals, with $\kappa \le \lambda$. Then we have the following diagram for the additivity, covering number, uniformity, shrinking number and cofinality of the ideals $\mathcal{N}_\kappa$ and $\mathcal{N}_\lambda$:

$$\begin{array}{ccccccccc}
 & & \operatorname{cov}\mathcal{N}_\lambda & — & \operatorname{cov}\mathcal{N}_\kappa & — & \operatorname{cf}\mathcal{N}_\kappa & — & \operatorname{cf}\mathcal{N}_\lambda & — & \lambda^\omega \\
 & & | & & | & & | & & | & & \\
 & & | & & | & & \operatorname{shr}\mathcal{N}_\kappa & — & \operatorname{shr}\mathcal{N}_\lambda & & \\
 & & | & & | & & | & & | & & \\
\omega_1 & — & \operatorname{add}\mathcal{N}_\lambda & — & \operatorname{add}\mathcal{N}_\kappa & — & \operatorname{non}\mathcal{N}_\kappa & — & \operatorname{non}\mathcal{N}_\lambda & &
\end{array}$$

(As in 522B, the cardinals here increase from bottom left to top right.)

proof For the inequalities relating two cardinals associated with the same ideal, see 511Jc; all we need to know is that $\mathcal{N}_\kappa$ and $\mathcal{N}_\lambda$ are proper ideals containing singletons. For the inequalities relating the cardinal functions of the two different ideals, use 521H; ν_κ is the image of ν_λ under the map $x \mapsto x{\restriction}\kappa : \{0,1\}^\lambda \to \{0,1\}^\kappa$, by 254Oa. Of course $\omega_1 \le \operatorname{add}\mathcal{N}_\lambda$. I leave the final inequality $\operatorname{cf}\mathcal{N}_\lambda \le \lambda^\omega$ for the moment, since this will be part of Theorem 523N below.

523C In the next few paragraphs I will set out what is known about the cardinals here. It will be convenient to begin with two easy lemmas.

Lemma Let I be any set, and $\mathcal{J}$ a family of subsets of I such that every countable subset of I is included in some member of $\mathcal{J}$. Then a subset A of $\{0,1\}^I$ belongs to $\mathcal{N}_I$ iff there is some $J \in \mathcal{J}$ such that $\{x{\restriction}J : x \in A\} \in \mathcal{N}_J$.

proof For $J \subseteq I$, $x \in \{0,1\}^I$ set $\pi_J(x) = x{\restriction}J \in \{0,1\}^J$. Then ν_J is the image measure $\nu_I\pi_J^{-1}$ (254Oa again), so $A \in \mathcal{N}_I$ whenever there is some $J \in \mathcal{J}$ such that $\pi_J[A] \in \mathcal{N}_J$. On the other hand, if $A \in \mathcal{N}_I$, there is a countable set $K \subseteq I$ such that $\pi_K[A] \in \mathcal{N}_K$ (254Od). Now there is a $J \in \mathcal{J}$ such that $K \subseteq J$, so that $\pi_J^{-1}[\pi_J[A]] \subseteq \pi_K^{-1}[\pi_K[A]] \in \mathcal{N}_I$ and $\pi_J[A] \in \mathcal{N}_J$.

523D Because the measures ν_I are homogeneous in a strong sense, we have the following facts which are occasionally useful.

Proposition Let κ be an infinite cardinal, and T the domain of ν_κ. For $A \subseteq \{0,1\}^\kappa$ write T_A for the subspace σ-algebra on A.

(a) If $E\subseteq\{0,1\}^\kappa$ is measurable and not negligible, then $(E,\mathfrak{T}_E,\mathcal{N}_\kappa\cap\mathcal{P}E)$ is isomorphic to $(\{0,1\}^\kappa,\mathfrak{T},\mathcal{N}_\kappa)$.

(b) If $\mathcal{E}\subseteq\mathcal{N}_\kappa$ and $\#(\mathcal{E})<\operatorname{cov}\mathcal{N}_\kappa$, then $(\nu_\kappa)_*(\bigcup\mathcal{E})=0$.

(c) If $A\subseteq\{0,1\}^\kappa$ is non-negligible, then there is a set $B\subseteq\{0,1\}^\kappa$, of full outer measure, such that $(A,\mathfrak{T}_A,\mathcal{N}_\kappa\cap\mathcal{P}A)$ is isomorphic to $(B,\mathfrak{T}_B,\mathcal{N}_\kappa\cap\mathcal{P}B)$.

(d) There is a set $A\subseteq\{0,1\}^\kappa$ with cardinal $\operatorname{non}\mathcal{N}_\kappa$ which has full outer measure.

proof (a) In fact the subspace measure on E is isomorphic to a scalar multiple of ν_I (344L).

(b) ? Otherwise, let $F\subseteq\bigcup\mathcal{E}$ be a non-negligible measurable set; then $\{F\cap E:E\in\mathcal{E}\}$ witnesses that $\operatorname{cov}(F,\mathcal{N}_\kappa\cap\mathcal{P}F)<\operatorname{cov}\mathcal{N}_\kappa$, which contradicts (a). **X**

(c) Let E be a measurable envelope of A. By (a), there is a bijection $f:E\to\{0,1\}^\kappa$ which is an isomorphism of the structures $(E,\mathfrak{T}_E,\mathcal{N}_\kappa\cap\mathcal{P}E)$ and $(\{0,1\}^\kappa,\mathfrak{T},\mathcal{N}_\kappa)$. Set $B=f[A]$. Then $f\restriction A$ is an isomorphism of the structures $(A,\mathfrak{T}_A,\mathcal{N}_\kappa\cap\mathcal{P}A)$ and $(B,\mathfrak{T}_B,\mathcal{N}_\kappa\cap\mathcal{P}B)$. Moreover, since A meets every member of $\mathfrak{T}_E\setminus\mathcal{N}_\kappa$, B meets every member of $\mathfrak{T}\setminus\mathcal{N}_\kappa$, that is, B has full outer measure.

(d) Let $A_0\subseteq\{0,1\}^\kappa$ be a non-negligible set of cardinal $\operatorname{non}\mathcal{N}_\kappa$. By (c), there is a set A of full outer measure which is isomorphic to A_0 in the sense described there; in particular, $\#(A)=\operatorname{non}\mathcal{N}_\kappa$.

523E Additivities Because the function $\kappa\mapsto\operatorname{add}\mathcal{N}_\kappa$ is non-increasing, it must stabilize, that is, there is some first κ_a such that $\operatorname{add}\mathcal{N}_\kappa=\operatorname{add}\mathcal{N}_{\kappa_a}$ for every $\kappa\ge\kappa_a$. But in fact it stabilizes almost immediately. If κ is any uncountable cardinal, then $\operatorname{add}\mathcal{N}_\kappa=\operatorname{add}\nu_\kappa=\omega_1$, by 521Jb. Thus among the additivities $\operatorname{add}\mathcal{N}_\kappa$, only $\operatorname{add}\mathcal{N}_\omega=\operatorname{add}\mathcal{N}$, the additivity of Lebesgue measure, can have any surprises for us.

523F Covering numbers Still on the left-hand side of the diagram, we again have a non-increasing function $\kappa\mapsto\operatorname{cov}\mathcal{N}_\kappa$, and a critical value κ_c after which it is constant. We can locate this value to some extent through the following simple fact. If $\theta=\operatorname{cov}\mathcal{N}_{\kappa_c}=\min\{\operatorname{cov}\mathcal{N}_\kappa:\kappa\text{ is a cardinal}\}$, then $\operatorname{cov}\mathcal{N}_\theta=\theta$. **P** Let κ be such that $\operatorname{cov}\mathcal{N}_\kappa=\theta$. For $I\subseteq\kappa$, set $\pi_I(x)=x\restriction I$ for $x\in\{0,1\}^\kappa$. Let $\mathcal{E}\subseteq\mathcal{N}_\kappa$ be a cover of $\{0,1\}^\kappa$ of cardinality θ. For each $E\in\mathcal{E}$, let $J_E\subseteq\kappa$ be a countable set such that $\pi_{J_E}[E]\in\mathcal{N}_{J_E}$. Set $I=\bigcup_{E\in\mathcal{E}}J_E$, so that $\#(I)\le\theta$ and $\pi_I[E]\in\mathcal{N}_I$ for every $E\in\mathcal{E}$. Then $\{\pi_I[E]:E\in\mathcal{E}\}$ is a cover of $\{0,1\}^I$ by at most $\operatorname{cov}\mathcal{N}_\kappa$ sets, and $\operatorname{cov}\mathcal{N}_I\le\operatorname{cov}\mathcal{N}_\kappa$. Since $(\{0,1\}^I,\mathcal{N}_I)$ is isomorphic to $(\{0,1\}^{\#(I)},\mathcal{N}_{\#(I)})$, we also have

$$\operatorname{cov}\mathcal{N}_\theta\le\operatorname{cov}\mathcal{N}_{\#(I)}\le\operatorname{cov}\mathcal{N}_\kappa\le\operatorname{cov}\mathcal{N}_\theta,$$

and $\operatorname{cov}\mathcal{N}_\theta=\operatorname{cov}\mathcal{N}_\kappa=\theta$. **Q**

What this means is that

$$\omega\le\kappa_c\le\theta\le\operatorname{cov}\mathcal{N}_{\kappa_c}\le\operatorname{cov}\mathcal{N}_\omega=\operatorname{cov}\mathcal{N}\le\mathfrak{c}.$$

Another way of putting the same idea is to say that

$$\text{if }\theta<\lambda\text{ then }\operatorname{cov}\mathcal{N}_\lambda\le\operatorname{cov}\mathcal{N}_\theta=\theta<\lambda$$

so that

$$\text{if }\lambda\text{ is a cardinal such that }\operatorname{cov}\mathcal{N}_\lambda\ge\lambda\text{ then }\operatorname{cov}\mathcal{N}_\kappa\ge\lambda\text{ for every }\kappa.$$

523G When the additivity of Lebesgue measure is large we have a further constraint on covering numbers.

Proposition (Kraszewski 01) If κ is a cardinal and $\operatorname{cov}\mathcal{N}_\kappa<\operatorname{add}\mathcal{N}$, then $\operatorname{cov}\mathcal{N}_\kappa\le\operatorname{cf}[\kappa]^{\le\omega}$.

proof As $\{0,1\}^\kappa$ is covered by negligible sets, κ is infinite. Let $\mathcal{E}$ be a subset of $\mathcal{N}_\kappa$ with cardinal $\operatorname{cov}\mathcal{N}_\kappa$ and union $\{0,1\}^\kappa$, and $\mathcal{J}$ a cofinal subset of $[\kappa]^\omega$ with cardinal $\operatorname{cf}[\kappa]^{\le\omega}$. For $J\in\mathcal{J}$ and $x\in\{0,1\}^\kappa$ set $\pi_J(x)=x\restriction J$, so that $\pi_J:\{0,1\}^\kappa\to\{0,1\}^J$ is inverse-measure-preserving. For $J\in\mathcal{J}$ set $\mathcal{E}_J=\{E:E\in\mathcal{E},\ \pi_J[E]\in\mathcal{N}_J\}$, $H_J=\bigcup\mathcal{E}_J$. Since

$$\#(\mathcal{E}_J)\le\#(\mathcal{E})=\operatorname{cov}\mathcal{N}_\kappa<\operatorname{add}\mathcal{N}=\operatorname{add}\mathcal{N}_\omega=\operatorname{add}\mathcal{N}_J,$$

$F_J=\bigcup\{\pi_J[E]:E\in\mathcal{E}_J\}\in\mathcal{N}_J$ and $H_J\subseteq\pi_J^{-1}[F_J]\in\mathcal{N}_\kappa$. Since $\bigcup_{J\in\mathcal{J}}\mathcal{E}_J=\mathcal{E}$ (523C) covers $\{0,1\}^\kappa$, $\{H_J:J\in\mathcal{J}\}$ covers $\{0,1\}^\kappa$ and $\operatorname{cov}\mathcal{N}_\kappa\le\#(\mathcal{J})=\operatorname{cf}[\kappa]^{\le\omega}$.

523H Uniformities On the other side of the diagram we have non-decreasing functions. To get upper bounds for $\operatorname{non}\mathcal{N}_\kappa$ we have the following method.

Lemma (Kraszewski 01) Suppose that I is a set and F a family of functions with domain I such that for every countable $J\subseteq I$ there is an $f\in F$ such that $f\restriction J$ is injective. Then

$$\operatorname{non}\mathcal{N}_I\le\max(\#(F),\sup_{f\in F}\operatorname{non}\mathcal{N}_{f[I]}).$$

proof If I is finite the result is trivial. Otherwise, for each $f\in F$ take a non-negligible subset A_f of $\{0,1\}^{f[I]}$ with cardinal $\operatorname{non}\mathcal{N}_{f[I]}$. Set $A=\{yf:f\in F,\,y\in A_f\}\subseteq\{0,1\}^I$. **?** If $A\in\mathcal{N}_I$, there is a countable set $J\subseteq I$ such that $\{x\restriction J:x\in A\}\in\mathcal{N}_J$. Let $f\in F$ be such that $f\restriction J$ is injective. Then we have a function $\phi:\{0,1\}^{f[I]}\to\{0,1\}^J$ defined by saying that $\phi(z)=zf\restriction J$ for every $z\in\{0,1\}^{f[I]}$, and (because $f\restriction J$ is injective) ϕ is inverse-measure-preserving for $\nu_{f[I]}$ and ν_J, so $\phi[A_f]$ cannot be ν_J-negligible. But if $y\in A_f$ then $\phi(y)(\xi)=y(f(\xi))$ for every $\xi\in J$, so $\phi[A_f]\subseteq\{x\restriction J:x\in A\}$, which is supposed to be negligible. **X**

Thus A is not negligible, and

$$\operatorname{non}\mathcal{N}_I\le\#(A)\le\max(\omega,\#(F),\sup_{f\in F}\#(A_f))=\max(\#(F),\sup_{f\in F}\operatorname{non}\mathcal{N}_{f[I]})$$

because we are supposing that I is infinite, so there is some $f\in F$ such that $f[I]$ is infinite.

523I Theorem (a) For any cardinal κ,

(i) $\operatorname{non}\mathcal{N}_\kappa\le\max(\operatorname{non}\mathcal{N},\operatorname{cf}[\kappa]^{\le\omega})$,

(ii) $\operatorname{non}\mathcal{N}_{\kappa^+}\le\max(\kappa^+,\operatorname{non}\mathcal{N}_\kappa)$,

(iii) $\operatorname{non}\mathcal{N}_{2^\kappa}\le\max(\mathfrak{c},\operatorname{cf}[\kappa]^{\le\omega})$,

(iv) $\operatorname{non}\mathcal{N}_{2^{\kappa^+}}\le\max(\kappa^+,\operatorname{non}\mathcal{N}_{2^\kappa})$.

(b) If $\operatorname{cf}\kappa>\omega$, then $\operatorname{non}\mathcal{N}_{\kappa^+}\le\max(\operatorname{cf}\kappa,\sup_{\lambda<\kappa}\operatorname{non}\mathcal{N}_\lambda)$.

proof (a) If κ is finite, all these are trivial; so suppose otherwise.

(i) Let $\mathcal{J}\subseteq[\kappa]^{\le\omega}$ be a cofinal set with cardinal $\operatorname{cf}[\kappa]^{\le\omega}$, and for $J\in\mathcal{J}$ let f_J be the identity function on J. Applying 523H with $F=\{f_J:J\in\mathcal{J}\}$ we get

$$\begin{aligned}\operatorname{non}\mathcal{N}_\kappa&\le\max(\#(\mathcal{J}),\sup_{J\in\mathcal{J}}\operatorname{non}\mathcal{N}_J)\\&\le\max(\operatorname{non}\mathcal{N}_\omega,\operatorname{cf}[\kappa]^{\le\omega})=\max(\operatorname{non}\mathcal{N},\operatorname{cf}[\kappa]^{\le\omega}).\end{aligned}$$

(ii) For each $\xi<\kappa^+$ choose a function $f_\xi:\kappa^+\to\kappa$ which is injective on ξ, and set $F=\{f_\xi:\xi<\kappa^+\}$. By 523H,

$$\operatorname{non}\mathcal{N}_{\kappa^+}\le\max(\#(F),\sup_{\xi<\kappa^+}\operatorname{non}\mathcal{N}_{f_\xi[\kappa^+]})\le\max(\kappa^+,\operatorname{non}\mathcal{N}_\kappa).$$

(iii) Take $\mathcal{J}$ as in (i). This time, for $J\in\mathcal{J}$, define $f_J:\mathcal{P}\kappa\to\mathcal{P}J$ by setting $f_J(A)=A\cap J$ for every $A\subseteq\kappa$. Applying 523H with $F=\{f_J:J\in\mathcal{J}\}$ we get

$$\begin{aligned}\operatorname{non}\mathcal{N}_{2^\kappa}=\operatorname{non}\mathcal{N}_{\mathcal{P}\kappa}&\le\max(\#(\mathcal{J}),\sup_{J\in\mathcal{J}}\operatorname{non}\mathcal{N}_{\mathcal{P}J})\le\max(\operatorname{cf}[\kappa]^{\le\omega},\operatorname{non}\mathcal{N}_\mathfrak{c})\\&\le\max(\operatorname{cf}[\kappa]^{\le\omega},\operatorname{non}\mathcal{N},\operatorname{cf}[\mathfrak{c}]^{\le\omega})=\max(\operatorname{cf}[\kappa]^{\le\omega},\mathfrak{c})\end{aligned}$$

(5A1F(e-iii)).

(iv) Set $f_\xi(A)=A\cap\xi$ for $\xi<\kappa^+$ and $A\subseteq\kappa^+$. If $\mathcal{J}\subseteq\mathcal{P}\kappa^+$ is countable, there is a $\xi<\kappa^+$ such that $A\cap\xi\ne A'\cap\xi$ for all distinct A, $A'\in\mathcal{J}$, that is, $f_\xi\restriction\mathcal{J}$ is injective. So 523H tells us that

$$\begin{aligned}\operatorname{non}\mathcal{N}_{2^{\kappa^+}}=\operatorname{non}\mathcal{N}_{\mathcal{P}(\kappa^+)}&\le\max(\kappa^+,\sup_{\xi<\kappa^+}\operatorname{non}\mathcal{N}_{f_\xi[\kappa^+]})\\&\le\max(\kappa^+,\sup_{\xi<\kappa^+}\operatorname{non}\mathcal{N}_{\mathcal{P}\xi})=\max(\kappa^+,\operatorname{non}\mathcal{N}_{2^\kappa}).\end{aligned}$$

(b)(i) If $\kappa=\theta^+$ where θ is an infinite cardinal, $\operatorname{non}\mathcal{N}_{\kappa^+}\le\max(\kappa,\operatorname{non}\mathcal{N}_\theta)$. **P** Choose an injective function $h_\zeta:\zeta\to\kappa$ for each $\zeta<\kappa^+$. For $\xi<\kappa$ define $f_\xi:\kappa^+\to\kappa$ by saying that

$$f_\xi(\zeta)=\min(\kappa\setminus\{f_\xi(\eta):\eta<\zeta,\,h_\zeta(\eta)\le\xi\})$$

for $\zeta<\kappa^+$. If $J\subseteq\kappa^+$ is countable, then $\xi=\sup_{\eta,\zeta\in J,\eta<\zeta}h_\zeta(\eta)$ is less than κ, and $f_\xi(\eta)\ne f_\xi(\zeta)$ for all distinct η, $\zeta\in J$. Applying 523H with $F=\{f_\xi:\xi<\kappa\}$ and using (a-ii) above, we get

$$\operatorname{non}\mathcal{N}_{\kappa^+}\le\max(\kappa,\operatorname{non}\mathcal{N}_\kappa)\le\max(\kappa,\operatorname{non}\mathcal{N}_\theta)=\max(\operatorname{cf}\kappa,\sup_{\lambda<\kappa}\operatorname{non}\mathcal{N}_\lambda).\ \mathbf{Q}$$

(ii) Now suppose that κ is an uncountable limit cardinal with uncountable cofinality. Again choose an injective function $h_\zeta:\zeta\to\kappa$ for each $\zeta<\kappa^+$. This time, let $K\subseteq\kappa$ be a cofinal set with cardinal $\operatorname{cf}\kappa$ consisting of cardinals, and for $\lambda\in K$ define $f_\lambda:\kappa^+\to\lambda^+$ by the formula

$$f_\lambda(\zeta)=\min(\lambda^+\setminus\{f_\lambda(\eta):\eta<\zeta,\,h_\zeta(\eta)\le\lambda\})$$

for $\zeta<\kappa^+$. If $J\subseteq\kappa^+$ is countable, then there is a $\lambda\in K$ such that $\lambda\ge\sup_{\eta,\zeta\in J,\eta<\zeta}h_\zeta(\eta)$, and $f_\lambda(\eta)\ne f_\lambda(\zeta)$ for all distinct η, $\zeta\in J$. Applying 523H with $F=\{f_\lambda:\lambda\in K\}$, we get

$$\operatorname{non}\mathcal{N}_{\kappa^+}\le\max(\#(F),\sup_{f\in F}\operatorname{non}\mathcal{N}_{f[\kappa^+]})\le\max(\operatorname{cf}\kappa,\sup_{\lambda<\kappa}\operatorname{non}\mathcal{N}_\lambda).$$

523J Corollary (Kraszewski 01) (a) $\operatorname{non}\mathcal{N}_{\omega_2}=\operatorname{non}\mathcal{N}_{\omega_1}=\operatorname{non}\mathcal{N}$.
(b) For any $n\in\mathbb{N}$, $\operatorname{non}\mathcal{N}_{\omega_{n+1}}\le\max(\omega_n,\operatorname{non}\mathcal{N})$.
(c) $\operatorname{non}\mathcal{N}_{2^{\omega_1}}=\operatorname{non}\mathcal{N}_{\mathfrak c}$.
(d) If $n\in\mathbb{N}$ then $\operatorname{non}\mathcal{N}_{2^{\omega_n}}\le\max(\omega_n,\operatorname{non}\mathcal{N}_{\mathfrak c})$.

proof (a) We have

$$\operatorname{non}\mathcal{N}=\operatorname{non}\mathcal{N}_\omega\le\operatorname{non}\mathcal{N}_{\omega_1}\le\operatorname{non}\mathcal{N}_{\omega_2}$$

(523B)

$$\le\max(\operatorname{cf}\omega_1,\operatorname{non}\mathcal{N}_\omega)$$

(523Ib)

$$=\operatorname{non}\mathcal{N}.$$

(b) Induce on n, using 523Ib for the inductive step.

(c) By 523I(a-iii), $\operatorname{non}\mathcal{N}_{2^{\omega_1}}\le\max(\omega_1,\operatorname{non}\mathcal{N}_{\mathfrak c})$; since

$$\omega_1\le\operatorname{non}\mathcal{N}_{\mathfrak c}\le\operatorname{non}\mathcal{N}_{2^{\omega_1}},$$

we have the result.

(d) Induce on n, using 523I(a-iii) or 523I(a-iv) for the inductive step.

523K Corollary (Burke n05) For any sets I, K let $\Upsilon_\omega(I,K)$ be the least cardinal of any family F of functions from I to K such that for every countable $J\subseteq I$ there is an $f\in F$ which is injective on J. (If $\#(K)<\min(\omega,\#(I))$ take $\Upsilon_\omega(I,K)=\infty$.) Then
(a) $\operatorname{non}\mathcal{N}_I\le\max(\Upsilon_\omega(I,K),\operatorname{non}\mathcal{N}_K)$ for all sets I and K;
(b) if $\kappa\ge\mathfrak c$ is a cardinal, then $\operatorname{non}\mathcal{N}_\kappa=\max(\Upsilon_\omega(\kappa,\mathfrak c),\operatorname{non}\mathcal{N}_{\mathfrak c})$.

proof (a) This is just a slightly weaker version of 523H.

(b) The point is that $\Upsilon_\omega(\kappa,\mathfrak c)\le\operatorname{non}\mathcal{N}_\kappa$. **P** Let $A\subseteq\{0,1\}^{\kappa\times\omega}$ be a non-negligible set of cardinal $\operatorname{non}\mathcal{N}_{\kappa\times\omega}$. For $x\in\{0,1\}^{\kappa\times\omega}$ define $f_x:\kappa\to\{0,1\}^\omega$ by setting $f_x(\xi)=\langle x(\xi,n)\rangle_{n\in\mathbb{N}}$ for each $\xi<\kappa$. If ξ, $\eta<\kappa$ are distinct, then $\{x:f_x(\xi)=f_x(\eta)\}$ is negligible, so if $J\subseteq\kappa$ is countable then $\{x:f_x\restriction J\text{ is injective}\}$ is conegligible and meets A. Accordingly $\{f_x:x\in A\}$ witnesses that

$$\Upsilon_\omega(\kappa,\mathfrak c)=\Upsilon_\omega(\kappa,\{0,1\}^\omega)\le\#(A)=\operatorname{non}\mathcal{N}_{\kappa\times\omega}=\operatorname{non}\mathcal{N}_\kappa.\ \mathbf{Q}$$

Since we already know that

$$\operatorname{non}\mathcal{N}_{\mathfrak c}\le\operatorname{non}\mathcal{N}_\kappa\le\max(\Upsilon_\omega(\kappa,\mathfrak c),\operatorname{non}\mathcal{N}_{\mathfrak c}),$$

we have the result.

523L On the other side we can find lower bounds which give a notion of the rate of growth of the numbers $\operatorname{non}\mathcal{N}_\kappa$ as κ increases.

Proposition (a) If λ and κ are infinite cardinals with $\kappa > 2^\lambda$, then $\operatorname{non}\mathcal{N}_\kappa > \lambda$.

(b) If κ is a strong limit cardinal of countable cofinality then $\operatorname{non}\mathcal{N}_\kappa > \kappa$.

proof (a) Let $A \subseteq \{0,1\}^\kappa$ be any set with cardinal at most λ. For $\xi < \kappa$ set $B_\xi = \{x : x \in A,\, x(\xi) = 1\}$. Because $\kappa > 2^{\#(A)}$, there is some $B \subseteq A$ such that $I = \{\xi : B_\xi = B\}$ is infinite. But what this means is that if $\xi \in I$ then $x(\xi) = 1$ for every $\xi \in B$ and $x(\xi) = 0$ for every $x \in A \setminus B$, and $A \subseteq \{x : x$ is constant on $I\}$ is negligible. As A is arbitrary, $\operatorname{non}\mathcal{N}_\kappa > \lambda$.

(b) By (a), $\operatorname{non}\mathcal{N}_\kappa > \lambda$ for every $\lambda < \kappa$, so $\operatorname{non}\mathcal{N}_\kappa \geq \kappa$; but also $\operatorname{cf}(\operatorname{non}\mathcal{N}_\kappa) \geq \operatorname{add}\mathcal{N}_\kappa$ (513C(b-ii)), so $\operatorname{non}\mathcal{N}_\kappa$ has uncountable cofinality and must be greater than κ.

523M Shrinking numbers As with $\operatorname{non}\mathcal{N}_\bullet$, the functions $\kappa \mapsto \operatorname{shr}\mathcal{N}_\kappa$ and $\kappa \mapsto \operatorname{shr}\mathcal{N}_\kappa$ are non-decreasing, by 521Hb. Some of the ideas used in 523I can be adapted to this context, but the pattern as a whole is rather different.

Proposition (a)(i) For any non-zero cardinals κ and λ,

$$\operatorname{shr}\mathcal{N}_\kappa \leq \max(\operatorname{cov}_{\mathrm{Sh}}(\kappa,\lambda,\omega_1,2), \sup_{\theta<\lambda}\operatorname{shr}\mathcal{N}_\theta).$$

(ii) For any infinite cardinal κ, $\operatorname{shr}\mathcal{N}_\kappa \leq \max(\operatorname{shr}\mathcal{N}, \operatorname{cf}[\kappa]^{\leq\omega})$.

(iii) If $\operatorname{cf}\kappa > \omega$, then $\operatorname{shr}\mathcal{N}_\kappa \leq \max(\kappa, \sup_{\theta<\kappa}\operatorname{shr}\mathcal{N}_\theta)$.

(b) For any infinite cardinal κ,

(i) $\operatorname{shr}\mathcal{N}_\kappa \geq \kappa$;

(ii) $\operatorname{cf}(\operatorname{shr}\mathcal{N}_\kappa) > \omega$;

(iii) $\operatorname{cf}(\operatorname{shr}^+\mathcal{N}_\kappa) > \kappa$.

Remark For the definition of $\operatorname{cov}_{\mathrm{Sh}}$, see 5A2Da.

proof (a)(i) If $\operatorname{cov}_{\mathrm{Sh}}(\kappa,\lambda,\omega_1,2) = \infty$ or κ is finite this is trivial. Otherwise, $\lambda \geq \omega_1$. Take a non-negligible $A \subseteq \{0,1\}^\kappa$. Let $\mathcal{J} \subseteq [\kappa]^{<\lambda}$ be a set with cardinal $\operatorname{cov}_{\mathrm{Sh}}(\kappa,\lambda,\omega_1,2)$ such that for every $I \in [\kappa]^{<\omega_1}$ there is a $\mathcal{D} \in [\mathcal{J}]^{<2}$ such that $I \subseteq \bigcup\mathcal{D}$, that is, there is a $J \in \mathcal{J}$ such that $I \subseteq J$. For each $J \in \mathcal{J}$, $A_J = \{x{\restriction}J : x \in A\}$ is non-negligible; let $B_J \subseteq A_J$ be a non-negligible set with cardinal at most $\operatorname{shr}\mathcal{N}_J$. Let $B \subseteq A$ be a set with cardinal at most $\max(\omega, \#(\mathcal{J}), \sup_{J\in\mathcal{J}}\operatorname{shr}\mathcal{N}_J)$ such that $B_J \subseteq \{x{\restriction}J : x \in B\}$ for every $J \in \mathcal{J}$. If $I \subseteq \kappa$ is countable, there is a $J \in \mathcal{J}$ such that $I \subseteq J$, so $\{x{\restriction}I : x \in B\} \supseteq \{y{\restriction}I : y \in B_J\}$ is non-negligible; it follows that B is non-negligible, while $\#(B) \leq \max(\operatorname{cov}_{\mathrm{Sh}}(\kappa,\lambda,\omega_1,2), \sup_{\theta<\lambda}\operatorname{shr}\mathcal{N}_\theta)$.

(ii) Taking $\lambda = \omega_1$ in (i),

$$\operatorname{shr}\mathcal{N}_\kappa \leq \max(\operatorname{cov}_{\mathrm{Sh}}(\kappa,\omega_1,\omega_1,2), \operatorname{shr}\mathcal{N}_\omega) = \max(\operatorname{cf}[\kappa]^{\leq\omega}, \operatorname{shr}\mathcal{N}).$$

(iii) Take $\lambda = \kappa$ in (i); as $[\kappa]^{\leq\omega} = \bigcup_{\xi<\kappa}[\xi]^{\leq\omega}$,

$$\operatorname{shr}\mathcal{N}_\kappa \leq \max(\operatorname{cov}_{\mathrm{Sh}}(\kappa,\kappa,\omega_1,2), \sup_{\theta<\kappa}\operatorname{shr}\mathcal{N}_\theta) = \max(\kappa, \sup_{\theta<\kappa}\operatorname{shr}\mathcal{N}_\theta).$$

(b)(i) Induce on κ. If $\kappa = \omega$ the result is trivial. For the inductive step to κ^+, consider the set

$$A = \{x : x \in \{0,1\}^{\kappa^+},\ \exists\, \xi < \kappa^+,\ x(\eta) = 0 \text{ for every } \eta \geq \xi\}.$$

Then the only set which includes A and is determined by coordinates in a countable set is $\{0,1\}^{\kappa^+}$, so A has full outer measure. On the other hand, if $B \subseteq A$ and $\#(B) \leq \kappa$, then there is some $\zeta < \kappa^+$ such that $x(\xi) = 0$ for every $x \in B$ and every $\xi \geq \zeta$, so B is negligible. Thus A witnesses that $\operatorname{shr}\mathcal{N}_{\kappa^+} \geq \kappa^+$. Because $\kappa \mapsto \operatorname{shr}\mathcal{N}_\kappa$ is non-decreasing (523B), the inductive step to limit cardinals κ is trivial.

(ii) ? Now suppose, if possible, that $\operatorname{cf}(\operatorname{shr}\mathcal{N}_\kappa) = \omega$. Then there is a sequence $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ of cardinals less than $\operatorname{shr}\mathcal{N}_\kappa$ with supremum $\operatorname{shr}\mathcal{N}_\kappa$. For each $n \in \mathbb{N}$ set $I_n = \kappa \times \{n\}$, and let $A_n \subseteq \{0,1\}^{I_n}$ be a non-negligible set such that every non-negligible subset of A_n has more than λ_n members. By 523Dc, there is a set $B_n \subseteq \{0,1\}^{I_n}$ of full outer measure such that every non-negligible subset of B_n has more than λ_n members. Set

$$B = \{x : x \in \{0,1\}^{\kappa\times\mathbb{N}},\, x\restriction I_n \in B_n \text{ for every } n \in \mathbb{N}\}.$$

Then the natural isomorphism between $\{0,1\}^{\kappa\times\mathbb{N}}$ and $\prod_{n\in\mathbb{N}}\{0,1\}^{I_n}$ identifies B with $\prod_{n\in\mathbb{N}} B_n$, so B has full outer measure in $\{0,1\}^{\kappa\times\mathbb{N}}$ (254Lb). There must therefore be a set $C \subseteq B$, of non-zero measure, such that $\#(C) \le \operatorname{shr}\mathcal{N}_\kappa$. Express C as $\bigcup_{n\in\mathbb{N}} C_n$ where $\#(C_n) \le \lambda_n$ for every n. Then there is an $n \in \mathbb{N}$ such that C_n is not negligible, in which case $D_n = \{x\restriction I_n : x \in C_n\}$ is non-negligible. But $D_n \subseteq B_n$ and $\#(D_n) \le \lambda_n$, so this is impossible. **X**

(iii) The argument of (i) shows that if κ is a successor cardinal, then $\operatorname{shr}^+\mathcal{N}_\kappa > \kappa$. So we need consider only the case in which κ is a limit cardinal. **?** If $\operatorname{cf}(\operatorname{shr}^+\mathcal{N}_\kappa) \le \kappa$, then there is a family $\langle\lambda_\xi\rangle_{\xi<\kappa}$ of cardinals less than $\operatorname{shr}^+\mathcal{N}_\kappa$ with supremum $\operatorname{shr}^+\mathcal{N}_\kappa$. I use the same method as in (ii). For each $\xi < \kappa$ set $I_\xi = \kappa\times\{\xi\}$, and let $B_\xi \subseteq \{0,1\}^{I_\xi}$ be a set of full outer measure such that every non-negligible subset of B_ξ has at least λ_ξ members. Set

$$B = \{x : x \in \{0,1\}^{\kappa\times\kappa},\, x\restriction I_\xi \in B_\xi \text{ for every } \xi < \kappa.$$

Then B has full outer measure in $\{0,1\}^{\kappa\times\kappa}$. There must therefore be a set $C \subseteq B$, of non-zero measure, such that $\#(C) < \operatorname{shr}^+\mathcal{N}_\kappa$. Let $\xi < \kappa$ be such that $\#(C) < \lambda_\xi$. Then $D = \{x\restriction I_\xi : x \in C\}$ is non-negligible. But $D \subseteq B_\xi$ and $\#(D_\xi) < \lambda_\xi$, so this is impossible. **X**

523N Cofinalities For the cardinals $\operatorname{cf}\mathcal{N}_\kappa$ the pattern from 523I(a-i) and 523Mb continues, and indeed we have an exact formula.

Theorem For any infinite cardinal κ,

$$\kappa \le \operatorname{cf}\mathcal{N}_\kappa = \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega}) \le \kappa^\omega.$$

proof (a) $\operatorname{cf}\mathcal{N}_\kappa \le \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega})$. **P** Let $\mathcal{J}$ be a cofinal family in $[\kappa]^\omega$ with cardinal $\operatorname{cf}[\kappa]^{\le\omega}$. For each $J \in \mathcal{J}$, write $\pi_J(x) = x\restriction J$ for $x \in \{0,1\}^\kappa$. Let $\mathcal{E}_J$ be a cofinal subset of $\mathcal{N}_J$ with cardinal $\operatorname{cf}\mathcal{N}_J = \operatorname{cf}\mathcal{N}_\omega = \operatorname{cf}\mathcal{N}$. Consider $\mathcal{E} = \{\pi_J^{-1}[E] : J \in \mathcal{J},\, E \in \mathcal{E}_J\}$. By 523C, $\mathcal{E}$ is cofinal with $\mathcal{N}_\kappa$, so that

$$\operatorname{cf}\mathcal{N}_\kappa \le \#(\mathcal{E}) \le \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega}). \quad \mathbf{Q}$$

(b) We know that $\operatorname{cf}[\kappa]^{\le\omega} \le \operatorname{cf}\mathcal{N}_\kappa$ (521Jb) and that $\operatorname{cf}\mathcal{N} = \operatorname{cf}\mathcal{N}_\omega \le \operatorname{cf}\mathcal{N}_\kappa$ (523B). So $\operatorname{cf}\mathcal{N}_\kappa = \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega})$.

(c) For the inequalities, note that $\omega \le \operatorname{cf}\mathcal{N}$ and if κ is uncountable then (in the language of 512Ba)

$$\operatorname{cf}[\kappa]^{\le\omega} \ge \operatorname{cov}(\kappa, \in, [\kappa]^{\le\omega}) = \kappa.$$

On the other side, $\operatorname{cf}\mathcal{N} \le \mathfrak{c} \le \kappa^\omega$ and $\operatorname{cf}[\kappa]^{\le\omega} \le \#([\kappa]^{\le\omega}) \le \kappa^\omega$.

523O Cofinalities of the cardinals In 523Mb I have shown that $\operatorname{shr}\mathcal{N}_\kappa$ has uncountable cofinality for infinite κ, and rather more about $\operatorname{shr}^+\mathcal{N}_\kappa$. From 513Cb we have a little information concerning the cofinalities of $\operatorname{add}\mathcal{N}_\kappa$, $\operatorname{cov}\mathcal{N}_\kappa$, $\operatorname{non}\mathcal{N}_\kappa$ and $\operatorname{cf}\mathcal{N}_\kappa$; but except when $\kappa = \omega$ we learn only that $\operatorname{cf}\mathcal{N}_\kappa$ and $\operatorname{non}\mathcal{N}_\kappa$ have uncountable cofinality, and that if $\operatorname{cov}\mathcal{N}_\kappa = \operatorname{cf}\mathcal{N}_\kappa$ then their common cofinality is at least $\operatorname{non}\mathcal{N}_\kappa$. This last remark can apply only to 'small' κ, since $\operatorname{cf}\mathcal{N}_\kappa \ge \kappa$ (if κ is infinite) and $\operatorname{cov}\mathcal{N}_\kappa \le \operatorname{cov}\mathcal{N}$.

523P The generalized continuum hypothesis In this chapter I am trying to present arguments in forms which show their full strength and are not tied to particular axioms beyond those of ZFC. However it is perhaps worth mentioning that in one of the standard universes the pattern is particularly simple.

Proposition Suppose that the generalized continuum hypothesis is true. Then, for any infinite cardinal κ,

$$\operatorname{add}\mathcal{N}_\kappa = \operatorname{add}\nu_\kappa = \operatorname{cov}\mathcal{N}_\kappa = \omega_1;$$

$$\begin{aligned}\operatorname{non}\mathcal{N}_\kappa &= \lambda \text{ if } \kappa = \lambda^+ \text{ where } \operatorname{cf}\lambda > \omega,\\ &= \kappa^+ \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa \text{ otherwise;}\end{aligned}$$

$$\begin{aligned}\operatorname{shr}\mathcal{N}_\kappa = \operatorname{cf}\mathcal{N}_\kappa &= \kappa^+ \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa \text{ otherwise;}\end{aligned}$$

$$\begin{aligned}\operatorname{shr}^+\mathcal{N}_\kappa = (\operatorname{shr}\mathcal{N}_\kappa)^+ &= \kappa^{++} \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

proof Since

$$\omega_1 \le \operatorname{add}\mathcal{N}_\kappa = \operatorname{add}\nu_\kappa \le \operatorname{cov}\mathcal{N}_\kappa \le \operatorname{cov}\mathcal{N} \le \mathfrak{c} = \omega_1,$$

the additivity and covering number are always ω_1.

If $\kappa = \lambda^+$ where $\operatorname{cf}\lambda > \omega$, then $\kappa > 2^\theta$ for every $\theta < \lambda$, so we have

$$\lambda \le \operatorname{non}\mathcal{N}_\kappa \le \max(\mathfrak{c}, \operatorname{cf}[\lambda]^{\le\omega}) = \lambda$$

(523La, 523I(a-iii), 5A6Ab). If $\kappa = \lambda^+$ where $\operatorname{cf}\lambda = \omega$, then

$$\lambda \le \operatorname{non}\mathcal{N}_\kappa \le \lambda^\omega \le 2^\lambda = \kappa;$$

but as $\operatorname{non}\mathcal{N}_\kappa$ has uncountable cofinality (513C(b-ii) again), $\operatorname{non}\mathcal{N}_\kappa$ must be κ. If κ is a limit cardinal, then $\kappa > 2^\theta$ for every $\theta < \kappa$, so

$$\kappa \le \operatorname{non}\mathcal{N}_\kappa \le \max(\omega_1, \operatorname{cf}[\kappa]^{\le\omega})$$

by 523I(a-i); if $\operatorname{cf}\kappa > \omega$ this is already enough to show that $\operatorname{non}\mathcal{N}_\kappa = \kappa$; if $\operatorname{cf}\kappa = \omega$ then $\operatorname{non}\mathcal{N}_\kappa$ cannot be κ so must be $\kappa^+ = \kappa^\omega$.

As for $\operatorname{shr}\mathcal{N}_\kappa$, if $\operatorname{cf}\kappa = \omega$, then

$$\kappa^+ \le \operatorname{shr}\mathcal{N}_\kappa \le \operatorname{cf}\mathcal{N}_\kappa = \max(\omega_1, \operatorname{cf}[\kappa]^{\le\omega}) \le 2^\kappa = \kappa^+$$

by 523M(b-ii) and 523N. If $\operatorname{cf}\kappa > \omega$ then

$$\kappa \le \operatorname{shr}\mathcal{N}_\kappa \le \operatorname{cf}\mathcal{N}_\kappa \le \kappa$$

by 523M(b-i), 523N and 5A6Ab. This deals with $\operatorname{shr}\mathcal{N}_\kappa$ and $\operatorname{cf}\mathcal{N}_\kappa$. For the augmented shrinking numbers, we know that if $\operatorname{cf}\kappa = \omega$ then $\operatorname{shr}\mathcal{N}_\kappa = \kappa^+$ is a successor cardinal so $\operatorname{shr}^+\mathcal{N}_\kappa = (\operatorname{shr}\mathcal{N}_\kappa)^+ = \kappa^{++}$, while if $\operatorname{cf}\kappa > \omega$ then

$$\begin{aligned}\operatorname{shr}\mathcal{N}_\kappa = \kappa &< \operatorname{shr}^+\mathcal{N}_\kappa\\ \text{(523M(b-iii))}\qquad &\le (\operatorname{shr}\mathcal{N}_\kappa)^+\end{aligned}$$

and $\operatorname{shr}^+\mathcal{N}_\kappa = (\operatorname{shr}\mathcal{N}_\kappa)^+ = \kappa^+$.

523X Basic exercises (a) Show that

$$(\mathcal{N}_\kappa, \not\ni, \{0,1\}^\kappa) \preccurlyeq_{\mathrm{GT}} ([\kappa]^{\le\omega}, \subseteq, [\kappa]^{\le\omega}) \ltimes (\mathcal{N}, \not\ni, \mathbb{R})$$

for every infinite cardinal κ. (See 512I for the definition of $\ltimes$.) Use this to prove 523I(a-i).

(b) Let κ be an infinite cardinal, and $\mathcal{J}$ a family of subsets of κ such that every countable subset of κ is included in some member of $\mathcal{J}$. Show that $\operatorname{non}\mathcal{N}_\kappa \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}}\operatorname{non}\mathcal{N}_J)$, $\operatorname{non}\mathcal{N}_{\mathcal{P}\kappa} \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}}\operatorname{non}\mathcal{N}_{\mathcal{P}J})$, $\operatorname{shr}\mathcal{N}_\kappa \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}}\operatorname{shr}\mathcal{N}_J)$ and $\operatorname{cf}\mathcal{N}_\kappa \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}}\operatorname{cf}\mathcal{N}_J)$.

(c) Show that

$$(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa) \preccurlyeq_{\mathrm{GT}} ([\kappa]^{\le\omega}, \subseteq, [\kappa]^{\le\omega}) \ltimes (\mathcal{N}, \subseteq, \mathcal{N})$$

for every infinite cardinal κ. Use this to prove 523N.

(d) Let (X,Σ,μ) be any probability space, and for each set I write $\mathcal{N}(\mu^I)$ for the null ideal of the product measure on X^I. Show that all the results of 523E-523I and 523L-523N are valid with $\mathcal{N}(\mu^I)$ in place of $\mathcal{N}_I$ and $\mathcal{N}(\mu^\omega)$ in place of $\mathcal{N}$, except that

—— in 523E the additivities may stabilize at ∞ rather than ω_1;

—— in 523F we can no longer be sure that $\operatorname{cov}\mathcal{N}(\mu^\omega)\le\mathfrak{c}$;

—— in 523I(a-iii) we need to write '$\operatorname{non}\mathcal{N}(\mu^{2^\kappa})\le\max(\mathfrak{c},\operatorname{non}\mathcal{N}(\mu^\omega),\operatorname{cf}[\kappa]^{\le\omega})$';

—— in 523L and 523Mb we have to assume that the measure algebra of μ is not $\{0,1\}$, so that the product measure $\mu^{\mathbb{N}}$ is atomless;

—— in 523N we can no longer be sure that $\operatorname{cf}\mathcal{N}(\mu^\omega)\le\kappa^\omega$.

523Y Further exercises (a) Set $\mathfrak{A}=\mathcal{P}\mathbb{R}/\mathcal{N}$. Show that $\mathfrak{c}\le c(\mathfrak{A})\le\pi(\mathfrak{A})\le 2^{\operatorname{shr}\mathcal{N}}$.

(b) Let κ be an infinite cardinal. Show that there is a family $\mathcal{J}\subseteq[\kappa]^{\le\omega}$ such that $\#(\mathcal{J})\le\operatorname{shr}\mathcal{N}_\kappa$ and every infinite subset of κ meets some member of $\mathcal{J}$ in an infinite set.

(c) Suppose that $\kappa\ge\omega$ and that $[\kappa]^{\le\omega}$ has bursting number at most $\operatorname{add}\mathcal{N}$. Show that $\mathcal{N}_\kappa\equiv_{\mathrm{T}}[\kappa]^{\le\omega}\times\mathcal{N}$.

(d) Show that

$$(\omega_1,\le,\omega_1)\ltimes(\omega_1,\le,\omega_1)\not\preccurlyeq_{\mathrm{GT}}(\omega_1,\le,\omega_1)\times(\omega_1,\le,\omega_1).$$

(e) For infinite cardinals κ, write $\mathcal{M}_\kappa$ for the ideal of meager subsets of $\{0,1\}^\kappa$. Show that under the same conventions as in 522B and 523B we have the diagrams

$$\begin{array}{ccccccccc}
 & & \operatorname{cov}\mathcal{M}_\lambda & — & \operatorname{cov}\mathcal{M}_\kappa & — & \operatorname{cf}\mathcal{M}_\kappa & — & \operatorname{cf}\mathcal{M}_\lambda & — & \lambda^\omega\\
 & & | & & | & & | & & | & & \\
 & & | & & | & & \operatorname{shr}\mathcal{M}_\kappa & — & \operatorname{shr}\mathcal{M}_\lambda & & \\
 & & | & & | & & | & & | & & \\
\omega_1 & — & \operatorname{add}\mathcal{M}_\lambda & — & \operatorname{add}\mathcal{M}_\kappa & — & \operatorname{non}\mathcal{M}_\kappa & — & \operatorname{non}\mathcal{M}_\lambda & &
\end{array}$$

and

$$\begin{array}{ccccccccccc}
 & & \operatorname{cov}\mathcal{N}_\kappa & — & \operatorname{non}\mathcal{M}_\kappa & — & \operatorname{cf}\mathcal{M}_\kappa & — & \operatorname{cf}\mathcal{N}_\kappa & — & \kappa^\omega\\
 & & | & & | & & | & & | & & \\
\omega_1 & — & \operatorname{add}\mathcal{N}_\kappa & — & \operatorname{add}\mathcal{M}_\kappa & — & \operatorname{cov}\mathcal{M}_\kappa & — & \operatorname{non}\mathcal{N}_\kappa & &
\end{array}$$

whenever $\omega\le\kappa\le\lambda$. Show moreover that all the results of 523E-523P have parallel forms referring to $\mathcal{M}_\kappa$.

(f) In the language of 523Ye, show that $\mathfrak{m}_{\mathrm{pc}\omega_1}\le\operatorname{cov}\mathcal{M}_\kappa$ for every infinite κ.

(g) Show that Ostaszewski's $\clubsuit$ (4A1M) implies that $\operatorname{cov}\mathcal{N}_{\omega_1}=\operatorname{cov}\mathcal{M}_{\omega_1}=\omega_1$.

523Z Problem Is there a proof in ZFC that $\operatorname{shr}\mathcal{N}_\kappa\ge\operatorname{cf}[\kappa]^{\le\omega}$ for every cardinal κ?

523 Notes and comments The basic diagram 523B is natural and easy to establish. Of course it leaves a great deal of room, especially on the right-hand side, where we have the increasing functions $\operatorname{non}\mathcal{N}_\bullet$, $\operatorname{shr}\mathcal{N}_\bullet$ and $\operatorname{cf}\mathcal{N}_\bullet$, and rather weak constraints

$$\lambda<\operatorname{non}\mathcal{N}_\kappa\le\operatorname{shr}\mathcal{N}_\kappa\le\operatorname{cf}\mathcal{N}_\kappa\le\kappa^\omega \text{ whenever } 2^\lambda<\kappa$$

to control them. However the generalized continuum hypothesis is sufficient to determine exact values for all the cardinals considered here (523P).

The combinatorics of $\operatorname{cf}[\kappa]^{\le\omega}$ and almost-disjoint families of functions are extremely complex, and depend in surprising ways on special axioms; I think it possible that the results of 523I-523J can be usefully extended. However 523N at least reduces the measure-theoretic problem of determining $\operatorname{cf}\mathcal{N}_\kappa$ to a standard, if difficult, question in infinitary combinatorics. I do not know if there are corresponding results concerning $\operatorname{non}\mathcal{N}_\kappa$ and $\operatorname{shr}\mathcal{N}_\kappa$ (see 523Kb and 523Z).

All the ideas in this section up to and including 523P can be applied to ideals of meager sets (523Ye) and indeed to other classes of ideals satisfying the fundamental lemma 523C; see Kraszewski 01.

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524 Radon measures

It is a remarkable fact that for a Radon measure the principal cardinal functions are determined by its measure algebra (524J), so can in most cases be calculated in terms of the cardinals of the last section (524P-524Q). The proof of this seems to require a substantial excursion involving not only measure algebras but also the Banach lattices $\ell^1(\kappa)$ and/or the κ-localization relation (524D, 524E). The same machinery gives us formulae for the cardinal functions of measurable algebras (524M). The results of §518 can be translated directly to give partial information on the Freese-Nation numbers of measurable algebras (524O). For covering number and uniformity, we can see from 521L that strictly localizable compact measures follow Radon measures. I know of no such general results for any other class of measure, but there are some bounds for cardinal functions of countably compact and quasi-Radon measures, which I give in 524R-524T.

524A Notation If (X,Σ,μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ. For any cardinal κ, ν_κ will be the usual measure on $\{0,1\}^\kappa$, T_κ its domain and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. As in §§522-523, I will write $\mathcal{N}_\kappa$ for $\mathcal{N}(\nu_\kappa)$ and $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$, so that $(\mathbb{R},\mathcal{N})$ and $(\{0,1\}^\omega,\mathcal{N}_\omega)$ are isomorphic (522Wa). If $\mathfrak{A}$ is any Boolean algebra, I write $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$ and $\mathfrak{A}^-$ for $\mathfrak{A}\setminus\{1\}$. If (A,R,B) is a supported relation, R' is the relation $\{(a,I):a\in R^{-1}[I]\}$ (see 512F). For any cardinal κ, $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$ will be the κ-localization relation (522K).

524B Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite Radon measure space with Maharam type κ. Then $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathfrak{B}_\kappa^-$.

proof (a) Suppose, to begin with, that $\mu X=1$ and $\kappa\ge\omega$. Let $\mathfrak{A}$ be the measure algebra of the Radon product measure $\tilde\lambda$ on $Y=X^{\mathbb{N}}$. Then $\mathfrak{A}\cong\mathfrak{B}_\kappa$. **P** By 417E(b-i), $\mathfrak{A}$ is isomorphic to the measure algebra of the usual product measure λ on Y, which by 334E is isomorphic to $\mathfrak{B}_\kappa$. **Q**

For $E\in\mathcal{N}(\mu)$, let $\langle F_{Ei}\rangle_{i\in\mathbb{N}}$ be a sequence of closed subsets of X such that $E\cap F_{Ei}=\emptyset$ and $\mu F_{Ei}\ge 1-2^{-i-1}$ for every $n\in\mathbb{N}$. Then

$$\tilde\lambda(\prod_{i\in\mathbb{N}}F_{Ei})=\lambda(\prod_{i\in\mathbb{N}}F_{Ei})\ge\prod_{i\in\mathbb{N}}(1-2^{-i-1})>0;$$

set

$$\phi(E)=(Y\setminus\prod_{i\in\mathbb{N}}F_{Ei})^\bullet\in\mathfrak{A}^-.$$

For $b\in\mathfrak{A}^-$ let $K_b\subseteq Y$ be a non-empty compact self-supporting set such that $K_b^\bullet\cap b=0$. Set $\pi_i(y)=y(i)$ for $i\in\mathbb{N}$ and $y\in Y$. Then each $\pi_i[K_b]\subseteq X$ is compact and $K_b\subseteq\prod_{i\in\mathbb{N}}\pi_i^{-1}[\pi_i[K_b]]$, so $\prod_{i\in\mathbb{N}}\mu\pi_i[K_b]>0$ and $\sup_{i\in\mathbb{N}}\mu\pi_i[K_b]=1$; set

$$\psi(b)=X\setminus\bigcup_{i\in\mathbb{N}}\pi_i[K_b]\in\mathcal{N}(\mu).$$

If $E\in\mathcal{N}(\mu)$ and $b\in\mathfrak{A}^-$ and $\phi(E)\subseteq b$ and $j\in\mathbb{N}$, then

$$K_b\setminus\pi_j^{-1}[F_{Ej}]\subseteq K_b\setminus\prod_{i\in\mathbb{N}}F_{Ei}$$

is negligible. As K_b is self-supporting, $K_b\setminus\pi_j^{-1}[F_{Ej}]$ is empty and $\pi_j[K_b]\subseteq F_{Ej}$. But this means that $\pi_j[K_b]\cap E=\emptyset$ for every $j\in\mathbb{N}$, so that $E\subseteq\psi(b)$.

This shows that ϕ is a Tukey function, so that $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathfrak{A}^-\cong\mathfrak{B}_\kappa^-$.

(b) If κ is finite, $\mathcal{N}(\mu)$ has a greatest member and the constant function with value 0 is a Tukey function from $\mathcal{N}(\mu)$ to $\mathfrak{B}_\kappa^-$ and the result is trivial. If κ is infinite and $\mu X\ne 1$, then, because μ is σ-finite and not trivial, there is a function $f:X\to\,]0,\infty[$ such that $\int fd\mu=1$ (215B(ix)). Let ν be the corresponding indefinite-integral measure; then ν is a Radon probability measure (416Sa) with the same measurable sets and the same negligible sets as μ (234L), so $\Sigma/\mathcal{N}(\nu)=\Sigma/\mathcal{N}(\mu)$ has Maharam type κ. In this case, (a) tells us that $\mathcal{N}(\mu)=\mathcal{N}(\nu)\preccurlyeq_{\mathrm{T}}\mathfrak{B}_{\kappa}^{-}$.

524C Lemma Let P be a partially ordered set such that $p\vee q=\sup\{p,q\}$ is defined for all p, $q\in P$. Suppose that ρ is a metric on P such that P is complete (as a metric space) and $\vee:P\times P\to P$ is uniformly continuous with respect to ρ. Let $Q\subseteq P$ be an open set, and $\kappa\ge d(Q)$ a cardinal. Then $(Q,\le',[Q]^{<\omega})\preccurlyeq_{\mathrm{GT}}(\ell^1(\kappa),\le,\ell^1(\kappa))$. If Q is upwards-directed, then $Q\preccurlyeq_{\mathrm{T}}\ell^1(\kappa)$.

proof (a) If Q is finite, then we can set $\phi(q)=0$ for every $q\in Q$, $\psi(x)=Q$ for every $x\in\ell^1(\kappa)$ and (ϕ,ψ) will be a Galois-Tukey connection from $(Q,\le',[Q]^{<\omega})$ to $(\ell^1(\kappa),\le,\ell^1(\kappa))$. So let us suppose that Q and κ are infinite.

(b) Let $\langle q_\xi\rangle_{\xi<\kappa}$ run over a dense subset of Q. For each $q\in Q$ let $m(q)\in\mathbb{N}$ be such that $\{p:p\in P,\ \rho(p,q)\le 2^{-m(q)}\}\subseteq Q$. For each $n\in\mathbb{N}$, let $\delta_n>0$ be such that $\rho(\sup I,\sup J)\le 2^{-n}$ whenever $\emptyset\ne I\subseteq J\subseteq P$ and $\#(J)\le 2^n$ and $\max_{q\in J}\min_{p\in I}\rho(p,q)\le 2\delta_n$; such exists because $\langle p_i\rangle_{i<k}\mapsto\sup_{i<k}p_i:P^k\to P$ is uniformly continuous whenever $k>0$, and in particular when $k=2^n$. Reducing the δ_n if necessary, we may suppose that $\delta_{n+1}\le\delta_n\le 2^{-n}$ for every n.

(c) Define $\phi:Q\to\ell^1(\kappa)$ as follows. Given $p\in Q$, choose a sequence $\langle\xi(p,n)\rangle_{n\in\mathbb{N}}$ in κ such that $\rho(p,q_{\xi(p,n)})\le\delta_{n+1}$ for every n. Take $\phi(p)\in\ell^1(\kappa)$ such that

$$\phi(p)(m(p))\ge 1,\quad \phi(p)(\xi(p,n))\ge 2^{-n}\text{ for every }n\in\mathbb{N}$$

(regarding $m(p)$ as a finite ordinal).

(d) Define $\psi:\ell^1(\kappa)\to[Q]^{<\omega}$ as follows. Given $x\in\ell^1(\kappa)$, set $K_n(x)=\{q_\xi:\xi<\kappa,\ x(\xi)\ge 2^{-n}\}$ for $n\in\mathbb{N}$. Then

$$\textstyle\sum_{n=0}^{\infty}2^{-n}\#(K_n(x))\le\sum_{\xi<\kappa}\sum\{2^{-n}:x(\xi)\ge 2^{-n}\}\le 2\|x\|_1<\infty,$$

so there is a $k(x)\in\mathbb{N}$ such that $x(n)<1$ for $n\in\omega\setminus k(x)$ and also $\#(K_n(x))\le 2^n$ for $n\ge k(x)$. Set $\tilde K(x)=K_{k(x)}(x)$. For $s\in\tilde K(x)$ set

$$I(x,s,k(x))=\{s\},\quad I(x,s,n+1)=\{q:q\in K_{n+1}(x),\ \rho(q,I(x,s,n))\le 2\delta_{n+1}\}$$

for $n\ge k(x)$, writing $\rho(q,I)$ for $\inf_{q'\in I}\rho(q,q')$. Because $\langle K_n(x)\rangle_{n\in\mathbb{N}}$ is non-decreasing, so is $\langle I(x,s,n)\rangle_{n\ge k(x)}$. Set $r_{xsn}=\sup I(x,s,n)$ in P for $n\ge k(x)$; then $\rho(r_{x,s,n+1},r_{xsn})\le 2^{-n-1}$ for every $n\ge k(x)$, by the choice of δ_{n+1}, so $r_{xs}=\lim_{n\to\infty}r_{xsn}$ is defined in P. Set $\psi(x)=Q\cap\{r_{xs}:s\in\tilde K(x)\}$.

(e) Now (ϕ,ψ) is a Galois-Tukey connection from $(Q,\le',[Q]^{<\omega})$ to $(\ell^1(\kappa),\le,\ell^1(\kappa))$. **P** Suppose that $p\in Q$ and $x\in\ell^1(\kappa)$ are such that $\phi(p)\le x$. Then $q_{\xi(p,n)}\in K_n(x)$ for every n, so $s=q_{\xi(p,k(x))}\in\tilde K(x)$. Also $q_{\xi(p,n)}\in I(x,s,n)$ for every $n\ge k(x)$, because

$$\rho(q_{\xi(p,n+1)},q_{\xi(p,n)})\le\delta_{n+2}+\delta_{n+1}\le 2\delta_{n+1}$$

for every n. So $q_{\xi(p,n)}\le r_{xsn}$ for every $n\ge k(x)$. It follows that

$$p\vee r_{xs}=\lim_{n\to\infty}q_{\xi(p,n)}\vee r_{xsn}=\lim_{n\to\infty}r_{xsn}=r_{xs}$$

and $p\le r_{xs}$.

By the choice of $k(x)$, we also have $\phi(p)(n)<1$ for $n\ge k(x)$, so that $m(p)<k(x)$. We therefore have

$$\rho(r_{xs},p)\le\rho(q_{\xi(p,k(x))},p)+\sum_{n=k(x)}^{\infty}\rho(r_{x,s,n+1},r_{xsn})$$

(because $s=q_{\xi(p,k(x))}$ is the unique member of $I(x,s,k(x))$, so is equal to $r_{x,s,k(x)}$)

$$\le\delta_{k(x)+1}+\sum_{n=k(x)}^{\infty}2^{-n-1}\le 2^{-k(x)-1}+2^{-k(x)}\le 2^{-m(p)}$$

and $r_{xs} \in Q$. So $p \le r_{xs} \in \psi(x)$ and $p \le' \psi(x)$. As p and x are arbitrary, (ϕ, ψ) is a Galois-Tukey connection. **Q**

(f) So $(Q, \le', [Q]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa))$, as claimed.

Finally, if Q is upwards-directed, then add $Q \ge \omega$, so $(Q, \le, Q) \equiv_{\mathrm{GT}} (Q, \le', [Q]^{<\omega})$ (513Id) and $(Q, \le, Q) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa))$, that is, $Q \preccurlyeq_{\mathrm{T}} \ell^1(\kappa)$.

524D Proposition If κ is any cardinal,

$$(\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa)).$$

proof If κ is finite then $\mathfrak{B}_\kappa^-$ is finite and the result is trivial. Otherwise, if we give $\mathfrak{B}_\kappa$ its measure metric ρ (323Ad), then it is a complete metric space in which $\cup$ is uniformly continuous (323Gc, 323B) and $\mathfrak{B}_\kappa^- = \mathfrak{B}_\kappa \setminus \{1\}$ is an open set. Now the topological density of $\mathfrak{B}_\kappa^-$ and $\mathfrak{B}_\kappa$ is κ, by 521E; so 524C gives the result.

524E Proposition Let κ be an infinite cardinal. Then

$$(\ell^1(\kappa), \le', [\ell^1(\kappa)]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa).$$

proof (a) For each $i \in \mathbb{N}$, let $\langle z_{i\xi}\rangle_{\xi<\kappa}$ run over a norm-dense subset of $\{x : x \in \ell^1(\kappa)^+,\ \|x\|_1 \le 4^{-i}\}$. Now there is a function $\phi : \ell^1(\kappa) \to \kappa^{\mathbb{N}}$ such that

for every $x \in \ell^1(\kappa)$, $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $x \le k\sum_{i=n}^{\infty} z_{i,\phi(x)(i)}$.

P Given $x \in \ell^1(\kappa)$, choose $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle \xi_n\rangle_{n\in\mathbb{N}}$, $\langle k_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Take $k_0 \ge 1$ such that $\|x^+\|_1 \le k_0$; set $x_0 = k_0^{-1}x^+$ and take $\xi_0 < \kappa$ such that $\|x_0 - z_{0\xi_0}\|_1 < \frac{1}{4}$. Given that $x_n \in \ell^1(\kappa)^+$, $\xi_n < \kappa$ are such that $\|x_n - z_{n\xi_n}\|_1 < 4^{-n-1}$, let $k_{n+1} \ge 1$ be such that $\|x_{n+1}\|_1 \le 4^{-n-1}$ where $x_{n+1} = (x_n - z_{n\xi_n})^+ + k_{n+1}^{-1}x^+$, and take $\xi_{n+1} < \kappa$ such that $\|x_{n+1} - z_{n+1,\xi_{n+1}}\|_1 < 4^{-n-2}$; continue. At the end of the process, set $\phi(x) = \langle \xi_n\rangle_{n\in\mathbb{N}}$.

Now, for any $n \in \mathbb{N}$, we have $x \le x^+ \le k_n x_n$. But we also have, for any $m \ge n$, $x_{m+1} \ge x_m - z_{m\xi_m}$, so that $x_n \le x_m + \sum_{i=n}^{m-1} z_{i\xi_i}$ for every $m \ge n$. Since $\|x_m\|_1 \le 4^{-m}$ for every m, $\lim_{m\to\infty} x_m = 0$ and

$$x \le k_n x_n \le k_n\sum_{i=n}^{\infty} z_{i,\phi(x)(i)}.$$

As x and n are arbitrary, ϕ is a suitable function. **Q**

(b) Define $\psi_0 : \mathcal{S}_\kappa \to \ell^1(\kappa)$ by setting $\psi_0(S) = \sum_{(i,\xi)\in S} z_{i\xi}$; because

$$\sum_{(i,\xi)\in S} \|z_{i\xi}\|_1 \le \sum_{i=0}^{\infty} 4^{-i}\#(S[\{i\}]) \le \sum_{i=0}^{\infty} 2^{-i}$$

is finite, $\psi_0(S)$ is well defined in $\ell^1(\kappa)$ for every $S \in \mathcal{S}_\kappa$ (4A4Ie). Now define $\psi : \mathcal{S}_\kappa \to [\ell^1(\kappa)]^{\le\omega}$ by setting $\psi(S) = \{k\psi_0(S) : k \in \mathbb{N}\}$ for $S \in \mathcal{S}_\kappa$.

(c) If $x \in \ell^1$ and $S \in \mathcal{S}_\kappa$ are such that $\phi(x) \subseteq^* S$, then $x \le' \psi(S)$. **P** Let $n \in \mathbb{N}$ be such that $(i, \phi(x)(i)) \in S$ for every $i \ge n$. Then there is a $k \in \mathbb{N}$ such that

$$x \le k\sum_{i=n}^{\infty} z_{i,\phi(x)(i)} \le k\sum_{(i,\xi)\in S} z_{i\xi} = k\psi_0(S) \in \psi(S). \textbf{ Q}$$

(d) Thus (ϕ, ψ) is a Galois-Tukey connection and

$$(\ell^1(\kappa), \le', [\ell^1(\kappa)]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa).$$

524F Lemma Let (X, Σ, μ) be a countably compact measure space with Maharam type κ.

(a) If μ is a Maharam-type-homogeneous probability measure, there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\bigcup_{\xi\in A} E_\xi$ has full outer measure for every uncountable $A \subseteq \kappa$.

(b) If μ is σ-finite, there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\bigcup_{\xi\in A} E_\xi$ is non-negligible for every uncountable $A \subseteq \kappa$.

proof Let $\mathfrak{A}$ be the measure algebra of (X, Σ, μ).

(a) If κ is countable, we can take $E_\xi = \emptyset$ for every ξ. Otherwise, $\mathfrak{A}$ is τ-generated by a stochastically independent family $\langle a_\xi \rangle_{\xi<\kappa}$ of elements of measure $\frac{1}{2}$, and for every $G \in \Sigma$ there is a smallest countable set $I_G \subseteq \kappa$ such that $G^\bullet$ is in the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in I_G\}$ (254Rd or 325Mb). For each $\xi < \kappa$ choose $F_\xi \in \Sigma$ such that $F_\xi^\bullet = a_\xi$. Let $\mathcal{K}$ be a countably compact class such that μ is inner regular with respect to $\mathcal{K}$.

Let $\langle J_\xi \rangle_{\xi<\kappa}$ be a disjoint family of subsets of κ all with cardinal ω_1. For each $\xi < \kappa$ choose $\langle K_{\xi n} \rangle_{n\in\mathbb{N}}$, $\langle \alpha_{\xi n} \rangle_{n\in\mathbb{N}}$ inductively, as follows. $\alpha_{\xi 0} = \min J_\xi$. Given $\alpha_{\xi n}$ and $\langle K_{\xi i} \rangle_{i<n}$, let $K_{\xi n} \in \mathcal{K}$ be such that $K_{\xi n} \cap F_{\alpha_{\xi n}} = \emptyset$ and $\mu(K_{\xi n}) \ge \frac{1}{2}(1-3^{-n-2})$; now let $\alpha_{\xi,n+1}$ be a member of J_ξ not belonging to $I_{K_{\xi i}} \cup \{\alpha_{\xi i}\}$ for any $i \le n$. Continue. Set

$$E_\xi = \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} K_{\xi m} \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} (X \setminus F_{\alpha_{\xi m}}),$$

so that E_ξ is negligible, because all the $\alpha_{\xi m}$ are different, so that $\langle F_{\alpha_{\xi m}} \rangle_{m\in\mathbb{N}}$ is stochastically independent.

Now suppose that $A \subseteq \kappa$ is uncountable, and that $F \subseteq X$ is measurable and not negligible. Let $K \in \mathcal{K}$ be such that $K \subseteq F$ and $\mu K > 0$; let $\xi \in A$ be such that $I_K \cap J_\xi = \emptyset$; let $n \in \mathbb{N}$ be such that $\mu K \ge 3^{-n-1}$. Set $G_m = K \cap \bigcap_{n\le i<m} K_{\xi i}$ for $m \ge n$. Then $I_{G_m} \subseteq I_K \cup \bigcup_{i<m} I_{K_{\xi i}}$ does not contain $\alpha_{\xi m}$, for any m. This means that

$$\mu G_{m+1} = \mu(G_m \cap K_{\xi m}) \ge \mu(G_m \setminus F_{\alpha_{\xi m}}) - \frac{3^{-m-2}}{2} = \frac{1}{2}(\mu G_m - 3^{-m-2})$$

for every $m \ge n$, and an easy induction shows that $\mu G_m \ge 3^{-m-1}$ for every m. But this tells us that every G_m is non-empty; because $\mathcal{K}$ is a countably compact class, $K \cap E_\xi \supseteq \bigcap_{m\ge n} G_m$ is non-empty, and F meets E_ξ.

As F is arbitrary, $\bigcup_{\xi\in A} E_\xi$ has full outer measure.

(b) For the general case, because μ is σ-finite, there is a countable partition of unity $\langle a_i \rangle_{i\in I}$ in $\mathfrak{A}$ such that all the principal ideals $\mathfrak{A}_{a_i}$ are totally finite and Maharam-type-homogeneous (use 332A), and we can find a partition $\langle X_i \rangle_{i\in I}$ of X into measurable sets such that $X_i^\bullet = a_i$ for each i. Moreover, the subspace measure μ_{X_i} on X_i is countably compact (451Db). Writing κ_i for the Maharam type of $\mathfrak{A}_{a_i}$, there is a family $\langle E_{i\xi} \rangle_{\xi<\kappa_i}$ of negligible subsets of X_i such that $\{\xi : \xi < \kappa_i,\, E_{i\xi} \subseteq E\}$ is countable for every negligible set E. (Apply (a) to a scalar multiple of μ_{X_i}.) Now we know from 332S that $\kappa = \sup_{i\in I} \kappa_i = \#(\{(i,\xi) : i \in I,\, \xi < \kappa_i\})$. On the other hand, for any negligible set $E \subseteq X$, $\{(i,\xi) : i \in I,\, \xi < \kappa_i,\, E_{i\xi} \subseteq E\}$ is countable. So if we re-enumerate $\langle E_{i\xi} \rangle_{i\in I,\xi<\kappa_i}$ as $\langle E_\xi \rangle_{\xi<\kappa}$ we shall have an appropriate family.

524G Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Maharam-type-homogeneous Radon probability space with Maharam type $\kappa \ge \omega$. Then $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$.

proof (Compare 522M.)

(a) By 524F, there is a family $\langle E_\xi \rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\{\xi : E_\xi \subseteq E\}$ is countable for every $E \in \mathcal{N}(\mu)$. Next, because the measure algebra of μ is isomorphic to the measure algebra of the usual measure on $[0,1]^{\mathbb{N}\times\kappa}$, there is a stochastically independent family $\langle G_{i\xi} \rangle_{i\in\mathbb{N},\xi<\kappa}$ in Σ such that $\mu G_{i\xi} = 2^{-i}$ for every $i \in \mathbb{N}$ and $\xi < \kappa$. For $f \in \kappa^{\mathbb{N}}$ set

$$\phi(f) = \bigcup_{n\in\mathbb{N}} E_{f(n)} \cup \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} G_{m,f(m)} \in \mathcal{N}(\mu).$$

(b) Take $E \in \mathcal{N}(\mu)$ and set $I_E = \{\xi : E_\xi \subseteq E\}$, so that I_E is countable. Define $\pi_E : X \to \{0,1\}^{\mathbb{N}\times I_E}$ by setting $\pi_E(x)(i,\xi) = 1$ if $x \in G_{i\xi}$, 0 otherwise. Then there is a non-empty compact self-supporting set K_E such that $\pi_E{\restriction}K_E$ is continuous. **P** Then π_E is measurable, therefore almost continuous (418J), and there is a non-negligible measurable set $H \subseteq X \setminus E$ such that $\pi_E{\restriction}H$ is continuous. Because μ is inner regular with respect to the compact self-supporting sets, there is a non-negligible compact self-supporting $K_E \subseteq H$, and this has the required property. **Q**

$\pi_E[K_E]$ is compact. Let $\langle W_n(E) \rangle_{n\in\mathbb{N}}$ run over the family of open-and-closed subsets W of $\{0,1\}^{\mathbb{N}\times I_E}$ meeting $\pi_E[K_E]$. Then $\pi_E^{-1}[W_n(E)]$ is a non-empty relatively open subset of K_E for every n; because K_E is self-supporting, $\pi_E^{-1}[W_n(E)]$ is never negligible. Set

$$J(E,n,i) = \{\xi : \xi \in I_E,\, \pi_E^{-1}[W_n(E)] \cap G_{i\xi} = \emptyset\}$$

for n, $i \in \mathbb{N}$. Because $\langle G_{i\xi} \rangle_{i\in\mathbb{N},\xi\in I_E}$ is stochastically independent,

$$\sum_{i=0}^{\infty} 2^{-i}\#(J(E,n,i)) = \sum\{\mu G_{i\xi} : i \in \mathbb{N},\, \xi \in I_E,\, G_{i\xi} \cap \pi_E^{-1}[W_n(E)] = \emptyset\}$$

is finite, by the Borel-Cantelli lemma (273K). For each n, let $k(E,n) \in \mathbb{N}$ be such that $2^{-i}\#(J(E,n,i)) \le 2^{-n-1}$ for $i \ge k(E,n)$, and set

$$\psi(E) = \bigcup_{n\in\mathbb{N}}\{(i,\xi) : i \ge k(E,n),\, \xi \in J(E,n,i)\} \subseteq \mathbb{N}\times\kappa.$$

Then

$$\begin{aligned}\#(\{\xi : (i,\xi) \in \psi(E)\} &\le \sum_{n\in\mathbb{N},k(E,n)\le i} \#(J(E,n,i))\\ &\le \sum_{n\in\mathbb{N},k(E,n)\le i} 2^{-n-1}2^i \le 2^i\end{aligned}$$

for every $i \in \mathbb{N}$, and $\psi(E) \in \mathcal{S}_\kappa$.

(c) Now (ϕ,ψ) is a Galois-Tukey connection from $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$ to $(\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))$. **P** Suppose that $f \in \kappa^{\mathbb{N}}$ and $E \in \mathcal{N}(\mu)$ are such that $\phi(f) \subseteq E$. Because $E_{f(n)} \subseteq \phi(f)$, $f(n) \in I_E$ for every $n \in \mathbb{N}$. Next, K_E does not meet $\phi(f)$, so $K_E \cap \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} G_{m,f(m)}$ is empty, that is,

$$\pi_E[K_E] \cap \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{w : w \in \{0,1\}^{\mathbb{N}\times I_E},\, w(m,f(m)) = 1\} = \emptyset.$$

By Baire's theorem, there is some $m \in \mathbb{N}$ such that

$$\pi_E[K_E] \cap \bigcup_{i\ge m}\{w : w \in \{0,1\}^{\mathbb{N}\times I_E},\, w(i,f(i)) = 1\}$$

is not dense in $\pi_E[K_E]$, and there is an $n \in \mathbb{N}$ such that

$$W_n(E) \cap \bigcup_{i\ge m}\{w : w \in \{0,1\}^{\mathbb{N}\times I_E},\, w(i,f(i)) = 1\} = \emptyset.$$

In this case, $f(i) \in J(E,n,i)$ for every $i \ge m$. But this means that $(i,f(i)) \in \psi(E)$ for every $i \ge \max(m,k(E,n))$, so that $f \subseteq^* \psi(E)$. As f and E are arbitrary, (ϕ,ψ) is a Galois-Tukey connection. **Q**

(d) Thus ϕ and ψ witness that $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))$, as claimed.

524H Corollary Let κ be an infinite cardinal, and μ a Maharam-type-homogeneous Radon probability measure with Maharam type κ. Then $(\mathfrak{B}_\kappa^+,\supseteq',[\mathfrak{B}_\kappa^+]^{\le\omega})$, $(\ell^1(\kappa),\le',[\ell^1(\kappa)]^{\le\omega})$, $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$ and $(\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))$ are Galois-Tukey equivalent.

proof By 512Gb, 524D and 524B,

$$(\mathfrak{B}_\kappa^-,\subseteq',[\mathfrak{B}_\kappa^-]^{<\omega_1}) \preccurlyeq_{\mathrm{GT}} (\mathfrak{B}_\kappa^-,\subseteq',[\mathfrak{B}_\kappa^-]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa),\le,\ell^1(\kappa)),$$

$$(\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathfrak{B}_\kappa^-,\subseteq,\mathfrak{B}_\kappa^-).$$

So

$$\begin{aligned}(\mathfrak{B}_\kappa^+,\supseteq',[\mathfrak{B}_\kappa^+]^{\le\omega}) &\cong (\mathfrak{B}_\kappa^-,\subseteq',[\mathfrak{B}_\kappa^-]^{\le\omega}) = (\mathfrak{B}_\kappa^-,\subseteq',[\mathfrak{B}_\kappa^-]^{<\omega_1})\\ &\preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa),\le',[\ell^1(\kappa)]^{<\omega_1})\end{aligned}$$

(512Gd)

$$= (\ell^1(\kappa),\le',[\ell^1(\kappa)]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$$

(524E)

$$\preccurlyeq_{\mathrm{GT}} (\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))$$

(524G)

$$\equiv_{\mathrm{GT}} (\mathcal{N}(\mu),\subseteq',[\mathcal{N}(\mu)]^{\le\omega})$$

(513Id again)

$$\preccurlyeq_{\mathrm{GT}} (\mathfrak{B}_\kappa^-,\subseteq',[\mathfrak{B}_\kappa^-]^{\le\omega})$$

by 512Gb again.

524I Corollary Let μ be a Maharam-type-homogeneous Radon probability measure with infinite Maharam type κ. Then

$$\operatorname{add}\mathcal{N}(\mu)=\operatorname{add}\mathcal{N}_\kappa=\operatorname{add}_\omega\ell^1(\kappa),$$

$$\operatorname{cf}\mathcal{N}(\mu)=\operatorname{cf}\mathcal{N}_\kappa=\operatorname{cf}\ell^1(\kappa).$$

proof By 524H and 512Db,

$$\begin{aligned}\operatorname{add}(\ell^1(\kappa),\le',[\ell^1(\kappa)]^{\le\omega})&=\operatorname{add}(\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))\\&=\operatorname{add}(\mathcal{N}(\nu_\kappa),\subseteq,\mathcal{N}(\nu_\kappa))=\operatorname{add}(\mathcal{N}_\kappa,\subseteq,\mathcal{N}_\kappa).\end{aligned}$$

But $\operatorname{add}(\ell^1(\kappa),\le',[\ell^1(\kappa)]^{\le\omega})=\operatorname{add}_\omega\ell^1(\kappa)$ (513Ia), while $\operatorname{add}(\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))=\operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{add}(\mathcal{N}_\kappa,\subseteq,\mathcal{N}_\kappa)=\operatorname{add}\mathcal{N}_\kappa$ (512Ea). So

$$\operatorname{add}_\omega\ell^1(\kappa)=\operatorname{add}\mathcal{N}(\mu)=\operatorname{add}\mathcal{N}_\kappa.$$

On the other side, 512Da tells us that

$$\operatorname{cov}(\mathcal{N}_\kappa,\subseteq,\mathcal{N}_\kappa)=\operatorname{cov}(\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))=\operatorname{cov}(\ell^1(\kappa),\le',[\ell^1(\kappa)]^{\le\omega}).$$

But

$$\operatorname{cov}(\mathcal{N}_\kappa,\subseteq,\mathcal{N}_\kappa)=\operatorname{cf}\mathcal{N}_\kappa,\quad\operatorname{cov}(\mathcal{N}(\mu),\subseteq,\mathcal{N}(\mu))=\operatorname{cf}\mathcal{N}(\mu)$$

(512Ea). Next, $\operatorname{cf}\ell^1(\kappa)>\omega$. **P** If $\langle x_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\ell^1(\kappa)$, then (because κ is infinite) $F_n=\{x:x\le x_n\}$ is nowhere dense (for the norm topology) for any $n\in\mathbb{N}$, so $\langle F_n\rangle_{n\in\mathbb{N}}$ cannot cover $\ell^1(\kappa)$ (4A2Ma) and $\{x_n:n\in\mathbb{N}\}$ cannot be cofinal. **Q** So 512Gf tells us that

$$\operatorname{cov}(\ell^1(\kappa),\le',[\ell^1(\kappa)]^{\le\omega})=\operatorname{cov}(\ell^1(\kappa),\le,\ell^1(\kappa))=\operatorname{cf}\ell^1(\kappa).$$

Putting these together,

$$\operatorname{cf}\mathcal{N}(\mu)=\operatorname{cf}\mathcal{N}_\kappa=\operatorname{cf}\ell^1(\kappa)$$

as required.

524J Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces with non-zero measure and isomorphic measure algebras.

(a) $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ are Tukey equivalent, so $\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)=\operatorname{add}\mathcal{N}(\nu)=\operatorname{add}\nu$ and $\operatorname{cf}\mathcal{N}(\mu)=\operatorname{cf}\mathcal{N}(\nu)$.

(b) $(X,\in,\mathcal{N}(\mu))$ and $(Y,\in,\mathcal{N}(\nu))$ are Galois-Tukey equivalent, so $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}(\nu)$ and $\operatorname{non}\mathcal{N}(\mu)=\operatorname{non}\mathcal{N}(\nu)$.

proof (a) Let $\mathfrak{A}$, $\mathfrak{B}$ be the measure algebras of μ and ν. Let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}^+$ such that all the principal ideals $\mathfrak{A}_{a_i}$ are homogeneous and totally finite, and $\langle b_i\rangle_{i\in I}$ a matching family in $\mathfrak{B}$, so that $\mathfrak{A}_{a_i}\cong\mathfrak{B}_{b_i}$ for every i. Because (X,Σ,μ) and (Y,T,ν) are strictly localizable (416B), there are decompositions $\langle X_i\rangle_{i\in I}$ and $\langle Y_i\rangle_{i\in I}$ of X, Y respectively such that $X_i^\bullet=a_i$ and $Y_i^\bullet=b_i$ for every i (322M). Write μ_{X_i}, ν_{Y_i} for the corresponding subspace measures; of course these are Radon measures (416Rb). Then $\mathcal{N}(\mu_{X_i})$ and $\mathcal{N}(\nu_{Y_i})$ are Tukey equivalent for every i. **P** If the common Maharam type of $\mathfrak{A}_{a_i}$ and $\mathfrak{B}_{b_i}$ is infinite, this is a consequence of 524H. If $\mathfrak{A}_{a_i}=\{0,a_i\}$, then μ_{X_i} is purely atomic and there is a single point x of X_i such that $\mu\{x\}=\mu X_i$ (414G). In this case $\mathcal{N}(\mu_{X_i})$ has a greatest member $X_i\setminus\{x\}$, and similarly $\mathcal{N}(\nu_{\kappa_i})$ has a greatest member, so they have Tukey equivalent cofinal subsets and are Tukey equivalent (513E(d-ii)). **Q**

Now $E\mapsto\langle E\cap X_i\rangle_{i\in I}$ is a partially-ordered-set isomorphism between $\mathcal{N}(\mu)$ and $\prod_{i\in I}\mathcal{N}(\mu_{X_i})$. Similarly, $\mathcal{N}(\nu)$ is isomorphic to $\prod_{i\in I}\mathcal{N}(\nu_{Y_i})$. It now follows from 513Eg that $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ are Tukey equivalent. Accordingly $\operatorname{add}\mathcal{N}(\mu)=\operatorname{add}\mathcal{N}(\nu)$ and $\operatorname{cf}\mathcal{N}(\mu)=\operatorname{cf}\mathcal{N}(\nu)$. By 521Ad, $\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{add}\nu=\operatorname{add}\mathcal{N}(\nu)$.

(b) Immediate from 521La, applied in both directions.

524K Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces with measure algebras $\mathfrak{A}$, $\mathfrak{B}$ respectively. If $\mathfrak{A}$ can be regularly embedded in $\mathfrak{B}$, then $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\nu)$.

proof As usual, write $\bar\mu$ and $\bar\nu$ for the functionals on $\mathfrak{A}$, $\mathfrak{B}$ respectively defined from μ and ν, and let $\pi:\mathfrak{A}\to\mathfrak{B}$ be a regular embedding, that is, an order-continuous injective Boolean homomorphism.

(a) Consider first the case in which μ is totally finite and π is measure-preserving for $\bar\mu$ and $\bar\nu$. Let $(\tilde X,\tilde{\mathfrak{T}},\tilde\Sigma,\tilde\mu)$ and $(\tilde Y,\tilde{\mathfrak{S}},\tilde{\mathrm{T}},\tilde\nu)$ be the Stone spaces of $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ respectively. Then π corresponds to a continuous function $f:\tilde Y\to\tilde X$ (312Q). By 418I, the image measure $\tilde\nu f^{-1}$ is a Radon measure on $\tilde X$. If $a\in\mathfrak{A}$ and $\widehat a$ is the corresponding open-and-closed set in $\tilde X$, then

$$\tilde\nu f^{-1}[\widehat a]=\tilde\nu(\widehat{\pi a})=\bar\nu(\pi a)=\bar\mu a=\tilde\mu\widehat a.$$

By 415H(v), $\tilde\nu f^{-1}=\tilde\mu$. By 521Hb, $\mathcal{N}(\tilde\mu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\tilde\nu)$. But now 524Ja tells us that

$$\mathcal{N}(\mu)\equiv_{\mathrm{T}}\mathcal{N}(\tilde\mu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\tilde\nu)\equiv_{\mathrm{T}}\mathcal{N}(\nu).$$

(b) Next, consider the case in which μ and ν are totally finite but π is not necessarily measure-preserving. As it is (sequentially) order-continuous, we have a measure μ' on X defined by saying that $\mu'E=\bar\nu(\pi E^\bullet)$ for $E\in\Sigma$, and $\mathcal{N}(\mu')=\mathcal{N}(\mu)$. Because μ' is absolutely continuous with respect to μ, it is an indefinite-integral measure over μ (234O) and is a Radon measure on X (416Sa again). Taking $\bar\mu'$ to be the corresponding functional on $\mathfrak{A}$, $(\mathfrak{A},\bar\mu')$ is the measure algebra of μ' and π is measure-preserving for $\bar\mu'$ and $\bar\nu$. So (a) tells us that

$$\mathcal{N}(\mu)=\mathcal{N}(\mu')\preccurlyeq_{\mathrm{T}}\mathcal{N}(\nu).$$

(c) Thirdly, suppose that μ is totally finite, but ν might not be. Set $\mathfrak{B}^f=\{b:b\in\mathfrak{B},\ \bar\nu b<\infty\}$. For $b\in\mathfrak{B}^f$, set $c_b=\sup\{a:a\in\mathfrak{A},\ b\cap\pi a=0\}$; then $b\cap\pi c_b=0$, because π is order-continuous. If $a\in\mathfrak{A}\setminus\{0\}$, there is a $b\in\mathfrak{B}^f$ such that $b\cap\pi a\ne0$, so that $a\not\subseteq c_b$. Accordingly $\sup_{b\in\mathfrak{B}^f}1\setminus c_b=1$ in $\mathfrak{A}$; as $\mathfrak{A}$ is ccc, there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}^f$ such that $\sup_{n\in\mathbb{N}}1\setminus c_{b_n}=1$, that is, $\bar\nu(a\cap\sup_{n\in\mathbb{N}}b_n)>0$ for every non-zero $a\in\mathfrak{A}$.

For each $n\in\mathbb{N}$, choose $F_n\in\mathrm{T}$ such that $F_n^\bullet=b_n$ in $\mathfrak{B}$, and set $Y'=\bigcup_{n\in\mathbb{N}}F_n$. The subspace measure $\nu_{Y'}$ is σ-finite, so there is a totally finite measure ν' on Y', an indefinite-integral measure over $\nu_{Y'}$, with the same null ideal as $\nu_{Y'}$ (use 215B(ix)). The measures $\nu_{Y'}$ and ν' are both Radon measures (416Rb, 416Sa). Setting $b=\sup_{n\in\mathbb{N}}b_n$ in $\mathfrak{B}$, the principal ideal $\mathfrak{B}_b$ can be identified with the measure algebra of $\nu_{Y'}$ (322I) and ν'. Moreover, the map $a\mapsto b\cap\pi a:\mathfrak{A}\to\mathfrak{B}_b$ is an injective order-continuous Boolean homomorphism. By (b) and 521Fa,

$$\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\nu')=\mathcal{N}(\nu_{Y'})\preccurlyeq_{\mathrm{T}}\mathcal{N}(\nu).$$

(d) For the general case, let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that $\bar\mu a_i$ is finite for every i, and set $b_i=\pi a_i$ for each i, so that $\langle b_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{B}$. As in the proof of 524J, we have corresponding partitions $\langle X_i\rangle_{i\in I}$, $\langle Y_i\rangle_{i\in I}$ of X, Y into measurable sets; as before, 322M tells us that $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ can be identified with $\prod_{i\in I}\mathcal{N}(\mu_{X_i})$ and $\prod_{i\in I}\mathcal{N}(\nu_{Y_i})$ respectively. Now, for each i, we can identify the principal ideals $\mathfrak{A}_{a_i}$, $\mathfrak{B}_{b_i}$ with the measure algebras of the subspace measures μ_{X_i} and ν_{Y_i}, and $\pi\restriction\mathfrak{A}_{a_i}$ is an order-continuous embedding of $\mathfrak{A}_{a_i}$ in $\mathfrak{B}_{b_i}$. So (c) tells us that $\mathcal{N}(\mu_{X_i})\preccurlyeq_{\mathrm{T}}\mathcal{N}(\nu_{Y_i})$. Accordingly

$$\mathcal{N}(\mu)\cong\prod_{i\in I}\mathcal{N}(\mu_{X_i})\preccurlyeq_{\mathrm{T}}\prod_{i\in I}\mathcal{N}(\nu_{Y_i})\cong\mathcal{N}(\nu)$$

(513Eg again), and the proof is complete.

524L So far we have been looking at cardinals defined from null ideals. Of course there is an equally important series based on measurable algebras, which turns out to be similarly strongly associated with the cardinal functions of the ideals $\mathcal{N}_\kappa$. I have already developed a good deal of the machinery in the arguments of this section. But for 'linking numbers' we need a new idea, which is most clearly expressed in the context of homogeneous algebras.

Proposition (Dow & Steprāns 94) Let κ be an infinite cardinal. Then for any $n\ge2$ the n-linking number $\mathrm{link}_n(\mathfrak{B}_\kappa)$ is the least λ such that $\kappa\le2^\lambda$.

proof Let λ be the least cardinal such that $\kappa\le2^\lambda$.

(a) By 514Cb, $\mathfrak{B}_\kappa$ is isomorphic, as partially ordered set, to a subset of $\mathcal{P}(\mathrm{link}(\mathfrak{B}_\kappa))$, so we must have

$$2^{\mathrm{link}(\mathfrak{B}_\kappa)} \ge \#(\mathfrak{B}_\kappa) \ge \kappa$$

and $\mathrm{link}(\mathfrak{B}_\kappa) \ge \lambda$. It follows at once that $\mathrm{link}_n(\mathfrak{B}_\kappa) \ge \lambda$ for every $n \ge 2$ (511Ia).

(b) Now let $n \ge 2$, and take an injective function $\phi : \kappa \to \{0,1\}^\lambda$. Let $\mathcal{C}$ be the family of measurable cylinders in $\{0,1\}^\kappa$, that is, sets of the form $\{x : x \in \{0,1\}^\kappa, x{\restriction}I = z\}$, where $I \subseteq \kappa$ is finite and $z \in \{0,1\}^I$. For each $E \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$ we can find disjoint finite sets I'_E, I''_E, $J_E \subseteq \kappa$ and $G_E \in \mathrm{T}_\kappa$ such that

setting $C_E = \{x : x \in \{0,1\}^\kappa, x(\xi) = 0$ for $\xi \in I'_E$ and $x(\xi) = 1$ for $\xi \in I''_E\}$, and $k_E = \#(I'_E) + \#(I''_E)$, $\nu_\kappa(C_E \setminus E) \le \frac{1}{4n}\nu_\kappa C_E = \frac{1}{4n} \cdot 2^{-k_E}$;

G_E is determined by coordinates in J_E and $\nu_\kappa(C_E \cap (E \triangle G_E)) \le \frac{1}{4n} \cdot 2^{-nk_E}$;

$\nu_\kappa G_E \ge 1 - \frac{1}{2n}$.

P By 254Fe, there is a set W, expressible as the union of finitely many measurable cylinders, such that $\nu_\kappa(E \triangle W) \le \frac{1}{5n}\nu_\kappa E$. Now $\nu_\kappa W \ge \frac{9}{10}\nu_\kappa E$ so $\nu_\kappa(W \setminus E) \le \frac{1}{4n}\nu_\kappa W$. W is determined by coordinates in a finite set, so is expressible as a disjoint union of non-empty measurable cylinders, and for at least one of these we must have $\nu_\kappa(C \setminus E) \le \frac{1}{4n}\nu_\kappa C$; take such a one for C_E. Express C_E as $\{x : x{\restriction}I_E = z_E\}$, where $I_E \subseteq \kappa$ is finite and $z_E \in \{0,1\}^{I_E}$, and set $I'_E = \{\xi : \xi \in I_E, z_E(\xi) = 0\}$ and $I''_E = \{\xi : \xi \in I_E, z_E(\xi) = 1\}$; then $\nu_\kappa C_E = 2^{-k_E}$ and $\nu_\kappa(C_E \setminus E) \le \frac{1}{4}n \cdot 2^{-k_E}$.

Next, take a set $W' \subseteq \{0,1\}^\kappa$, determined by coordinates in a finite subset J of κ, such that $\nu_\kappa(E \triangle W') \le \frac{1}{4n} \cdot 2^{-nk_E}$. Set

$$G_E = \{x : x \in \{0,1\}^\kappa, \exists\, y \in W' \cap C_E, x{\restriction}\kappa \setminus I_E = y{\restriction}\kappa \setminus I_E\},$$

so that G_E is determined by coordinates in $J_E = J \setminus I_E$ and $G_E \cap C_E = W' \cap C_E$; accordingly

$$\nu_\kappa(C_E \cap (E \triangle G_E)) = \nu_\kappa(C_E \cap (E \triangle W')) \le \nu_\kappa(E \triangle W') \le \frac{1}{4n} \cdot 2^{-nk_E}.$$

Note that G_E and C_E are stochastically independent, so that

$$\begin{aligned}\nu_\kappa C_E(1 - \nu_\kappa G_E) = \nu_\kappa(C_E \setminus G_E) &\le \nu_\kappa(C_E \setminus E) + \nu_\kappa(C_E \cap (E \setminus G_E)) \\ &\le \frac{1}{4n}\nu_\kappa C_E + \frac{1}{4n}(\nu_\kappa C_E)^n \le \frac{1}{2n}\nu_\kappa C_E\end{aligned}$$

and $\nu_\kappa G_E \ge 1 - \frac{1}{2n}$. **Q**

(c) Let Q be the set of all quadruples (k, U, V, W) where $k \in \mathbb{N}$ and U, V, W are disjoint open-and-closed subsets of $\{0,1\}^\lambda$ in its usual topology. For $q = (k, U, V, W) \in Q$, set

$$\mathcal{E}_q = \{E : E \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa, k_E = k, \phi[I'_E] \subseteq U, \phi[I''_E] \subseteq V, \phi[J_E] \subseteq W\}.$$

For any $E \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$, I'_E, I''_E and J_E, as chosen in (b) above, are disjoint finite sets, so $\phi[I'_E]$, $\phi[I''_E]$ and $\phi[J_E]$ also are, and there is a $q \in Q$ such that $E \in \mathcal{E}_q$. Now if $q = (k, U, V, W) \in Q$ and $E_i \in \mathcal{E}_q$ for $i < n$, then $\nu_\kappa(\bigcap_{i<n} E_i) > 0$. **P** Set $I' = \bigcup_{i<n} I'_{E_i}$, $I'' = \bigcup_{i<n} I''_{E_i}$ and $J = \bigcup_{i<n} J_{E_i}$. Then $\phi[I'] \subseteq U$, $\phi[I''] \subseteq V$ and $\phi[J] \subseteq W$, so that I', I'' and J must be disjoint. Set

$$C = \bigcap_{i<n} C_{E_i} = \{x : x \in \{0,1\}^\kappa, x(\xi) = 0 \text{ for } \xi \in I', x(\xi) = 1 \text{ for } \xi \in I''\};$$

then $\nu_\kappa C = 2^{-\#(I' \cup I'')} \ge 2^{-nk}$. Next, setting $G = \bigcap_{i<n} G_{E_i}$,

$$\nu_\kappa G \ge 1 - \sum_{i=0}^{n-1}(1 - \nu_\kappa G_{E_i}) \ge \frac{1}{2},$$

and G is stochastically independent of C, so that $\nu_\kappa(C \cap G) \ge 2^{-nk-1}$. Finally,

$$\nu_\kappa(C \cap G \setminus E_i) \le \nu_\kappa(C_{E_i} \cap G_{E_i} \setminus E_i) \le \frac{1}{4n} \cdot 2^{-nk}$$

for each i, so

$$\nu_\kappa(C\cap G\setminus\bigcap_{i<n}E_i)\le 2^{-nk-2}<\nu_\kappa(C\cap G)$$

and $\nu_\kappa(\bigcap_{i<n}E_i)>0$. **Q**

(d) This means that if we set $A_q=\{E^\bullet:E\in\mathcal{E}_q\}$ for each $q\in Q$, then every A_q is an n-linked set in $\mathfrak{B}_\kappa$ and $\bigcup_{q\in Q}A_q=\mathfrak{B}_\kappa^+$. Because $\{0,1\}^\lambda$ is a compact topological space with a subbase with cardinal $\lambda\ge\omega$, it has λ open-and-closed sets and $\#(Q)=\lambda$. So $\langle A_q\rangle_{q\in Q}$ witnesses that $\operatorname{link}_n(\mathfrak{B}_\kappa)\le\lambda$, and the proof is complete.

524M Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Let K be the set of infinite cardinals κ such that $\mathfrak{A}$ has a homogeneous principal ideal with Maharam type κ.

(a)
$$\begin{aligned}\#(\mathfrak{A})&=2^{c(\mathfrak{A})}\text{ if }\mathfrak{A}\text{ is finite,}\\&=\tau(\mathfrak{A})^\omega\text{ if }\mathfrak{A}\text{ is ccc and infinite.}\end{aligned}$$

(b)
$$\begin{aligned}\operatorname{wdistr}(\mathfrak{A})&=\infty\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\operatorname{add}\mathcal{N}\text{ if }K=\{\omega\},\\&=\omega_1\text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\pi(\mathfrak{A})&=c(\mathfrak{A})\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\max(c(\mathfrak{A}),\operatorname{cf}\mathcal{N},\sup_{\kappa\in K}\operatorname{cf}[\kappa]^{\le\omega})\text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\mathfrak{m}(\mathfrak{A})&=\infty\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\min_{\kappa\in K}\operatorname{cov}\mathcal{N}_\kappa\text{ otherwise.}\end{aligned}$$

(e)
$$\begin{aligned}d(\mathfrak{A})&=c(\mathfrak{A})\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\max(c(\mathfrak{A}),\sup_{\kappa\in K}\operatorname{non}\mathcal{N}_\kappa)\text{ otherwise.}\end{aligned}$$

(f) For $2\le n<\omega$,

$$\begin{aligned}\operatorname{link}_n(\mathfrak{A})&=c(\mathfrak{A})\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\max(c(\mathfrak{A}),\min\{\lambda:\tau(\mathfrak{A})\le 2^\lambda\})\text{ otherwise.}\end{aligned}$$

proof The case $\mathfrak{A}=\{0\}$ is trivial, so I shall assume henceforth that $\mathfrak{A}\ne\{0\}$. Let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}^+$ such that all the principal ideals $\mathfrak{A}_{a_i}$ are homogeneous and totally finite. For each $i\in I$, set $\kappa_i=\tau(\mathfrak{A}_{a_i})$, so that $\mathfrak{A}_{a_i}\cong\mathfrak{B}_{\kappa_i}$, and let (Z_i,λ_i) be the Stone space of $(\mathfrak{A}_{a_i},\bar\mu{\restriction}\mathfrak{A}_{a_i})$. Let $(\widehat{\mathfrak{A}},\tilde\mu)$ be the localization of $(\mathfrak{A},\bar\mu)$ (322Q). $\mathfrak{A}$ can be identified with an order-dense Boolean subalgebra of $\widehat{\mathfrak{A}}$, so that $\langle a_i\rangle_{i\in I}$ is still a partition of unity in $\widehat{\mathfrak{A}}$. Because $\mathfrak{A}^f=\widehat{\mathfrak{A}}^f$ (322P), $\mathfrak{A}_{a_i}$ is still a principal ideal of $\widehat{\mathfrak{A}}$, and $\widehat{\mathfrak{A}}$ can be identified with the simple product $\prod_{i\in I}\mathfrak{A}_{a_i}$ (315F).

(a) This is elementary if $\mathfrak{A}$ is finite (see 511Ic). If $\mathfrak{A}$ is infinite, then 515Ma tells us that $\#(\mathfrak{A})=\tau(\mathfrak{A})^\omega$.

(b)

$$\operatorname{wdistr}(\mathfrak{A})=\operatorname{wdistr}(\widehat{\mathfrak{A}})$$

(514Ee)

$$=\min_{i\in I}\operatorname{wdistr}(\mathfrak{A}_{a_i})$$

(514Ef)

$$=\min_{i\in I}\operatorname{wdistr}(\mathfrak{B}_{\kappa_i})=\min_{i\in I}\operatorname{add}(\mathcal{N}(\lambda_i))$$

(514Be, because $\mathcal{N}(\lambda_i)$ is the ideal of nowhere dense subsets of Z, by 322R)

$$=\min_{i\in I}\operatorname{add}(\mathcal{N}_{\kappa_i})$$

(524Ja)

$$= \infty \text{ if } K = \emptyset,$$
$$= \operatorname{add}\mathcal{N} \text{ if } K = \{\omega\},$$
$$= \omega_1 \text{ otherwise}$$

(523E).

(c)(i) Consider first an algebra $\mathfrak{B}_\kappa$, where $\kappa \geq \omega$. Then $\operatorname{ci}\mathfrak{B}_\kappa^+ > \omega$. **P** If $\langle b_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{B}_\kappa^+$, then (because $\mathfrak{B}_\kappa$ is atomless) we can choose $c_n \subseteq b_n$ such that $0 < \bar{\nu}_\kappa c_n \leq 2^{-n-2}$ for each $n \in \mathbb{N}$. Set $c = \sup_{n\in\mathbb{N}} c_n$, $b = 1 \setminus c$; then $b \neq 0$ and $b_n \not\subseteq b$ for every n, so $\{b_n : n \in \mathbb{N}\}$ is not coinitial with $\mathfrak{B}_\kappa^+$. **Q**

It follows that

$$\operatorname{ci}\mathfrak{B}_\kappa^+ = \operatorname{cov}(\mathfrak{B}_\kappa^+, \supseteq, \mathfrak{B}_\kappa^+) = \operatorname{cov}(\mathfrak{B}_\kappa^+, \supseteq', [\mathfrak{B}_\kappa^+]^{\leq\omega})$$

(512Gf)

$$= \operatorname{cov}(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa)$$

(524H)

$$= \operatorname{cf}\mathcal{N}_\kappa.$$

(ii) If $\mathfrak{A}$ is purely atomic, then $\mathfrak{A}_{a_i} = \{0, a_i\}$ for every i, and $\pi(\mathfrak{A}) = \#(I) = c(\mathfrak{A})$. Otherwise,

$$\max(c(\mathfrak{A}), \sup_{i\in I} \pi(\mathfrak{A}_{a_i})) \leq \pi(\mathfrak{A})$$

(514Da, 514Ed)

$$\leq \max(\omega, c(\mathfrak{A}), \sup_{i\in I} \pi(\mathfrak{A}_{a_i}))$$

(514Ef)

$$= \max(c(\mathfrak{A}), \sup_{\kappa\in K} \pi(\mathfrak{B}_\kappa)) = \max(c(\mathfrak{A}), \sup_{\kappa\in K} \operatorname{cf}\mathcal{N}_\kappa)$$

(by (i))

$$= \max(c(\mathfrak{A}), \operatorname{cf}\mathcal{N}, \sup_{\kappa\in K} \operatorname{cf}[\kappa]^{\leq\omega}))$$

by 523N.

(d) If $\mathfrak{A}$ is purely atomic, then $\mathfrak{m}(\mathfrak{A}) = \infty$ (511If). Otherwise,

$$\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}(\widehat{\mathfrak{A}})$$

(517Id)

$$= \min_{i\in I} \mathfrak{m}(\mathfrak{A}_{a_i}) = \min_{i\in I} n(Z_i)$$

(517N)

$$= \min_{i\in I} \operatorname{cov}\mathcal{N}(\lambda_i)$$

(again because $\mathcal{N}(\lambda_i)$ is the ideal of nowhere dense subsets of Z_i)

$$= \min_{i\in I} \operatorname{cov}\mathcal{N}_{\kappa_i}$$

(524Jb)

$$= \min_{\kappa\in K} \operatorname{cov}\mathcal{N}_\kappa,$$

as claimed.

(e)(i) I note first that $d(\mathfrak{A}_{a_i}) = \operatorname{non}\mathcal{N}_{\kappa_i}$ for each i. **P** Let $A \in \mathcal{P}Z_i \setminus \mathcal{N}(\lambda_i)$ be a set with cardinal $\operatorname{non}\mathcal{N}(\lambda_i)$. Then $H = \operatorname{int}\overline{A}$ is not empty. Let $a \in \mathfrak{A}_{a_i}^+$ be such that the corresponding open-and-closed set $\widehat{a}$

is included in H. Then $\widehat{a}$ can be identified with the Stone space of $\mathfrak{A}_a$ (312T); because $\mathfrak{A}_{a_i}$ is homogeneous, and $A\cap\widehat{a}$ is dense in $\widehat{a}$,

$$d(\mathfrak{A}_{a_i})=d(\mathfrak{A}_a)=d(\widehat{a})$$

(514Bd)

$$\le\#(A\cap\widehat{a})\le\operatorname{non}\mathcal{N}(\lambda_i)=\operatorname{non}\mathcal{N}_{\kappa_i}$$

(524Jb)

$$\le d(Z_i)$$

(because $\mathcal{N}(\lambda_i)$ is the ideal of nowhere dense subsets of Z_i, so surely contains no dense set)

$$=d(\mathfrak{A}_{a_i})$$

by 514Bd again. **Q**

(ii) If $\mathfrak{A}$ is purely atomic, $d(\mathfrak{A})=c(\mathfrak{A})$. Otherwise,

$$\max(c(\mathfrak{A}),\sup_{i\in I}d(\mathfrak{A}_{a_i}))\le d(\mathfrak{A})$$

(514Da, 514Ed)

$$=d(\widehat{\mathfrak{A}})$$

(514Ee)

$$\le\max(\omega,c(\mathfrak{A}),\sup_{i\in I}d(\mathfrak{A}_{a_i}))$$

(514Ef)

$$=\max(c(\mathfrak{A}),\sup_{i\in I}\operatorname{non}\mathcal{N}_{\kappa_i})=\max(c(\mathfrak{A}),\sup_{\kappa\in K}\operatorname{non}\mathcal{N}_{\kappa}).$$

(f) If $\mathfrak{A}$ is purely atomic, this is elementary, since any linked subset of $\mathfrak{A}^+$ can contain at most one atom. Otherwise, set

$$\theta=\max(c(\mathfrak{A}),\min\{\lambda:\tau(\mathfrak{A})\le 2^\lambda\}),\quad\theta'=\operatorname{link}_n(\mathfrak{A}).$$

For any $i\in I$, $\kappa_i\le\tau(\mathfrak{A})$ (514Ed), so $\kappa_i\le 2^\theta$ and $\operatorname{link}_n(\mathfrak{A}_{a_i})=\operatorname{link}_n(\mathfrak{B}_{\kappa_i})\le\theta$ (524L; of course the case $\kappa_i=0$ is trivial here). Accordingly

$$\theta'=\operatorname{link}_n(\widehat{\mathfrak{A}})$$

(514Ee)

$$\le\max(\omega,c(\mathfrak{A}),\sup_{i\in I}\operatorname{link}_n(\mathfrak{A}_{a_i}))$$

(514Ef)

$$\le\theta.$$

On the other hand, $c(\mathfrak{A})\le\theta'$ (514Da). For each $i\in I$, $\operatorname{link}_n(\mathfrak{A}_i)\le\theta'$ (514Ed), so $\kappa_i\le 2^{\theta'}$ (524L, in the other direction). Let A_i be a τ-generating subset of $\mathfrak{A}_{a_i}$ with cardinal κ_i. Now the order-closed subalgebra of $\mathfrak{A}$ generated by $A=\{a_i:i\in I\}\cup\bigcup_{i\in I}A_i$ is $\mathfrak{A}$, so

$$\tau(\mathfrak{A})\le\#(A)=\max(c(\mathfrak{A}),\sup_{i\in I}\kappa_i)\le\max(\theta',2^{\theta'})=2^{\theta'}.$$

But this means that $\theta\le\theta'$ and the two are equal.

Remark For the corresponding calculation of $\tau(\mathfrak{A})$, when $(\mathfrak{A},\bar\mu)$ is localizable, see 332S.

524N Corollary (a) If (X,Σ,μ) is a semi-finite locally compact measure space, with $\mu X>0$, then $\operatorname{cov}\mathcal{N}(\mu)\ge\mathfrak{m}_{\sigma\text{-linked}}$.

(b) If $\mathfrak{A}$ is any measurable algebra, then $\mathfrak{m}(\mathfrak{A})\ge\mathfrak{m}_{\sigma\text{-linked}}$.

proof (a) Because μ is semi-finite and $\mu X>0$, there is an $E\in\Sigma$ such that $0<\mu E<\infty$. The subspace measure μ_E on E is compact, so $\nu=\frac{1}{\mu E}\mu_E$ is a compact probability measure. Set $\kappa=\max(\omega,\tau(\nu))$. Because ν is a compact measure, there is a function $f:\{0,1\}^\kappa\to E$ which is inverse-measure-preserving for ν_κ and ν (343Cd). Now

$$\mathfrak{m}_{\sigma\text{-linked}}\le\mathfrak{m}(\mathfrak{B}_{\mathfrak{c}})$$

(because $\mathfrak{B}_{\mathfrak{c}}$ is σ-linked, by 524Mf)

$$=\operatorname{cov}\mathcal{N}_{\mathfrak{c}}$$

(524Md)

$$\le\operatorname{cov}\mathcal{N}_\kappa$$

(523F)

$$\le\operatorname{cov}\mathcal{N}(\nu)$$

(521Ha)

$$=\operatorname{cov}\mathcal{N}(\mu_E)\le\operatorname{cov}\mathcal{N}(\mu)$$

(521Fb).

(b) This is now immediate from 524Md.

524O Freese-Nation numbers I spell out those facts about Freese-Nation numbers of measure algebras which can be read off from the results in §518.

Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be an infinite measure algebra. Then $\mathrm{FN}(\mathfrak{A})\ge\mathrm{FN}(\mathcal{P}\mathbb{N})$.

(b) Let $\mathfrak{A}$ be a measurable algebra.

(i) $\mathrm{FN}(\mathfrak{A})\le\mathfrak{c}^+$.

(ii) If $\tau(\mathfrak{A})\le\mathfrak{c}$ then $\mathrm{FN}(\mathfrak{A})\le\mathrm{FN}(\mathcal{P}\mathbb{N})$.

(iii) If

(α) $\operatorname{cf}([\lambda]^{\le\omega})\le\lambda^+$ for every cardinal $\lambda\le\tau(\mathfrak{A})$,

(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda\le\tau(\mathfrak{A})$ of countable cofinality,

then $\mathrm{FN}(\mathfrak{A})\le\mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(c) Suppose that the continuum hypothesis and $\mathrm{CTP}(\omega_{\omega+1},\omega_\omega)$ are both true. If $\mathfrak{A}$ is a measurable algebra, then

$$\begin{aligned}\mathrm{FN}(\mathfrak{A})&=\mathfrak{c}=\omega_1\text{ if }\omega\le\tau(\mathfrak{A})<\omega_\omega,\\&=\mathfrak{c}^+=\omega_2\text{ otherwise}.\end{aligned}$$

proof (a) This is a special case of 518Ca.

(b)(i) Consider first the case $\mathfrak{A}=\mathfrak{B}_\kappa$ for some cardinal κ. For $I\subseteq\kappa$, let $\mathfrak{C}_I$ be the closed subalgebra of $\mathfrak{B}_\kappa$ consisting of those $a\in\mathfrak{B}_\kappa$ expressible in the form $E^\bullet$ for some measurable $E\subseteq\{0,1\}^\kappa$ determined by coordinates in I. For $a\in\mathfrak{B}_\kappa$, there is a smallest subset I_a of λ such that $a\in\mathfrak{C}_I$ (325M again); I_a is always countable.

For each $a\in\mathfrak{B}_\kappa$, set

$$f(a)=\{b:I_b\subseteq I_a\}.$$

Then $\#(f(a))\le\mathfrak{c}$. If $a\subseteq b$, then there is a $c\in\mathfrak{B}_\kappa$ such that $a\subseteq c\subseteq b$ and $I_c\subseteq I_a\cap I_b$ (325M(b-ii)). So f is a Freese-Nation function. This shows that $\mathrm{FN}(\mathfrak{B}_\kappa)\le\mathfrak{c}^+$.

In general, $\mathfrak{A}$ is either $\{0\}$ or isomorphic to a closed subalgebra of $\mathfrak{B}_\kappa$ where $\kappa=\max(\omega,\tau(\mathfrak{A}))$, so $\mathrm{FN}(\mathfrak{A})\le\mathrm{FN}(\mathfrak{B}_\kappa)\le\mathfrak{c}^+$ by 518Cc.

(ii) $\mathfrak{A}$ is σ-linked (524Mf), so 518D(iii) tells us that $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(iii) If $\mathfrak{B} \subseteq \mathfrak{A}$ is a countably generated order-closed subalgebra, then $\mathrm{FN}(\mathfrak{B}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$, by (ii); so 518I tells us that $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(c) If $\tau(\mathfrak{A}) < \omega_\omega$ then $\mathrm{cf}[\lambda]^{\le\omega} = \lambda$ for $\omega_1 \le \lambda \le \tau(\mathfrak{A})$ (5A1F(e-iv)), so we can use (a) and (b-iii); otherwise use (b-i) and 518K.

524P The Maharam classification If the cardinal functions of a Radon measure space are determined by its measure algebra, there ought to be some way of calculating them directly from the classification of measure algebras in §332. In many cases this is straightforward.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, and $\mathfrak{A}$ its measure algebra. Let K be the set of infinite cardinals κ such that the Maharam-type-κ component of $\mathfrak{A}$ is non-zero.

(a)
$$\begin{aligned}\mathrm{add}\,\mu = \mathrm{add}\,\mathcal{N}(\mu) &= \infty \text{ if } K = \emptyset,\\ &= \mathrm{add}\,\mathcal{N} \text{ if } K = \{\omega\},\\ &= \omega_1 \text{ otherwise.}\end{aligned}$$

(b)
$$\begin{aligned}\pi(\mu) = \pi(\mathfrak{A}) &= c(\mathfrak{A}) \text{ if } K = \emptyset,\\ &= \max(c(\mathfrak{A}), \mathrm{cf}\,\mathcal{N}, \sup_{\kappa\in K} \mathrm{cf}[\kappa]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\mathrm{cov}\,\mathcal{N}(\mu) &= 1 \text{ if } \mathfrak{A} = \{0\},\\ &= \infty \text{ if } \mathfrak{A} \text{ has an atom,}\\ &= \mathrm{cov}\,\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\mathrm{non}\,\mathcal{N}(\mu) &= \infty \text{ if } \mathfrak{A} = \{0\},\\ &= 1 \text{ if } \mathfrak{A} \text{ has an atom,}\\ &= \mathrm{non}\,\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(e)
$$\begin{aligned}\mathrm{shr}\,\mathcal{N}(\mu) &= 0 \text{ if } \mathfrak{A} = \{0\},\\ &= 1 \text{ if } \mathfrak{A} \text{ has an atom,}\\ &\ge \mathrm{shr}\,\mathcal{N} \text{ otherwise.}\end{aligned}$$

(f) If μ is σ-finite,

$$\begin{aligned}\mathrm{cf}\,\mathcal{N}(\mu) &= 1 \text{ if } K = \emptyset,\\ &= \max(\mathrm{cf}\,\mathcal{N}, \mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

proof If $\mu X = 0$ all these results are trivial, so let us suppose henceforth that $\mu X > 0$. As in part (a) of the proof of 524J, there is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that the subspace measures μ_{X_i} are all Maharam-type-homogeneous and non-zero. Note that $\max(\omega,\#(I)) = \max(\omega, c(\mathfrak{A}))$ (332E). For each $i \in I$, let κ_i be the Maharam type of μ_{X_i}.

(a) By 521Ad, $\mathrm{add}\,\mu = \mathrm{add}\,\mathcal{N}(\mu)$. The map $E \mapsto \langle E \cap X_i\rangle_{i\in I}$ identifies $\mathcal{N}(\mu)$, as partially ordered set, with the product of the family $\langle \mathcal{N}(\mu_{X_i})\rangle_{i\in I}$. So $\mathrm{add}\,\mathcal{N}(\mu) = \min_{i\in I} \mathrm{add}\,\mathcal{N}(\mu_{X_i})$ (511Hg). Now if $i \in I$ and $\kappa_i = 0$, X_i is an atom of (X,Σ,μ), so there is an $x_i \in X_i$ such that $\mu(X_i \setminus \{x_i\}) = 0$ (414G again). In this case, $X_i \setminus \{x_i\}$ is the largest member of $\mathcal{N}(\mu_{X_i})$ and $\mathrm{add}\,\mathcal{N}(\mu_{X_i}) = \infty$. If κ_i is infinite, then $\mathrm{add}\,\mathcal{N}(\mu_{X_i}) = \mathrm{add}\,\mathcal{N}_{\kappa_i}$, by 524I applied to a scalar multiple of μ_{X_i}. So $\mathrm{add}\,\mathcal{N}(\mu) = \min_{\kappa\in K} \mathrm{add}\,\mathcal{N}_\kappa$, interpreting this as ∞ if $K = \emptyset$. But we know from 523E that $\mathrm{add}\,\mathcal{N}_\kappa = \omega_1$ if $\kappa > \omega$, while of course $\mathrm{add}\,\mathcal{N}_\omega = \mathrm{add}\,\mathcal{N}$. It follows at once that

$$\begin{aligned}\mathrm{add}\,\mathcal{N}(\mu) = \min_{i\in I} \mathrm{add}\,\mathcal{N}(\mu_{X_i}) &= \infty \text{ if } K = \emptyset,\\ &= \mathrm{add}\,\mathcal{N} \text{ if } K = \{\omega\},\\ &= \omega_1 \text{ otherwise.}\end{aligned}$$

(b) By 521Dd, $\pi(\mu)=\pi(\mathfrak{A})$; and 524Mc gives us the formula for $\pi(\mathfrak{A})$.

(c) If $\mathcal{E}$ is a cover of X by negligible sets, and $i\in I$, then $\{E\cap X_i:E\in\mathcal{E}\}$ is a cover of X_i by negligible sets; thus $\operatorname{cov}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{cov}\mathcal{N}(\mu_{X_i})$. By 524Jb, $\operatorname{cov}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{cov}\mathcal{N}_{\kappa_i}$. If any of the κ_i is zero, that is, if $\mathfrak{A}$ has an atom, this is ∞, and we can stop.

Otherwise, for each $i\in I$,

$$\operatorname{cov}\mathcal{N}(\mu_{X_i})=\operatorname{cov}\mathcal{N}_{\kappa_i}\le\operatorname{cov}\mathcal{N}_{\min K}=\lambda$$

say, by 523B. So we have a family $\langle E_{i\xi}\rangle_{\xi<\lambda}$ of negligible subsets of X_i covering X_i; setting $E_\xi=\bigcup_{i\in I}E_{i\xi}$ for each ξ, we have a family $\langle E_\xi\rangle_{\xi<\lambda}$ in $\mathcal{N}(\mu)$ covering X, so $\operatorname{cov}\mathcal{N}(\mu)\le\operatorname{cov}\mathcal{N}_{\min K}$. But we already know that

$$\operatorname{cov}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{cov}\mathcal{N}_{\kappa_i}\ge\operatorname{cov}\mathcal{N}_{\min K},$$

so $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}_{\min K}$.

(d) A set $A\subseteq X$ is non-negligible iff $A\cap X_i$ is non-negligible for some $i\in I$. It follows at once that $\operatorname{non}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{non}\mathcal{N}(\mu_{X_i})$. If any of the X_i is an atom, it contains a point of non-zero measure, so that $\operatorname{non}\mathcal{N}(\mu)=1$. If $\kappa_i\ge\omega$ for every i, then we have

$$\operatorname{non}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{non}\mathcal{N}_{\kappa_i}=\operatorname{non}\mathcal{N}_{\min K}$$

by 524Jb and 523B again.

(e) If $\mathfrak{A}$ is purely atomic, then μ is point-supported, so $\operatorname{shr}\mathcal{N}(\mu)=1$. Otherwise, let E be a measurable set of non-zero finite measure such that the subspace measure μ_E is atomless; let ν be the normalized subspace measure $\dfrac{1}{\mu E}\mu_E$; then ν, like μ_E, is a Radon measure. By 343Cb, there is a function $f:E\to\{0,1\}^\omega$ which is inverse-measure-preserving for ν and ν_ω; because $\{0,1\}^\omega$ is separable and metrizable, νf^{-1} is a Radon measure (451O, or 418I-418J) and must be equal to ν_ω (416Eb). By 521Fd and 521Hb,

$$\operatorname{shr}\mathcal{N}(\mu)\ge\operatorname{shr}\mathcal{N}(\mu_E)=\operatorname{shr}\mathcal{N}(\nu)\ge\operatorname{shr}\mathcal{N}(\nu_\omega)=\operatorname{shr}\mathcal{N}.$$

(f)(i) If $K=\emptyset$ then (a) tells us that $\mathcal{N}(\mu)$ has a greatest member, so that $\operatorname{cf}\mathcal{N}(\mu)=1$.

(ii) Now suppose that K is not empty. Then 524Fb tells us that there is a family $\langle E_\xi\rangle_{\xi<\tau(\mathfrak{A})}$ in $\mathcal{N}(\mu)$ such that $\{\xi:E_\xi\subseteq E\}$ is countable for every $E\in\mathcal{N}(\mu)$. In this case, $J\mapsto\bigcup_{\xi\in J}E_\xi:[\tau(\mathfrak{A})]^{\le\omega}\to\mathcal{N}(\mu)$ is a Tukey function, so $\operatorname{cf}\mathcal{N}(\mu)\ge\operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}$. At the same time, there is an $i\in I$ such that $\kappa_i\ge\omega$. The identity map from $\mathcal{N}(\mu_{X_i})$ to $\mathcal{N}(\mu)$ is a Tukey function; but this means that

$$\operatorname{cf}\mathcal{N}(\mu)\ge\operatorname{cf}\mathcal{N}(\mu_{X_i})=\operatorname{cf}\mathcal{N}_{\kappa_i}$$

(524I again)

$$\ge\operatorname{cf}\mathcal{N}_\omega=\operatorname{cf}\mathcal{N}$$

(523B). Thus $\operatorname{cf}\mathcal{N}(\mu)\ge\max(\operatorname{cf}\mathcal{N},\operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega})$.

(iii) In the other direction, we know from 524H (again, applied to a scalar multiple of μ_{X_i}) that $(\mathcal{N}(\mu_{X_i}),\subseteq,\mathcal{N}(\mu_{X_i})\equiv_{\mathrm{GT}}(\kappa_i^{\mathbb{N}},\subseteq^*,\mathcal{S}_{\kappa_i})$ whenever κ_i is infinite. Now $\tau(\mathfrak{A})\ge\kappa_i$, so the maps

$$\text{identity: }\kappa_i^{\mathbb{N}}\to\tau(\mathfrak{A})^{\mathbb{N}},\quad S\mapsto S\cap(\mathbb{N}\times\kappa_i):\mathcal{S}_{\tau(\mathfrak{A})}\to\mathcal{S}_{\kappa_i}$$

form a Galois-Tukey connection from $(\kappa_i^{\mathbb{N}},\subseteq^*,\mathcal{S}_{\kappa_i})$ to $(\tau(\mathfrak{A})^{\mathbb{N}},\subseteq^*,\mathcal{S}_{\tau(\mathfrak{A})})$. Accordingly we have

$$\begin{aligned}(\mathcal{N}(\mu_{X_i}),\subseteq,\mathcal{N}(\mu_{X_i}))&\equiv_{\mathrm{GT}}(\kappa_i^{\mathbb{N}},\subseteq^*,\mathcal{S}_{\kappa_i})\\&\preccurlyeq_{\mathrm{GT}}(\tau(\mathfrak{A})^{\mathbb{N}},\subseteq^*,\mathcal{S}_{\tau(\mathfrak{A})})\equiv_{\mathrm{GT}}(\mathcal{N}_{\tau(\mathfrak{A})},\subseteq,\mathcal{N}_{\tau(\mathfrak{A})}),\end{aligned}$$

and $\mathcal{N}(\mu_{X_i})\preccurlyeq_{\mathrm{T}}\mathcal{N}_{\tau(\mathfrak{A})}$.

The arguments quoted assume that κ_i is infinite; but of course it is still true that $\mathcal{N}(\mu_{X_i})\preccurlyeq_{\mathrm{T}}\mathcal{N}_{\tau(\mathfrak{A})}$ when $\kappa_i=0$, since then any constant function from $\mathcal{N}(\mu_{X_i})$ to $\mathcal{N}_{\tau(\mathfrak{A})}$ is a Tukey function. It follows that

$$\mathcal{N}(\mu) \cong \prod_{i\in I} \mathcal{N}(\mu_{X_i}) \preccurlyeq_{\mathrm{T}} \mathcal{N}^I_{\tau(\mathfrak{A})}$$

(513Eg once more).

(iv) At this point observe that as we are assuming that $K \neq \emptyset$, $\tau(\mathfrak{A})$ is infinite; and as μ is supposed to be σ-finite, I is countable. So we can find a disjoint family $\langle F_i\rangle_{i\in I}$ of measurable subsets of $\{0,1\}^{\tau(\mathfrak{A})}$ such that all the subspace measures $(\nu_{\tau(\mathfrak{A})})_{F_i}$ are isomorphic to scalar multiples of $\nu_{\tau(\mathfrak{A})}$. (Take $F_i = \{x : x(n_i) = 1$, $x(m) = 0$ for $m < n_i\}$ where $i \mapsto n_i : I \to \mathbb{N}$ is injective.) In this case, the map

$$\langle E_i\rangle_{i\in I} \mapsto \bigcup_{i\in I} E_i : \prod_{i\in I} \mathcal{N}((\nu_{\tau(\mathfrak{A})})_{F_i}) \to \mathcal{N}(\nu_{\tau(\mathfrak{A})})$$

is a Tukey function, while $\mathcal{N}^I_{\tau(\mathfrak{A})}$ is isomorphic to $\prod_{i\in I} \mathcal{N}((\nu_{\tau(\mathfrak{A})})_{F_i})$. Putting these together,

$$\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}^I_{\tau(\mathfrak{A})} \cong \prod_{i\in I} \mathcal{N}((\nu_{\tau(\mathfrak{A})})_{F_i}) \preccurlyeq_{\mathrm{T}} \mathcal{N}_{\tau(\mathfrak{A})}.$$

It follows that

$$\mathrm{cf}\,\mathcal{N}(\mu) \le \mathrm{cf}\,\mathcal{N}_{\tau(\mathfrak{A})} = \max(\mathcal{N}, \mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega}).$$

So we have inequalities in both directions and $\mathrm{cf}\,\mathcal{N}(\mu) = \max(\mathcal{N}, \mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega})$, as claimed.

***524Q** I do not know how to calculate $\mathrm{cf}\,\mathcal{N}(\mu)$ for non-σ-finite Radon measures μ without special assumptions. In the presence of GCH, however, we have the following result.

Proposition Suppose that the generalized continuum hypothesis is true. Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and $(\mathfrak{A}, \bar\mu)$ its measure algebra. For each cardinal κ, write e_κ for the Maharam-type-κ component of $\mathfrak{A}$, and $\mathfrak{C}_\kappa$ for the principal ideal of $\mathfrak{A}$ generated by $\sup_{\kappa'>\kappa} e_{\kappa'}$; set $\lambda = \sup\{\kappa : e_\kappa \neq 0\}$. Then $\mathrm{cf}\,\mathcal{N}(\mu) = \max(c(\mathfrak{C}_0)^+, \lambda^+)$ unless $\lambda > c(\mathfrak{C}_0)$ and there is some $\gamma < \lambda$ such that $\mathrm{cf}\,\lambda > c(\mathfrak{C}_\gamma)$, in which case $\mathrm{cf}\,\mathcal{N}(\mu) = \lambda$.

proof (a) Write

$$\begin{aligned} \theta &= \lambda \text{ if } \lambda > c(\mathfrak{C}_0) \text{ and } \mathrm{cf}\,\lambda > \min_{\gamma<\lambda} c(\mathfrak{C}_\gamma), \\ &= \max(\lambda^+, c(\mathfrak{C}_0)^+) \text{ otherwise.} \end{aligned}$$

If μ is purely atomic, it is point-supported, so $\lambda = 0$ and $\mathfrak{C}_0 = \{0\}$ and $\theta = 1 = \mathrm{cf}\,\mathcal{N}(\mu)$. So let us suppose henceforth that μ is not purely atomic, that is, $\mathfrak{C}_0 \neq \{0\}$ and $\lambda \ge \omega$. As in the proofs of 524J and 524P, there is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that the subspace measures μ_{X_i} are all Maharam-type-homogeneous and non-zero. Let κ_i be the Maharam type of μ_{X_i} for each i, so that $\lambda = \sup_{i\in I} \kappa_i$. Now $\mathcal{N}(\mu) \cong \prod_{i\in I} \mathcal{N}(\mu_{X_i})$ (see the proof of 524Ja). For $i \in I$, $\mathrm{cf}\,\mathcal{N}(\mu_{X_i}) = 1$ if $\kappa_i = 0$, and otherwise is $\max(\mathrm{cf}\,\mathcal{N}, \mathrm{cf}[\kappa_i]^{\le\omega}) = \max(\omega_1, \mathrm{cf}[\kappa_i]^{\le\omega})$ (524Ja, 523N). By 5A6Ab,

$$\begin{aligned} \mathrm{cf}\,\mathcal{N}(\mu_{X_i}) &= 1 \text{ if } \kappa_i = 0, \\ &= \kappa_i \text{ if } \mathrm{cf}\,\kappa_i > \omega, \\ &= \kappa_i^+ \text{ if } \mathrm{cf}\,\kappa_i = \omega. \end{aligned}$$

(b) For each cardinal κ, set $J_\kappa = \{i : i \in I,\ \mathrm{cf}\,\mathcal{N}(\mu_{X_i}) > \kappa\}$, and set

$$\begin{aligned} \lambda_1 = \sup_{i\in I} \mathrm{cf}\,\mathcal{N}(\mu_{X_i}) &= \lambda^+ \text{ if there is an } i \in I \text{ such that } \kappa_i = \lambda \text{ and } \mathrm{cf}\,\kappa_i = \omega, \\ &= \lambda \text{ otherwise.} \end{aligned}$$

Then 513J tells us that if $\lambda_1 > \#(J_1)$ and there is some $\gamma < \lambda_1$ such that $\mathrm{cf}\,\lambda_1 > \#(J_\gamma)$, then $\mathrm{cf}\,\mathcal{N}(\mu) = \lambda_1$, and that otherwise $\mathrm{cf}\,\mathcal{N}(\mu) = \max(\#(J_1)^+, \lambda_1^+)$. As we are supposing that μ is not purely atomic, $c(\mathfrak{C}_0) \ge \omega$ and $c(\mathfrak{C}_0) = \max(\omega, \#(J_1))$; also $\lambda^+ \ge \lambda_1 \ge \lambda \ge \omega$.

case 1 Suppose $\lambda_1 \le \#(J_1)$. Then J_1 is infinite, so $c(\mathfrak{C}_0) = \#(J_1) \ge \lambda$, and

$$\mathrm{cf}\,\mathcal{N}(\mu) = \#(J_1)^+ = c(\mathfrak{C}_0)^+ = \theta$$

as required.

case 2 Suppose $\lambda_1 > \max(\lambda, \#(J_1))$. Then there must be some $i \in I$ such that $\operatorname{cf}\mathcal{N}(\mu_{X_i}) > \lambda \ge \omega$, in which case $\kappa_i = \lambda$ has countable cofinality and $\lambda_1 = \lambda^+$. In this case, $\operatorname{cf}\lambda_1 = \lambda_1 > \#(J_1)$, so $\operatorname{cf}\mathcal{N}(\mu) = \lambda_1$. If $\gamma < \lambda$, then $\mathfrak{C}_\gamma$ is non-trivial, and $\operatorname{cf}\lambda = \omega \le c(\mathfrak{C}_\gamma)$; so

$$\theta = \max(\lambda^+, \#(J_1)^+) = \lambda_1 = \operatorname{cf}\mathcal{N}(\mu).$$

case 3 Suppose $\lambda_1 = \lambda > \#(J_1)$ has countable cofinality. In this case we must have $\kappa_i < \lambda_1$ for every i, so $\#(J_\gamma) \ge \omega = \operatorname{cf}\lambda_1$ for every $\gamma < \lambda_1$, and $\operatorname{cf}\mathcal{N}(\mu) = \lambda_1^+$. At the same time, $\operatorname{cf}\lambda = \omega \le c(\mathfrak{C}_\gamma)$ for every $\gamma < \lambda$, so

$$\theta = \max(\lambda^+, \#(J_1)^+) = \max(\lambda_1^+, \#(J_1)^+) = \operatorname{cf}\mathcal{N}(\mu).$$

case 4 Suppose $\lambda_1 = \lambda > \#(J_1)$ has uncountable cofinality. In this case we have $\lambda > \max(\omega, \#(J_1)) = c(\mathfrak{C}_0)$, so

$$\begin{aligned}\operatorname{cf}\mathcal{N}(\mu) = \lambda_1 &\iff \#(J_\gamma) < \operatorname{cf}\lambda_1 \text{ for some } \gamma < \lambda_1\\ &\iff \max(\omega, \#(J_\gamma)) < \operatorname{cf}\lambda_1 \text{ for some } \gamma < \lambda_1\\ &\iff c(\mathfrak{C}_\gamma) < \operatorname{cf}\lambda \text{ for some } \gamma < \lambda\\ &\iff \theta = \lambda \iff \theta = \lambda_1,\end{aligned}$$

and otherwise

$$\operatorname{cf}\mathcal{N}(\mu) = \lambda_1^+ = \max(\lambda^+, c(\mathfrak{C}_0)^+) = \theta.$$

Thus $\operatorname{cf}\mathcal{N}(\mu) = \theta$ in all cases.

524R The results above show that most of the most important cardinal functions of measurable algebras and Radon measures are readily calculable from the cardinal functions of the ideals $\mathcal{N}_\kappa$ studied in §523. There are no such simple formulae for other classes of space such as compact or quasi-Radon measures (524Xj, 524Xk). However I can give a handful of partial results, as follows.

Proposition Let (X, Σ, μ) be a countably compact σ-finite measure space with Maharam type κ. Then $[\kappa]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$. Consequently $\operatorname{cf}[\kappa]^{\le\omega} \le \operatorname{cf}\mathcal{N}(\mu)$, and if κ is uncountable then $\operatorname{add}\mathcal{N}(\mu) = \omega_1$ and $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N}_\kappa$.

proof If $\langle E_\xi \rangle_{\xi<\kappa}$ is a family as in 524Fb, then $I \mapsto \bigcup_{\xi \in I} E_\xi : [\kappa]^{\le\omega} \to \mathcal{N}(\mu)$ is a Tukey function, if both $[\kappa]^{\le\omega}$ and $\mathcal{N}(\mu)$ are given their natural partial orderings of inclusion. By 513Ee, $\operatorname{cf}[\kappa]^{\le\omega} \le \operatorname{cf}\mathcal{N}(\mu)$ and $\operatorname{add}[\kappa]^{\le\omega} \ge \operatorname{add}\mathcal{N}(\mu)$. But if κ is uncountable, $\operatorname{add}[\kappa]^{\le\omega} = \omega_1$ so $\operatorname{add}\mathcal{N}(\mu)$ is also ω_1. At the same time, $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N}$ (521K), so

$$\operatorname{cf}\mathcal{N}(\mu) \ge \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega}) = \operatorname{cf}\mathcal{N}_\kappa.$$

524S In a different direction, there is something we can say about quasi-Radon measures.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, with $\mu X > 0$, and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a quasi-Radon measure space such that the measure algebras of μ and ν are isomorphic. Then

(a) $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$, so $\operatorname{add}\nu = \operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mu$ and $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}(\mu)$;

(b) $(Y, \in, \mathcal{N}(\nu)) \preccurlyeq_{\mathrm{GT}} (X, \in, \mathcal{N}(\mu))$, so $\operatorname{cov}\mathcal{N}(\nu) \le \operatorname{cov}\mathcal{N}(\mu)$ and $\operatorname{non}\mathcal{N}(\nu) \ge \operatorname{non}\mathcal{N}(\mu)$.

proof (a) Let $(Z, \mathfrak{U}, \Lambda, \lambda)$ be the Stone space of the measure algebra $\mathfrak{B}$ of (Y, T, ν), and $R \subseteq Z \times Y$ the relation described in 415Q/416V, so that $R^{-1}[F] \in \mathcal{N}(\lambda)$ for every $F \in \mathcal{N}(\nu)$. Let $W \subseteq Z$ be the union of the open sets of finite measure. Then the subspace measure λ_W is a Radon measure and its measure algebra is isomorphic to the measure algebras of ν and μ (411Pf).

Now $F \mapsto W \cap R^{-1}[F] : \mathcal{N}(\nu) \to \mathcal{N}(\lambda_W)$ is a Tukey function. **P?** Otherwise, there is a family $\mathcal{A} \subseteq \mathcal{N}(\nu)$ such that $\bigcup\mathcal{A} \notin \mathcal{N}(\nu)$ but $\{W \cap R^{-1}[A] : A \in \mathcal{A}\}$ is bounded above in $\mathcal{N}(\lambda_W)$. Because W is conegligible, $B = \bigcup_{A\in\mathcal{A}} R^{-1}[A]$ is negligible in Z. Let $E \in \mathrm{T}$ be a measurable envelope of $\bigcup\mathcal{A}$ (213J/213L). Then the open-and-closed set $E^* \subseteq Z$ corresponding to $E^\bullet \in \mathfrak{B}$ is not negligible; as λ is inner regular with respect to the open-and-closed sets (411Pb), there must be a non-empty open-and-closed set $V \subseteq E^*$ which is disjoint from $\bigcup_{A\in\mathcal{A}} R^{-1}[A]$. Express V as F^* where $F \in \mathrm{T}$. Then $R[V] = R[F^*]$ is disjoint from $\bigcup\mathcal{A}$. But $R[F^*]$ is measurable and $F \setminus R[F^*]$ is negligible (415Qb), while $F \setminus E$ must also be negligible, so $E \cap R[F^*]$ is a non-negligible measurable subset of $E \setminus \bigcup\mathcal{A}$, which is impossible. **XQ**

This shows that $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\lambda_W)$. But λ_W and μ are Radon measures with isomorphic non-zero measure algebras, so $\mathcal{N}(\lambda_W) \equiv_{\mathrm{T}} \mathcal{N}(\mu)$ (524J) and $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$. Accordingly $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}(\mu)$

(b) This is a special case of 521La.

524T Corollary Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a quasi-Radon measure space, and $\mathfrak{B}$ its measure algebra. Let K be the set of infinite cardinals κ such that the Maharam-type-κ component of $\mathfrak{B}$ is non-zero.

(a)
$$\begin{aligned}\operatorname{add}\nu = \operatorname{add}\mathcal{N}(\nu) &= \infty \text{ if } K = \emptyset,\\ &\ge \operatorname{add}\mathcal{N} \text{ if } K = \{\omega\}.\end{aligned}$$

(b)
$$\begin{aligned}\pi(\nu) = \pi(\mathfrak{B}) &= c(\mathfrak{B}) \text{ if } K = \emptyset,\\ &= \max(c(\mathfrak{B}), \operatorname{cf}\mathcal{N}, \sup_{\kappa\in K} \operatorname{cf}[\kappa]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\operatorname{cov}\mathcal{N}(\nu) &= 1 \text{ if } \mathfrak{B} = \{0\},\\ &= \infty \text{ if } \mathfrak{B} \text{ has an atom},\\ &\le \operatorname{cov}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\operatorname{non}\mathcal{N}(\nu) &= \infty \text{ if } \mathfrak{B} = \{0\},\\ &= 1 \text{ if } \mathfrak{B} \text{ has an atom},\\ &\ge \operatorname{non}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(e) If ν is σ-finite,
$$\begin{aligned}\operatorname{cf}\mathcal{N}(\nu) &= 1 \text{ if } K = \emptyset,\\ &\le \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\tau(\mathfrak{B})]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

proof Parts (a), (c), (d) and (e) are mostly a matter of putting 524P and 524S together. If there are atoms for μ, they may no longer include singletons of non-zero measure; but they do include minimal non-negligible closed sets, so there are non-negligible singletons and $\operatorname{cov}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\mu)$ are ∞ and 1 respectively. As for (b), the proof of 524Pb still works.

524U There is an natural calculation which I shall want to call on later.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra. Then there is a Radon probability measure on $\{0,1\}^{\tau(\mathfrak{A})}$ with measure algebra isomorphic to $(\mathfrak{A}, \bar\mu)$.

proof Write κ for $\tau(\mathfrak{A})$.

(a) If $\mathfrak{A}$ is finite, it is isomorphic to $\mathcal{P}I$ where I is the set of atoms of $\mathfrak{A}$. Now $\#(I) \le 2^\kappa$ so we have an injection $f : I \to \{0,1\}^\kappa$. Let μ be the point-supported probability measure on $\{0,1\}^\kappa$ such that $\mu\{f(a)\} = \bar\mu a$ for every $a \in I$; this works.

(b) Otherwise, κ is infinite. By Maharam's theorem, we have a partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that, for each $i \in I$, either a_i is an atom or the principal ideal $\mathfrak{A}_{a_i}$ is homogeneous with Maharam type $\kappa_i \ge \omega$, and in the latter case $(\mathfrak{A}_{a_i}, \bar\mu\restriction\mathfrak{A}_{a_i})$ is isomorphic to the measure algebra of $\epsilon_i\nu_{\kappa_i}$, where I write $\epsilon_i = \bar\mu a_i$ for each $i \in I$. If a_i is an atom, let μ_i be a point-supported measure concentrated at a single point of $\{0,1\}^\kappa$ and with mass ϵ_i. Otherwise, $\kappa_i = \tau(\mathfrak{A}_{a_i}) \le \kappa$; let $f_i : \{0,1\}^{\kappa_i} \to \{0,1\}^\kappa$ be a continuous injection and set $\mu_i = \epsilon_i\nu_{\kappa_i}f_i^{-1}$ where $\nu_{\kappa_i}f_i^{-1}$ is the image measure. Then μ_i is a Radon probability measure on $\{0,1\}^\kappa$ with measure algebra isomorphic to $(\mathfrak{A}_{a_i}, \bar\mu\restriction\mathfrak{A}_{a_i})$.

Now take an injection $g : I \to \{0,1\}^{\mathbb{N}}$. Define a measure μ on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^\kappa$ by setting

$$\mu W = \textstyle\sum_{i\in I} \mu_i W[\{g(i)\}]$$

for those sets $W \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^\kappa$ for which the sum is defined. It is easy to check that μ is a complete topological probability measure which is inner regular with respect to the compact sets, that is, is a Radon probability measure. Also, setting $E_i = \{g(i)\} \times \{0,1\}^\kappa$, the subspace measure μ_{E_i} is isomorphic to μ_i so has measure algebra isomorphic to $(\mathfrak{A}_{a_i}, \bar{\mu}\restriction\mathfrak{A}_{a_i})$; as $\langle E_i\rangle_{i\in I}$ is a partition of a μ-conegligible set, the measure algebra of μ is isomorphic to the simple product of the measure algebras of μ_{E_i}, that is, to the simple product $\prod_{i\in I}(\mathfrak{A}_{a_i}, \bar{\mu}\restriction\mathfrak{A}_{a_i}) \cong (\mathfrak{A}, \bar{\mu})$.

As κ is infinite, $\{0,1\}^{\mathbb{N}} \times \{0,1\}^\kappa$ is homeomorphic to $\{0,1\}^\kappa$ and we can copy μ onto a Radon probability measure on $\{0,1\}^\kappa$ with measure algebra isomorphic to $(\mathfrak{A}, \bar{\mu})$.

524X Basic exercises (a) Suppose that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and that $\kappa = \mathrm{link}_n(\mathfrak{A})$, where $2 \le n < \omega$. Show that there are families $\langle A_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}\setminus\{0\}$ and $\langle\epsilon_\xi\rangle_{\xi<\kappa}$ in $]0,1]$ such that $\bar{\mu}(\inf I) \ge \epsilon_\xi$ whenever $I \in [A_\xi]^n$ and $\bigcup_{\xi<\kappa} A_\xi = \mathfrak{A}\setminus\{0\}$. (*Hint*: proof of 524L.)

(b) Let (X, Σ, μ) be a semi-finite measure space with measure algebra $\mathfrak{A}$, and $\mathcal{A}$ a family of non-negligible (not necessarily measurable) subsets of X such that every non-negligible member of Σ includes a member of $\mathcal{A}$. Show that $\#(\mathcal{A}) \ge \pi(\mathfrak{A})$.

(c) Show that if κ is uncountable, there is no function $f : [0,1]^\kappa \to \{0,1\}^\kappa$ which is almost continuous and inverse-measure-preserving for the usual measures on these spaces. (*Hint*: if $K \subseteq [0,1]^\kappa$ is a zero set, any continuous function from K to $\{0,1\}^\kappa$ is determined by coordinates in a countable set.)

(d) Let $I^{\|}$ be the split interval and μ its usual measure (343J). Show that there are $f : \{0,1\}^\omega \to I^{\|}$ and $g : I^{\|} \to \{0,1\}^\omega$ such that $\mu = \nu_\omega f^{-1}$ and $\nu_\omega = \mu g^{-1}$. (*Hint*: let $A \subseteq [0,1]$ be a non-measurable set; define $f_0 : [0,1]^2 \to I^{\|}$ by setting $f_0(x,y) = y^+$ if $x \in A$, y^- otherwise.)

(e) Let (Z, μ) be the Stone space of $(\mathfrak{B}_\omega, \bar{\nu}_\omega)$. Show that there is no $f : \{0,1\}^\omega \to Z$ such that $\mu = \nu_\omega f^{-1}$. (*Hint*: use 515J and 322Ra to show that every non-negligible measurable subset of Z has cardinal $2^{\mathfrak{c}}$.)

(f) Let X be a Hausdorff space with a compact topological probability measure μ with Maharam type κ, and suppose that $w(X) < \mathrm{cov}\,\mathcal{N}_\kappa$. (i) Show that there is an equidistributed sequence for μ. (*Hint*: 491Eb.) (ii) Show that if μ is strictly positive then X is separable.

(g) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon probability space with a strong lifting, and (Z, ν) the Stone space of its measure algebra. Show that $\mathrm{shr}\,\mathcal{N}(\mu) \le \mathrm{shr}\,\mathcal{N}(\nu)$ and $\mathrm{shr}^+\,\mathcal{N}(\mu) \le \mathrm{shr}^+\,\mathcal{N}(\nu)$. (*Hint*: 453Mb.)

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and K the set of infinite cardinals κ such that the Maharam-type-κ component of its measure algebra $\mathfrak{A}$ is non-zero. Show that

$$\min\{\#(A) : A \subseteq X \text{ has full outer measure}\} = \sup(\{c(\mathfrak{A})\} \cup \{\mathrm{non}\,\mathcal{N}_\kappa : \kappa \in K\}).$$

(i) Show that for any σ-ideal $\mathcal{I}$ of sets there is a compact probability measure μ such that $\mathcal{I} = \mathcal{N}(\mu)$. (*Hint*: set $X = \bigcup\mathcal{I} \cup \{x_0\}$.)

(j) Show that for any non-zero measurable algebra $\mathfrak{B}$ and any cardinal κ, there is a complete compact probability measure μ such that the measure algebra of μ is isomorphic to $\mathfrak{B}$, $\mathrm{add}\,\mathcal{N}(\mu) = \omega_1$ and $\mathrm{cf}\,\mathcal{N}(\mu) \ge \kappa$. (*Hint*: 524Xi.)

(k) Suppose that $\mathrm{non}\,\mathcal{N}_{\mathfrak{c}} = \mathrm{cov}\,\mathcal{N}_{\mathfrak{c}} = \mathrm{cf}\,\mathfrak{c} = \mathfrak{c}$. Show that there is a quasi-Radon probability measure μ with Maharam type $\mathfrak{c}$ such that $\mathrm{add}\,\mathcal{N}(\mu) = \mathfrak{c}$.

524Y Further exercises (a) Show that if $m \ge 2$ and $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of σ-m-linked Boolean algebras, with $\#(I) \le \mathfrak{c}$, then the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is σ-m-linked.

(b) Let $\mathfrak{A}$ be a measurable algebra with Maharam type λ. Show that there is a family $\mathcal{V} \subseteq [\lambda]^{\le\mathfrak{c}}$, cofinal with $[\lambda]^{\le\mathfrak{c}}$, such that $\#(\{A \cap V : V \in \mathcal{V}\}) < \mathrm{FN}^*(\mathfrak{A})$ for every countable set $A \subseteq \lambda$.

(c) For a Boolean algebra $\mathfrak{A}$ and a cardinal θ, write $\psi_\theta(\mathfrak{A})$ for the smallest size of any subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ such that $d(\mathfrak{C}) \ge \theta$. (If $\theta > d(\mathfrak{A})$ set $\psi_\theta(\mathfrak{A}) = \infty$.) (i) Show that if Z is the Stone space of $\mathfrak{A}$, $\mathcal{I}$ is the ideal of nowhere dense sets in Z, and $\theta \ge 2$ then $\psi_\theta(\mathfrak{A}) \le \operatorname{cov}([Z]^{<\theta}, \subseteq, \mathcal{I})$. (ii) Show that if (X, Σ, μ) is a Maharam-type-homogeneous compact probability space with Maharam type κ, and θ is uncountable, then

$$\psi_\theta(\mathfrak{B}_\kappa) = \operatorname{cov}([X]^{<\theta}, \subseteq, \mathcal{N}(\mu)) = \operatorname{add}(\Sigma \setminus \mathcal{N}(\mu), \mathtt{meet}, [X]^{<\theta}),$$

where $\mathtt{meet}$ is the relation $\{(A, B) : A \cap B \ne \emptyset\}$. (*Hint*: start with $\mu = \nu_\kappa$.) (iii) Show that if (X, Σ, μ) is a semi-finite locally compact measure space with measure algebra $\mathfrak{A}$ then $\psi_{\omega_1}(\mathfrak{A}) \le \operatorname{cf}([\operatorname{cov}\mathcal{N}(\mu)]^{\le\omega})$. (iv) Show that if (X, Σ, μ) is any probability space, with measure algebra $\mathfrak{A}$, and λ is the product probability measure on $X^{\mathbb{N}}$, then $\operatorname{cov}\mathcal{N}(\lambda) \le \psi_{\omega_1}(\mathfrak{A})$. (v) Show that $\psi_{\operatorname{add}\mathcal{M}}(\mathfrak{B}_\omega) \le \operatorname{non}\mathcal{M}$, where $\mathcal{M}$ is the ideal of meager subsets of $\mathbb{R}$.

524Z Problems (a) Let (Z, μ) be the Stone space of $(\mathfrak{B}_\omega, \bar\nu_\omega)$. Is $\operatorname{shr}\mathcal{N}(\mu)$ necessarily equal to $\operatorname{shr}\mathcal{N}$?

(b) Can there be a quasi-Radon probability measure μ with Maharam type greater than $\mathfrak{c}$ such that $\operatorname{add}\mathcal{N}(\mu) > \omega_1$?

524 Notes and comments The ideas of this section are derived primarily from Bartoszyński 84, Fremlin 84b and Fremlin 91. Of course it is not necessary to pass through both $\ell^1(\kappa)$ and the κ-localization relation $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$. I bring $\ell^1(\kappa)$ into the argument (following Bartoszyński 84) because it will be useful when we come to look at other structures in later in the chapter, and $\mathcal{S}_\kappa$ because it echoes the ideas of §522. But note that 524G seems to need a new idea (the family $\langle E_\xi\rangle_{\xi<\kappa}$ from 524F) not required in 522M.

The difficulties of the work above arise from the fact that while there are many inverse-measure-preserving functions between Radon measure spaces, immediately linking covering numbers and uniformities, there are far fewer continuous inverse-measure-preserving functions; for instance, there is no almost continuous inverse-measure-preserving function from the unit interval to the split interval, let alone to the Stone space of its measure algebra. And the straightforward Tukey functions between the ideals $\mathcal{N}_\kappa$ of §523 depend on measures being images of each other, which is something we can rely on only when our functions are almost continuous. (But see 524Xd.) I do not know of any direct construction of a Tukey function from the null ideal of the Stone space of the Lebesgue measure algebra to $\mathcal{N}$, for instance. This is why there is nearly nothing about shrinking numbers in this section (see 524Za).

There is a significant gap in the calculations in 524P; for the cofinality of the null ideal I need to assume that the measure is σ-finite. I have no useful general recipe for $\operatorname{cf}\mathcal{N}(\mu)$, valid in ZFC, when μ is a non-σ-finite Radon measure. The point is that although we can identify $\mathcal{N}(\mu)$ with the product of a family $\mathcal{N}(\mu_{X_i})$ of partially ordered sets to which the arguments of this section apply (524Q), this is not in itself enough to determine its cofinality in the absence of special axioms.

Version of 11.9.13

525 Precalibers

I continue the discussion of precalibers in §516 with results applying to measure algebras. I start with connexions between measure spaces and precalibers of their measure algebras (525B-525C). The next step is to look at measure-precalibers. Elementary facts are in 525D-525G. When we come to ask which cardinals are precalibers of which measure algebras, there seem to be real difficulties; partial answers, largely based on infinitary combinatorics, are in 525I-525O. 525P is a note on a particular pair of cardinals. Finally, 525T deals with precaliber triples (κ, κ, k) where k is finite; I approach it through a general result on correlations in uniformly bounded families of random variables (525S).

525A Notation If (X, Σ, μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ. For any set I, ν_I will be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}_I = \mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. In this context, set $e_i = \{x : x \in \{0,1\}^I, x(i) = 1\}^\bullet$ in $\mathfrak{B}_I$ for $i \in I$. Then $\langle e_i\rangle_{i\in I}$ is a stochastically independent family of elements of measure $\frac{1}{2}$ in $\mathfrak{B}_I$, and $\{e_i : i \in I\}$ τ-generates $\mathfrak{B}_I$; I will say that $\langle e_i\rangle_{i\in I}$ is the **standard generating family** in $\mathfrak{B}_I$.

525B Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and $\mathfrak{A}$ its measure algebra. Then the downwards precaliber triples of the partially ordered set $(\Sigma \setminus \mathcal{N}(\mu), \subseteq)$ are just the precaliber triples of the Boolean algebra $\mathfrak{A}$.

proof Put 521Dd and 516C together.

525C Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra.

(a) A pair (κ, λ) of cardinals is a precaliber pair of $\mathfrak{A}$ iff whenever $\langle E_\xi \rangle_{\xi<\kappa}$ is a family in $\Sigma \setminus \mathcal{N}(\mu)$ there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi\}) \ge \lambda$.

(b) A pair (κ, λ) of cardinals is a measure-precaliber pair of $(\mathfrak{A}, \bar{\mu})$ iff whenever $\langle E_\xi \rangle_{\xi<\kappa}$ is a family in $\Sigma \setminus \mathcal{N}(\mu)$ such that $\inf_{\xi<\kappa} \mu E_\xi > 0$ then there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi\}) \ge \lambda$.

(c) Suppose that $\kappa \ge \operatorname{sat}(\mathfrak{A})$ is an infinite regular cardinal. Then the following are equiveridical:

(i) κ is a precaliber of $\mathfrak{A}$;

(ii) $\mu_*(\bigcup_{\xi<\kappa} E_\xi) = 0$ whenever $\langle E_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family in $\mathcal{N}(\mu)$;

(iii) whenever $\langle A_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family of sets such that $\bigcup_{\xi<\kappa} A_\xi = X$, then there is some $\xi < \kappa$ such that A_ξ has full outer measure in X.

proof (a)(i) Suppose that (κ, λ) is a precaliber pair of $\mathfrak{A}$ and $\langle E_\xi \rangle_{\xi<\kappa}$ is a family in $\Sigma \setminus \mathcal{N}(\mu)$. For each $\xi < \kappa$, let $K_\xi \subseteq E_\xi$ be a non-negligible compact set. Then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{K_\xi^\bullet : \xi \in \Gamma\}$ is centered in $\mathfrak{A}$. But in this case $\{X\} \cup \{K_\xi : \xi \in \Gamma\}$ has the finite intersection property, and must have non-empty intersection. If x is any point of this intersection, $\{\xi : x \in E_\xi\}$ includes Γ and has size at least λ.

(ii) Suppose that whenever $\langle E_\xi \rangle_{\xi<\kappa}$ is a family in $\Sigma \setminus \mathcal{N}(\mu)$ there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi\}) \ge \lambda$. Because μ is complete and strictly localizable (416B), it has a lifting $\psi : \mathfrak{A} \to \Sigma$ (341K). Let $\langle a_\xi \rangle_{\xi<\kappa}$ be a family in $\mathfrak{A} \setminus \{0\}$; then there is an $x \in X$ such that $\Gamma = \{\xi : x \in \psi a_\xi\}$ has cardinal at least λ. But now $\{\psi a_\xi : \xi \in \Gamma\}$ is centered in Σ so $\{a_\xi : \xi \in \Gamma\}$ is centered in $\mathfrak{A}$. As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, (κ, λ) is a precaliber pair of $\mathfrak{A}$.

(b) We can use exactly the same argument, provided that in part (i) we make sure that $\mu K_\xi \ge \frac{1}{2}\mu E_\xi$, so that $\inf_{\xi<\kappa} \bar{\mu} K_\xi^\bullet > 0$.

(c)(i)⇒(iii) Suppose that κ is a precaliber of $\mathfrak{A}$ and $\langle A_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family of sets with union X. **?** If no A_ξ has full outer measure, then we can choose, for each $\xi < \kappa$, a non-negligible compact set $K_\xi \subseteq X \setminus A_\xi$. Because κ is a precaliber of $\mathfrak{A}$, there is a set $\Gamma \in [\kappa]^\kappa$ such that $\{K_\xi^\bullet : \xi \in \Gamma\}$ is centered. Now $\{K_\xi : \xi \in \Gamma\}$ has the finite intersection property and there is some $x \in \bigcap_{\xi\in\Gamma} K_\xi$, in which case $x \notin \bigcup_{\xi\in\Gamma} A_\xi$. But since Γ must be cofinal with κ, $\bigcup_{\xi\in\Gamma} A_\xi = X$. **X** As $\langle A_\xi \rangle_{\xi<\kappa}$ is arbitrary, (iii) is true.

(iii)⇒(ii) Suppose that (iii) is true, and that $\langle E_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family in $\mathcal{N}(\mu)$. **?** If $\bigcup_{\xi<\kappa} E_\xi$ has non-zero inner measure, let $E \subseteq \bigcup_{\xi<\kappa} E_\xi$ be a non-negligible measurable set. Set $A_\xi = E_\xi \cup (X \setminus E)$ for each ξ; then $\langle A_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family with union X, so there is some ξ such that A_ξ has full outer measure. But $E \setminus E_\xi$ is a non-negligible measurable set disjoint from A_ξ. **X** As $\langle E_\xi \rangle_{\xi<\kappa}$ is arbitrary, (ii) is true.

(ii)⇒(i) Let Z be the Stone space of $\mathfrak{A}$ and ν its usual measure (411P). Because μ has a lifting, there is an inverse-measure-preserving function $f : X \to Z$ (341P).

Let $\langle F_\xi \rangle_{\xi<\kappa}$ be a non-decreasing family of nowhere dense subsets of Z. Then they are all ν-negligible (411Pa), so $\langle f^{-1}[F_\xi] \rangle_{\xi<\kappa}$ is a non-decreasing family in $\mathcal{N}(\mu)$ and $\mu_*(\bigcup_{\xi<\kappa} f^{-1}[F_\xi]) = 0$. But this means that if $G = \operatorname{int}(\bigcup_{\xi<\kappa} F_\xi)$, $\nu G = \mu f^{-1}[G] = 0$ and G is empty. By 516Rb, κ is a precaliber of $\mathfrak{A}$.

525D Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) Any precaliber triple of $\mathfrak{A}$ is a measure-precaliber triple of $(\mathfrak{A}, \bar{\mu})$.

(b) If $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A}, \bar{\mu})$ and κ has uncountable cofinality, then $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

(c) If κ is a measure-precaliber of $(\mathfrak{A}, \bar{\mu})$, so is $\operatorname{cf}\kappa$.

proof (a) is immediate from the definitions in 511E.

(b) If $\langle a_\xi\rangle_{\xi<\kappa}$ is any family in $\mathfrak{A}^+$, then there is a $\delta>0$ such that $\Gamma=\{\xi:\bar\mu a_\xi\ge\delta\}$ has cardinal κ; and now there is a $\Gamma'\in[\Gamma]^\lambda$ such that $\{a_\xi:\xi\in I\}$ has a non-zero lower bound for every $I\in[\Gamma']^{<\theta}$.

(c) The point is that κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ iff it is a precaliber of the supported relation $(A_\delta,\supseteq,\mathfrak{A}^+)$ for every $\delta>0$, where $A_\delta=\{a:a\in\mathfrak{A},\ \bar\mu a\ge\delta\}$; so this is just a special case of 516Bd.

525E Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and κ an infinite cardinal. Then κ is a precaliber of $\mathfrak{A}$ iff either $\mathfrak{A}$ is finite or κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ and $\operatorname{cf}\kappa>\omega$.

(b) An infinite cardinal κ is a precaliber of every measurable algebra iff it is a measure-precaliber of every probability algebra and has uncountable cofinality.

proof (a) If κ is a precaliber of $\mathfrak{A}$, of course κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$. Also $\operatorname{cf}\kappa$ is a precaliber of $\mathfrak{A}$ (516Bd again), so $\operatorname{cf}\kappa\ge\operatorname{sat}(\mathfrak{A})$ (516Ja); and if $\mathfrak{A}$ is infinite, $\operatorname{cf}\kappa>\omega$.

If $\mathfrak{A}$ is finite, then any infinite cardinal is a precaliber of $\mathfrak{A}$ (516Lc). If κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ and $\operatorname{cf}\kappa>\omega$, then κ is a precaliber of $\mathfrak{A}$ by 525Db.

(b) Recall that an algebra $\mathfrak{A}$ is 'measurable' iff either $\mathfrak{A}=\{0\}$ or there is a functional $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a probability algebra (391B). So the result follows directly from (a).

525F Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra.

(a) ω is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

(b) If $\omega\le\kappa<\mathfrak{m}(\mathfrak{A})$, then κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

proof (a) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}}\bar\mu a_n=\delta>0$. Set $a=\inf_{n\in\mathbb{N}}\sup_{m\ge n}a_m$; then $\bar\mu a=\inf_{n\in\mathbb{N}}\bar\mu(\sup_{m\ge n}a_m)\ge\delta>0$, so $a\ne0$. If $0\ne b\subseteq a$ and $n\in\mathbb{N}$, there is an $m\ge n$ such that $b\cap a_m\ne0$. We can therefore choose inductively a strictly increasing sequence $\langle n_i\rangle_{i\in\mathbb{N}}$ such that $a\cap\inf_{j\le i}a_{n_j}\ne0$ for every i, so that $\langle a_{n_i}\rangle_{i\in\mathbb{N}}$ is centered. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ω is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

(b) If $\kappa=\omega$, this is (a). Otherwise, let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}$ with $\inf_{\xi<\kappa}\bar\mu a_\xi=\delta>0$. Set

$$c=\inf_{J\subseteq\kappa,\#(J)<\kappa}\sup_{\xi\in\kappa\setminus J}a_\xi;$$

then

$$\bar\mu c=\inf_{J\subseteq\kappa,\#(J)<\kappa}\bar\mu(\sup_{\xi\in\kappa\setminus J}a_\xi)\ge\delta.$$

Choose $\langle I_\xi\rangle_{\xi<\kappa}$ inductively so that, for each $\xi<\kappa$, I_ξ is a countable subset of $\kappa\setminus\bigcup_{\eta<\xi}I_\eta$ and $c\subseteq\sup_{\eta\in I_\xi}a_\eta$.

For $\xi<\kappa$, set

$$Q_\xi=\{b:0\ne b\subseteq c,\ \exists\,\eta\in I_\xi,\ b\subseteq a_\eta\}.$$

Then Q_ξ is coinitial with $\mathfrak{A}_c^+$. Because $\kappa<\mathfrak{m}(\mathfrak{A})\le\mathfrak{m}(\mathfrak{A}_c)$, there is a centered $R\subseteq\mathfrak{A}_c^+$ meeting every Q_ξ. Now

$$\Gamma=\{\eta:\eta<\kappa,\ \exists\,b\in R,\ b\subseteq a_\eta\}$$

meets every I_ξ so has cardinal κ, and $\{a_\eta:\eta\in\Gamma\}$ is centered. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

525G As is surely to be expected, questions about precalibers of measurable algebras can generally be reduced to questions about precalibers of the algebras $\mathfrak{B}_\kappa$. Some of these can be quickly answered in terms of the cardinals examined earlier in this chapter.

Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra. Let K be the set of infinite cardinals κ' such that the Maharam-type-κ' component of $\mathfrak{A}$ is non-zero (cf. 524M). If κ, λ and θ are cardinals, of which κ is infinite, then $(\kappa,\lambda,{<\theta})$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$ iff it is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$ for every $\kappa'\in K$.

(b) Suppose that $\omega\le\kappa<\operatorname{cov}\mathcal{N}_{\kappa'}$. Then κ is a measure-precaliber of $\mathfrak{B}_{\kappa'}$.

(c) For any cardinal κ', ω_1 is a precaliber of $\mathfrak{B}_{\kappa'}$ iff $\operatorname{cov}\mathcal{N}_{\kappa'}>\omega_1$.

(d) If κ, κ' are cardinals such that $\operatorname{non}\mathcal{N}_{\kappa'}<\operatorname{cf}\kappa$, then κ is a precaliber of $\mathfrak{B}_{\kappa'}$.

proof (a)(i) Suppose that $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$, $\kappa'\in K$ and $\langle b_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{B}_{\kappa'}$ with $\inf_{\xi<\kappa}\bar\nu_{\kappa'}b_\xi=\delta>0$. Let $a\in\mathfrak{A}$ be such that the principal ideal $\mathfrak{A}_a$ is homogeneous with Maharam type κ', so that there is an isomorphism $\pi:\mathfrak{B}_{\kappa'}\to\mathfrak{A}_a$ with $\frac{1}{\bar\mu a}\bar\mu(\pi b)=\bar\nu_{\kappa'}b$ for every $b\in\mathfrak{B}_{\kappa'}$ (331L). Then $\inf_{\xi<\kappa}\bar\mu(\pi b_\xi)=\delta\bar\mu a>0$, so there is a $\Gamma\in[\kappa]^\lambda$ such that $\inf_{\xi\in I}\pi b_\xi$ and $\inf_{\xi\in I}b_\xi$ are non-zero for every $I\in[\Gamma]^{<\theta}$. As $\langle b_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$.

(ii) Suppose that $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$ for every $\kappa'\in K$ and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ with $\inf_{\xi<\kappa}\bar\mu a_\xi=\delta>0$. Let $D\subseteq\mathfrak{A}\setminus\{0\}$ be a partition of unity in $\mathfrak{A}$ such that all the principal ideals $\mathfrak{A}_d$, for $d\in D$, are homogeneous. Let $C\subseteq D$ be a finite set such that $\sum_{d\in D\setminus C}\bar\mu d\le\frac12\delta$. Then for every $\xi<\kappa$ there is a $c\in C$ such that $\bar\mu(a_\xi\cap c)\ge\frac12\delta\bar\mu c$, so (because κ is infinite) there are $c\in C$ and $\Gamma_0\in[\kappa]^\kappa$ such that $\bar\mu(a_\xi\cap c)\ge\frac12\delta\bar\mu c$ for every $\xi\in\Gamma_0$. If c is an atom then $\inf_{\xi\in I}a_\xi\supseteq c$ is non-zero for every $I\subseteq\Gamma_0$. Otherwise, the Maharam type κ' of $\mathfrak{A}_c$ belongs to K. Let $\pi:\mathfrak{B}_{\kappa'}\to\mathfrak{A}_c$ be an isomorphism with $\bar\mu(\pi b)=\bar\mu c\cdot\bar\nu_{\kappa'}b$ for every $b\in\mathfrak{B}_{\kappa'}$. Set $b_\xi=\pi^{-1}(a_\xi\cap c)$; then $\bar\nu_{\kappa'}b_\xi\ge\frac12\delta$ for every $\xi\in\Gamma_0$. There is therefore a $\Gamma\in[\Gamma_0]^\lambda$ such that $\inf_{\xi\in I}b_\xi$ and $\inf_{\xi\in I}a_\xi$ are non-zero for every $I\in[\Gamma]^{<\theta}$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$.

(b) We have $\operatorname{cov}\mathcal{N}_{\kappa'}=\mathfrak{m}(\mathfrak{B}_{\kappa'})$ (524Md), so we can use 525Fb.

(c) If $\operatorname{cov}\mathcal{N}_{\kappa'}>\omega_1$ then (b) tells us that ω_1 is a precaliber of $\mathfrak{B}_{\kappa'}$. If $\operatorname{cov}\mathcal{N}_{\kappa'}=\omega_1$, let $\langle E_\xi\rangle_{\xi<\omega_1}$ be a cover of $\{0,1\}^{\kappa'}$ by negligible sets; then $\langle\bigcup_{\eta<\xi}E_\eta\rangle_{\xi<\omega_1}$ is a non-decreasing family in $\mathcal{N}_{\kappa'}$ with union of non-zero inner measure, so 525Cc tells us that ω_1 is not a precaliber of $\mathfrak{B}_{\kappa'}$.

(d) If κ' is finite this is elementary. Otherwise, $d(\mathfrak{B}_{\kappa'})=\operatorname{non}\mathcal{N}_{\kappa'}$ (524Me). By 516Lc, κ is a precaliber of $\mathfrak{B}_{\kappa'}$.

525H The structure of $\mathfrak{B}_I$ Several of the arguments below will depend on the following ideas. Let I be any set and $\langle e_i\rangle_{i\in I}$ the standard generating family in $\mathfrak{B}_I$. If $a\in\mathfrak{B}_I$, there is a smallest countable set $J\subseteq I$ such that a belongs to the closed subalgebra $\mathfrak{C}_J$ of $\mathfrak{B}_I$ generated by $\{e_i:i\in J\}$ (254Rd, 325Mb). (Of course $\mathfrak{C}_J$ is canonically isomorphic to $\mathfrak{B}_J$; see 325Ma.)

Now suppose that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is a family in $\mathfrak{B}_I$, that for each $\xi\in\Gamma$ we are given a set $I_\xi\subseteq I$ such that $a_\xi\in\mathfrak{C}_{I_\xi}$, and that $J\subseteq I$ is such that $I_\xi\cap I_\eta\subseteq J$ for all distinct ξ, $\eta\in\Gamma$. Then $\langle a_\xi\rangle_{\xi\in\Gamma}$ is relatively stochastically independent over $\mathfrak{C}_J$. **P** $\langle\mathfrak{C}_{I_\xi\setminus J}\rangle_{\xi\in\Gamma}$ is stochastically independent, because $\langle I_\xi\setminus J\rangle_{\xi\in\Gamma}$ is disjoint; moreover, $\mathfrak{C}_J$ is independent of $\mathfrak{C}_{I\setminus J}\supseteq\bigcup_{\xi\in\Gamma}\mathfrak{C}_{I_\xi\setminus J}$, and $\mathfrak{C}_{I_\xi\cup J}$ is the closed subalgebra generated by $\mathfrak{C}_{I_\xi\setminus J}\cup\mathfrak{C}_J$ for each ξ. So 458Lg tells us that $\langle\mathfrak{C}_{I_\xi\cup J}\rangle_{\xi\in\Gamma}$ is relatively stochastically independent over $\mathfrak{C}_J$; *a fortiori*, $\langle a_\xi\rangle_{\xi\in\Gamma}$ is relatively stochastically independent over $\mathfrak{C}_J$. **Q** It follows that if $\Delta\subseteq\Gamma$ is finite and $\inf_{\xi\in\Delta}\operatorname{upr}(a_\xi,\mathfrak{C}_J)\ne0$, then $\inf_{\xi\in\Delta}a_\xi\ne0$ (458Lf); in particular, if $\langle\operatorname{upr}(a_\xi,\mathfrak{C}_J)\rangle_{\xi\in\Gamma}$ is centered, so is $\langle a_\xi\rangle_{\xi\in\Gamma}$.

525I Theorem (a)(i) If $\kappa>0$ and $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$, then it is a measure-precaliber triple of every probability algebra.

(ii) If $\kappa>0$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{B}_\kappa$, then it is a precaliber triple of every measurable algebra.

(b) Suppose that $\operatorname{cf}\kappa\ge\omega_2$. If (κ,λ) is a precaliber pair of $\mathfrak{B}_{\kappa'}$ for every $\kappa'<\kappa$, then it is a precaliber pair of every measurable algebra.

(c) Suppose that $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\omega,\bar\nu_\omega)$ and that κ' is such that $\operatorname{cf}[\kappa']^{\le\omega}<\operatorname{cf}\kappa$. Then $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$.

proof (a)(i) Let $(\mathfrak{A},\bar\mu)$ be any probability algebra and $\langle a_\xi\rangle_{\xi<\kappa}$ a family in $\mathfrak{A}^+$ such that $\inf_{\xi<\kappa}\bar\mu a_\xi>0$. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi:\xi<\kappa\}$. Then $(\mathfrak{B},\bar\mu\restriction\mathfrak{B})$ is a probability algebra with Maharam type at most κ, so is isomorphic to a closed subalgebra of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ (332N). Since $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ it is a measure-precaliber triple of $(\mathfrak{B},\bar\mu\restriction\mathfrak{B})$ (cf. 516Sb), and there is a $\Gamma\in[\kappa]^\lambda$ such that $\{a_\xi:\xi\in I\}$ is bounded below in $\mathfrak{B}^+$ and therefore in $\mathfrak{A}^+$ for every $I\in[\Gamma]^{<\theta}$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$.

(ii) The same argument applies, deleting the phrase '$\inf_{\xi<\kappa}\bar\mu a_\xi>0$', since if $\mathfrak{A}$ is a measurable algebra other than $\{0\}$ there is a functional $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a probability algebra.

(b) By (a-ii), it is enough to prove that (κ,λ) is a precaliber pair of $\mathfrak{B}_\kappa$. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\kappa^+$. For each $I\subseteq\kappa$, let $\mathfrak{C}_I$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_i:i\in I\}$, as in 525H. Then for each $\xi<\kappa$ we have a countable set $I_\xi\subseteq\kappa$ such that $a_\xi\in\mathfrak{C}_{I_\xi}$. Because $\operatorname{cf}\kappa\ge\omega_2$, there are a $\Gamma\in[\kappa]^\kappa$ and a $J\in[\kappa]^{<\kappa}$ such that $I_\xi\cap I_\eta\subseteq J$ for all distinct ξ, $\eta\in\Gamma$ (5A1J(a-i)). Because $\#(J)<\kappa$, (κ,λ) is a precaliber pair of $\mathfrak{B}_J\cong\mathfrak{C}_J$, so there is a $\Gamma'\in[\Gamma]^\lambda$ such that $\langle\operatorname{upr}(a_\xi,\mathfrak{C}_J)\rangle_{\xi\in\Gamma'}$ is centered. It follows that $\langle a_\xi\rangle_{\xi\in\Gamma'}$ is centered (525H). As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, we have the result.

(c) Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_{\kappa'}$ such that $\bar\nu_{\kappa'}a_\xi\ge\delta>0$ for every $\xi<\kappa$. Fix a cofinal family $\mathcal{J}$ in $[\kappa']^{\le\omega}$ with cardinal less than $\operatorname{cf}\kappa$. For each $\xi<\kappa$ let $J_\xi\in\mathcal{J}$ be such that $a_\xi\in\mathfrak{C}_{J_\xi}$, where this time $\mathfrak{C}_{J_\xi}$ is interpreted as a subalgebra of $\mathfrak{B}_{\kappa'}$. Then there must be some $J\in\mathcal{J}$ such that $A=\{\xi:J_\xi=J\}$ has cardinal κ. Now $(\mathfrak{C}_J,\bar\nu_{\kappa'}\restriction\mathfrak{C}_J)$ is isomorphic to a subalgebra of $(\mathfrak{B}_\omega,\bar\nu_\omega)$, so has $(\kappa,\lambda,<\theta)$ as a measure-precaliber triple, and there is a $\Gamma\in[A]^\lambda$ such that $\{a_\xi:\xi\in I\}$ has a non-zero lower bound for every $I\in[\Gamma]^{<\theta}$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\lambda,<\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$.

525J Corollary Suppose that κ is an infinite cardinal and $\kappa<\operatorname{cov}\mathcal{N}_\kappa$. Then κ is a measure-precaliber of every probability algebra.

proof By 525Gb, κ is a measure-precaliber of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$; by 525Ia, it is a measure-precaliber of every probability algebra.

525K Proposition Let $\kappa>\operatorname{non}\mathcal{N}_\omega$ be a regular cardinal such that $\operatorname{cf}[\lambda]^{\le\omega}<\kappa$ for every $\lambda<\kappa$ (e.g., $\kappa=\mathfrak{c}^+$, $(\mathfrak{c}^+)^+$, etc.; or $\kappa=\omega_2$ if $\operatorname{non}\mathcal{N}_\omega=\omega_1$). Then κ is a precaliber of every measurable algebra.

proof The point is that κ is a precaliber of $\mathfrak{B}_\lambda$ for every $\lambda<\kappa$. **P** If λ is finite, this is trivial. Otherwise,

$$d(\mathfrak{B}_\lambda)=\operatorname{non}\mathcal{N}_\lambda\le\max(\operatorname{non}\mathcal{N}_\omega,\operatorname{cf}[\lambda]^{\le\omega})<\kappa=\operatorname{cf}\kappa$$

by 524Me and 523I(a-i); it follows that κ is a precaliber of $\mathfrak{B}_\lambda$ (516Lc once more). **Q**

By 525Ib, κ is a precaliber of all measurable algebras.

525L If $\kappa>\mathfrak{c}$ is not a strong limit cardinal we can do a little better than 525K.

Proposition (Džamonja & Plebanek 04) Suppose that λ and κ are infinite cardinals such that $\lambda^\omega<\operatorname{cf}\kappa\le\kappa\le 2^\lambda$, where λ^ω is the cardinal power. Then κ is a precaliber of every measurable algebra.

proof By 525Eb and 525I(a-ii), it is enough to show that κ is a precaliber of $\mathfrak{B}_\kappa$. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\kappa\setminus\{0\}$. Let $\theta:\mathfrak{B}_\kappa\to\mathrm{T}_\kappa$ be a lifting, and for each $\xi<\kappa$ let K_ξ be a non-empty closed subset of θa_ξ which is determined by coordinates in a countable set I_ξ. We may suppose that each I_ξ is infinite; let $h_\xi:\mathbb{N}\to I_\xi$ be a bijection, and set $g_\xi(x)=xh_\xi$ for $x\in\{0,1\}^\kappa$, so that $g_\xi:\{0,1\}^\kappa\to\{0,1\}^{\mathbb{N}}$ is continuous and $K_\xi=g_\xi^{-1}[g_\xi[K_\xi]]$. As $\mathfrak{c}<\operatorname{cf}\kappa$, there is an $L\subseteq\{0,1\}^{\mathbb{N}}$ such that $\Gamma_0=\{\xi:\xi<\kappa,\ g_\xi[K_\xi]=L\}$ has cardinal κ.

Because $\kappa\le 2^\lambda$, there is an $f:\kappa\times\lambda^\omega\to\mathbb{N}$ such that whenever $\langle\xi_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of κ there is an $\eta<\lambda^\omega$ such that $f(\xi_n,\eta)=n$ for every n (5A1Fg). For each $\eta<\lambda^\omega$, set $A_\eta=\{\xi:\xi<\kappa,\ f(h_\xi(n),\eta)=n$ for every $n\in\mathbb{N}\}$; then $\bigcup_{\eta<\lambda^\omega}A_\eta=\kappa$, while $\operatorname{cf}\kappa>\lambda^\omega$, so there is an $\eta^*<\lambda^\omega$ such that $\Gamma=\Gamma_0\cap A_{\eta^*}$ has κ members.

Fix $z\in L$. For ξ, $\eta\in\Gamma$ and i, $j\in\mathbb{N}$,

$$h_\xi(i)=h_\eta(j)\Longrightarrow i=f(h_\xi(i),\eta^*)=f(h_\eta(j),\eta^*)=j.$$

So we can find an $x\in\{0,1\}^\kappa$ such that $x(h_\xi(i))=z(i)$ whenever $\xi\in\Gamma$ and $i\in\mathbb{N}$; that is, $g_\xi(x)=z$ for every $\xi\in\Gamma$. But this means that $x\in g_\xi^{-1}[L]=K_\xi$ for every $\xi\in\Gamma$. It follows that whenever $I\in[\Gamma]^{<\omega}$ then $\bigcap_{\xi\in I}\theta a_\xi\ne\emptyset$ and $\inf_{\xi\in I}a_\xi\ne 0$; that is, that $\{a_\xi:\xi\in\Gamma\}$ is centered. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a precaliber of $\mathfrak{B}_\kappa$.

525M Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and κ an infinite cardinal such that $\operatorname{cf}\kappa$ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ and $\lambda^\omega<\kappa$ for every $\lambda<\kappa$. Then κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

proof If $\kappa=\operatorname{cf}\kappa$, we can stop; so henceforth I will suppose that κ is singular.

(a) Suppose first that $\mathfrak{A} = \mathfrak{B}_I$ for some set I; let $\langle e_i\rangle_{i\in I}$ be the standard generating family in $\mathfrak{B}_I$. If κ is regular, the result is trivial. Otherwise, let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}^+$ such that $\inf_{\xi<\kappa}\bar\nu_I a_\xi = \delta > 0$. There is a strictly increasing family $\langle\kappa_\alpha\rangle_{\alpha<\operatorname{cf}\kappa}$ of regular uncountable cardinals with supremum κ such that $\kappa_0 > \operatorname{cf}\kappa$ and if $\alpha < \operatorname{cf}\kappa$ and $\lambda < \kappa_\alpha$ then $\lambda^\omega < \kappa_\alpha$. **P** All we need to know is that if $\theta < \kappa$ there is a regular uncountable cardinal θ' such that $\theta \le \theta' < \kappa$ and $\lambda^\omega < \theta'$ whenever $\lambda < \theta'$; and $\theta' = (\theta^\omega)^+$ has this property. **Q**

For each $\xi < \kappa$, let $I_\xi \subseteq I$ be a countable set such that a_ξ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\{e_i : i \in I_\xi\}$. By the Δ-system Lemma (5A1J(a-ii)), there is for each $\alpha < \operatorname{cf}\kappa$ a set $\Gamma_\alpha \subseteq \kappa_{\alpha+1}\setminus\kappa_\alpha$ such that $\#(\Gamma_\alpha) = \kappa_{\alpha+1}$ and $\langle I_\xi\rangle_{\xi\in\Gamma_\alpha}$ is a Δ-system with root J_α say. Set $J = \bigcup_{\alpha<\operatorname{cf}\kappa} J_\alpha$, so that $\#(J) \le \operatorname{cf}\kappa$, and

$$\Gamma'_\alpha = \{\xi : \xi \in \Gamma_\alpha,\ (I_\xi \setminus J_\alpha) \cap (J \cup \bigcup_{\eta<\kappa_\alpha} I_\eta) = \emptyset\};$$

then $\#(\Gamma'_\alpha) = \kappa_{\alpha+1}$ for every $\alpha < \operatorname{cf}\kappa$, and $I_\xi \cap I_\eta \subseteq J$ for all distinct ξ, $\eta \in \Gamma' = \bigcup_{\alpha<\operatorname{cf}\kappa}\Gamma'_\alpha$. Let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{e_i : i \in J\}$. For $\xi \in \Gamma'$, set $b_\xi = \operatorname{upr}(a_\xi, \mathfrak{C}_J)$. By 515Ma,

$$\#(\mathfrak{C}_J) \le \max(\omega, \#(J))^\omega < \kappa_{\alpha+1} = \operatorname{cf}\kappa_{\alpha+1},$$

there is for each $\alpha < \operatorname{cf}\kappa$ a $c_\alpha \in \mathfrak{C}_J$ such that $\Gamma''_\alpha = \{\xi : \xi \in \Gamma'_\alpha,\ b_\xi = c_\alpha\}$ has cardinal $\kappa_{\alpha+1}$. Note that

$$\bar\nu_I c_\alpha = \bar\nu_I b_\xi \ge \bar\nu_I a_\xi \ge \delta$$

whenever $\alpha < \operatorname{cf}\kappa$ and $\xi \in \Gamma''_\alpha$.

Now recall that we are supposing that $\operatorname{cf}\kappa$ is a measure-precaliber of $\mathfrak{A}$. So there is a $\Delta \in [\operatorname{cf}\kappa]^{\operatorname{cf}\kappa}$ such that $\{c_\alpha : \alpha \in \Delta\}$ is centered in $\mathfrak{A}$. Now $\Gamma'' = \bigcup_{\alpha\in\Delta}\Gamma''_\alpha$ has cardinal κ, and $\langle b_\xi\rangle_{\xi\in\Gamma''}$ is centered. It follows that $\langle a_\xi\rangle_{\xi\in\Gamma''}$ is centered (525H).

As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $\mathfrak{A}$.

(b) For the general case, observe that by Maharam's theorem (332B) $\mathfrak{A}$ is isomorphic to the simple product $\prod_{k\in K}\mathfrak{A}_{d_k}$ of a countable family of homogeneous principal ideals, where $\langle d_k\rangle_{k\in K}$ is a partition of unity in $\mathfrak{A}$. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa}\bar\mu a_\xi = \delta > 0$. Let $L \subseteq K$ be a finite set such that $\sum_{k\in K\setminus L}\bar\mu d_k = \delta' < \delta$. Then there is some $k \in L$ such that

$$\Gamma_k = \{\xi : \xi < \kappa,\ \bar\mu(a_\xi \cap d_k) \ge \tfrac{\delta-\delta'}{\#(L)}\}$$

has cardinal κ. Since $\operatorname{cf}\kappa$ is a measure-precaliber of $\mathfrak{A}$, it is also a measure-precaliber of $\mathfrak{A}_{d_k}$ (cf. 516Sc). Since $(\mathfrak{A}_{d_k}, \bar\mu\restriction\mathfrak{A}_{d_k})$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_I, \bar\nu_I)$ for some I, (a) tells us that κ is a measure-precaliber of $\mathfrak{A}_{d_k}$. There is therefore a set $\Gamma \in [\Gamma_k]^\kappa$ such that $\langle a_\xi \cap d_k\rangle_{\xi\in\Gamma}$ and $\langle a_\xi\rangle_{\xi\in\Gamma}$ are centered. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $\mathfrak{A}$.

525N Proposition (Argyros & Tsarpalias 82) Let κ be either ω or a strong limit cardinal of countable cofinality, and suppose that $2^\kappa = \kappa^+$. Then κ^+ is not a precaliber of $\mathfrak{B}_\kappa$.

proof By 523Lb, $\operatorname{non}\mathcal{N}_\kappa > \kappa$. So if we enumerate $\{0,1\}^\kappa$ as $\langle x_\xi\rangle_{\xi<\kappa^+}$ and set $E_\xi = \{x_\eta : \eta < \xi\}$ for $\xi < \kappa^+$, $\langle E_\xi\rangle_{\xi<\kappa^+}$ is an increasing family in $\mathcal{N}_\kappa$ with union $\{0,1\}^\kappa$. By 525Cc, κ^+ is not a precaliber of $\mathfrak{B}_\kappa$.

525O As in 523P, GCH decides the most important questions.

Proposition Suppose that the generalized continuum hypothesis is true.

(a) An infinite cardinal κ is a measure-precaliber of every probability algebra iff $\operatorname{cf}\kappa$ is not the successor of a cardinal of countable cofinality.

(b) An infinite cardinal κ is a precaliber of every measurable algebra iff $\operatorname{cf}\kappa$ is neither ω nor the successor of a cardinal of countable cofinality.

proof (a)(i) If κ is a measure-precaliber of every probability algebra, so is $\operatorname{cf}\kappa$ (525Dc). By 525N, $\operatorname{cf}\kappa$ cannot be the successor of a cardinal of countable cofinality.

(ii) Now suppose that $\operatorname{cf}\kappa$ is not the successor of a cardinal of countable cofinality. If $\kappa = \omega$, then certainly κ is a measure-precaliber of every probability algebra (525Fa). Otherwise, $\kappa > \lambda^\omega$ for every $\lambda < \kappa$

and $\mathrm{cf}\,\kappa > \lambda^\omega$ for every $\lambda < \mathrm{cf}\,\kappa$ (5A6Ac). By 525K, $\mathrm{cf}\,\kappa$ is a measure-precaliber of every probability algebra; by 525M, so is κ.

(b) Put (a) and 525Eb together.

***525P** As in 522U, the Freese-Nation number of $\mathcal{P}\mathbb{N}$ is relevant to the questions here.

Proposition $(\mathfrak{m}_{\mathrm{countable}}, \mathrm{FN}^*(\mathcal{P}\mathbb{N}))$ is not a precaliber pair of $\mathfrak{B}_\omega$.

proof By 518D(iv), the Freese-Nation number of the topology of $\{0,1\}^\omega$ is $\mathrm{FN}(\mathcal{P}\mathbb{N})$; the regular Freese-Nation numbers are therefore also equal. We know that $\mathfrak{m}_{\mathrm{countable}}$ is the covering number of the meager ideal of $\mathbb{R}$ (522Sa), and therefore also of the meager ideal of $\{0,1\}^\omega$ (522Wb) and of the nowhere dense ideal of $\{0,1\}^\omega$. By 518E, there is a set $A \subseteq \{0,1\}^\omega$, with cardinal $\mathfrak{m}_{\mathrm{countable}}$, such that $\#(A \cap F) < \mathrm{FN}^*(\mathcal{P}\mathbb{N})$ for every nowhere dense set $F \subseteq \{0,1\}^\omega$.

Fix a nowhere dense compact set $K \subseteq \{0,1\}^\omega$ of non-zero measure. For each $x \in A$, set $a_x = (K+x)^\bullet$ in $\mathfrak{B}_\omega$, where $+$ here is the usual group operation corresponding to the identification $\{0,1\}^\omega \cong \mathbb{Z}_2^\omega$. Then every a_x is non-zero. If $B \subseteq A$ and $\{a_x : x \in B\}$ is centered, then $\{K + x : x \in B\}$ has the finite intersection property, so there is a y in its intersection; now $B \subseteq A \cap (K + y)$, and $K + y$ is nowhere dense, so $\#(B) < \mathrm{FN}^*(\mathcal{P}\mathbb{N})$. Thus $\langle a_x\rangle_{x\in A}$ has no centered subfamily with cardinal $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$ and witnesses that $(\mathfrak{m}_{\mathrm{countable}}, \mathrm{FN}^*(\mathcal{P}\mathbb{N}))$ is not a precaliber pair of $\mathfrak{B}_\omega$.

525Q I turn now to some results which may be interpreted as information on precaliber triples in which the third cardinal is *finite*.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, $\langle u_n\rangle_{n\in\mathbb{N}}$ a $\|\;\|_2$-bounded sequence in $L^2 = L^2(\mathfrak{A}, \bar\mu)^+$, and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. Suppose that $p \in [0,\infty[$ is such that $\sup_{n\in\mathbb{N}} \|u_n^p\|_2$ is finite, and set $v = \lim_{n\to\mathcal{F}} u_n$, $w = \lim_{n\to\mathcal{F}} u_n^p$, the limits being taken for the weak topology in L^2. Then $v^p \le w$.

proof Of course the positive cone of L^2 is closed for the weak topology so $v \ge 0$ and we can speak of v^p. **?** If $v^p \not\le w$, there are α, $\beta \ge 0$ such that $\alpha^p > \beta$ and

$$a = [\![v > \alpha]\!] \setminus [\![w > \beta]\!] \ne 0.$$

Let $b \subseteq a$ be such that $0 < \bar\mu b < \infty$ and consider $u = \frac{1}{\bar\mu b}\chi b$. Then, setting $q = p/(p-1)$ (of course $p > 1$),

$$(\alpha\bar\mu b)^p \le (\int_b v)^p = \lim_{n\to\mathcal{F}}(\int u_n \times \chi b \times \chi b)^p \le \lim_{n\to\mathcal{F}}(\|u_n \times \chi b\|_p \|\chi b\|_q)^p$$

(by Hölder's inequality, 244Eb)

$$= \lim_{n\to\mathcal{F}} (\bar\mu b)^{p/q} \int_b u_n^p = (\bar\mu b)^{p-1} \int_b w \le \beta(\bar\mu b)^p < \alpha^p(\bar\mu b)^p,$$

which is absurd. **X** So $v^p \le w$.

525R Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\langle u_n\rangle_{n\in\mathbb{N}}$ a $\|\;\|_\infty$-bounded sequence in $L^\infty(\mathfrak{A}, \bar\mu)^+$ such that $\delta = \inf_{n\in\mathbb{N}} \int u_n > 0$. Let $k_0, \dots, k_m$ be strictly positive integers with sum k. Suppose that $\gamma < \delta^k$.

(a) There are integers $n_0 < n_1 < \ldots < n_m$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$.

(b) In fact, there is an infinite set $I \subseteq \mathbb{N}$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$ whenever $n_0, \dots, n_m$ belong to I and $n_0 < n_1 < \ldots < n_m$.

proof (a) Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. For each $j \le m$, let v_j be the limit $\lim_{n\to\mathcal{F}} u_n^{k_j}$ for the weak topology on $L^2(\mathfrak{A}, \bar\mu)$; let v be the limit $\lim_{n\to\mathcal{F}} u_n$. By 525Q,

$$\int \prod_{j=0}^m v_j \ge \int \prod_{j=0}^m v^{k_j} = \int v^k = (\|\chi 1\|_q \|v\|_k)^k$$

(where $q = \frac{k}{k-1}$, or ∞ if $k = 1$)

$$\geq \Bigl(\int v \times \chi 1\Bigr)^k$$

(by Hölder's inequality again, if $k > 1$)

$$= \Bigl(\int v\Bigr)^k = \lim_{n\to\mathcal{F}} \Bigl(\int u_n\Bigr)^k \geq \delta^k > \gamma.$$

(Or use 244Xd to show more directly that $\int v^k \geq (\int v)^k$.) We can therefore choose $n_0, \dots, n_m$ inductively so that

$$\int \prod_{j=0}^{s} u_{n_j}^{k_j} \times \prod_{j=s+1}^{m} v_j > \gamma$$

for each $s \leq m$ (interpreting the final product $\prod_{j=m+1}^{m} v_j$ as $\chi 1$), since when we come to choose n_s we shall be able to use any member of

$$\{n : n > n_j \text{ for } j < s,\ \int u_n^{k_s} \times \prod_{j=0}^{s-1} u_{n_j}^{k_j} \times \prod_{j=s+1}^{m} v_j > \gamma\},$$

which belongs to $\mathcal{F}$ so is not empty. At the end of the induction we shall have a sequence $n_0 < \dots < n_m$ such that $\int \prod_{j=0}^{m} u_{n_j}^{k_j} \geq \gamma$, as required.

(b) Let $\mathcal{J} \subseteq [\mathbb{N}]^{m+1}$ be the family of all sets of the form $\{n_0, \dots, n_m\}$ where $n_0 < \dots < n_m$ and $\int \prod_{j=0}^{m} u_{n_j}^{k_j} \geq \gamma$. By (a), applied to subsequences of $\langle u_n\rangle_{n\in\mathbb{N}}$, every infinite subset of $\mathbb{N}$ includes some member of $\mathcal{J}$. By Ramsey's theorem (4A1G), there is an infinite $I \subseteq \mathbb{N}$ such that $[I]^{m+1} \subseteq \mathcal{J}$, which is what we need.

525S Theorem (Fremlin 88) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and κ an infinite cardinal. Let $\langle u_\xi\rangle_{\xi<\kappa}$ be a $\|\,\|_\infty$-bounded family in $L^\infty(\mathfrak{A})^+$. Set $\delta = \inf_{\xi<\kappa} \int u_\xi$. Then for any $k \in \mathbb{N}$ and $\gamma < \delta^{k+1}$ there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=0}^{k} u_{\xi_i} \geq \gamma$ for all $\xi_0, \dots, \xi_k \in \Gamma$.

proof (a) It will be helpful to note straight away that it will be enough to consider the case $(\mathfrak{A}, \bar\mu) = (\mathfrak{B}_I, \bar\nu_I)$ for some set I. **P** There is always a $(\mathfrak{B}_I, \bar\nu_I)$ in which $(\mathfrak{A}, \bar\mu)$ can be embedded. In this case, $L^\infty(\mathfrak{A})$ can be identified, as f-algebra, with a subspace of $L^\infty(\mathfrak{B}_I)$, and the embedding respects integrals. So we can regard $\langle u_\xi\rangle_{\xi<\kappa}$ as a family in $L^\infty(\mathfrak{B}_I)$ and perform all calculations there. **Q**

At the same time, the case $\delta = 0$ is trivial, so let us suppose henceforth that $\delta > 0$.

(b) Next, having fixed on a suitable set I, let $\langle e_i\rangle_{i\in I}$ be the standard generating family in $\mathfrak{B}_I$, and for $J \subseteq I$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_i : i \in J\}$; following 325N, I will say that a member of $\mathfrak{C}_J$ is 'determined by coordinates in J'. For $J \subseteq I$ let $P_J : L^1(\mathfrak{B}_I, \bar\nu_I) \to L^1(\mathfrak{C}_J, \bar\nu_I\restriction\mathfrak{C}_J)$ be the conditional expectation operator. Note that if J, $K \subseteq I$ then $P_J P_K = P_{J\cap K}$ (254Ra/458M(iii)).

It will be useful to start by looking at a particular subset W of $L^\infty(\mathfrak{B}_I)$, being the set of linear combinations $\sum_{i=0}^{n} \alpha_i \chi c_i$ where every α_i is rational and every c_i is determined by coordinates in a finite set. Now $P_J[W] \subseteq W$ for every $J \subseteq I$. **P** If $c \in \mathfrak{C}_K$ where $K \subseteq I$ is finite, then

$$P_J(\chi c) = P_J P_K(\chi c) = P_{J\cap K}(\chi c) = \sum_{d \text{ is an atom of } \mathfrak{C}_{J\cap K}} \frac{\bar\nu_I(c\cap d)}{\bar\nu_I d} \chi d \in W.$$

As P_J is linear, this is enough. **Q** Observe also that if $K \subseteq I$ is finite, then $P_K[W]$ is countable, being the set of rational linear combinations of $\{\chi c : c \in \mathfrak{C}_K\}$.

(c) Suppose, therefore, that we have a set I, a $\|\,\|_\infty$-bounded family $\langle u_\xi\rangle_{\xi<\kappa}$ in $L^\infty(\mathfrak{B}_I)^+$ with $\inf_{\xi<\kappa} \int u_\xi = \delta > 0$, a $k \in \mathbb{N}$ and a $\gamma < \delta^{k+1}$. To begin with, let us suppose further that

(α) $u_\xi \leq \chi 1$ for every $\xi < \kappa$,

(β) $u_\xi \in W$, as described in (b), for each $\xi < \kappa$;

for each $\xi < \kappa$, let $I_\xi \in [I]^{<\omega}$ be such that $P_{I_\xi} u_\xi = u_\xi$.

(i) Suppose that $\kappa = \omega$. Because there are only finitely many sequences $k_0, \dots, k_m$ of strictly positive integers with sum equal to $k + 1$, we can use 525Rb a finite number of times to find an infinite $\Gamma \subseteq \omega$ such that $\int \prod_{j=0}^{m} u_{n_j}^{k_j} \geq \gamma$ whenever $\sum_{j=0}^{m} k_j = k + 1$ and $n_0 < \dots < n_m$ belong to Γ. But in this case we surely have $\int \prod_{i=0}^{k} u_{n_i} \geq \gamma$ for all $n_0, \dots, n_k \in \Gamma$.

(ii) Next, suppose that $\kappa > \omega$ is regular. By the Δ-system Lemma (4A1Db) there is a $\Delta \in [\kappa]^\kappa$ such that $\langle I_\xi \rangle_{\xi\in\Delta}$ is a Δ-system with root J say. Since $P_J[W]$ is countable, there is a v such that $\Gamma = \{\xi : \xi \in \Delta,\, P_J u_\xi = v\}$ has cardinal κ. Of course

$$\int v = \int u_\xi \ge \delta$$

for every $\xi \in \Gamma$.

By 458Lg again, $\langle \mathfrak{C}_{I_\xi} \rangle_{\xi\in\Delta}$ is relatively independent over $\mathfrak{C}_J$. Now suppose that $\xi_0, \dots , \xi_k$ belong to Γ. Then

$$\int \prod_{i=0}^{k} u_{\xi_i} \ge \int \prod_{i=0}^{k} P_J u_{\xi_i}$$

(458Lh)

$$= \int v^{k+1} \ge \Bigl(\int v\Bigr)^{k+1}$$

(as in the proof of 525Ra)

$$\ge \delta^{k+1} \ge \gamma,$$

and this is what we need to know.

(iii) Finally, suppose that $\kappa > \mathrm{cf}\,\kappa \ge \omega$. Set $\lambda = \mathrm{cf}\,\kappa$ and let $\langle \kappa_\zeta \rangle_{\zeta<\lambda}$ be a strictly increasing family of regular cardinals greater than λ and with supremum κ. For each $\zeta < \lambda$ let $\Delta_\zeta \subseteq \kappa_{\zeta+1} \setminus \kappa_\zeta$ be a set with cardinal $\kappa_{\zeta+1}$ such that $\langle I_\xi \rangle_{\xi\in\Delta_\zeta}$ is a Δ-system with root J_ζ say. Set $J = \bigcup_{\zeta<\lambda} J_\zeta$; note that $\#(J) \le \lambda < \kappa_{\zeta+1}$ for every ζ, so that

$$\Delta'_\zeta = \{\xi : \xi \in \Delta_\zeta,\, (I_\xi \setminus J_\zeta) \cap (J \cup \textstyle\bigcup_{\eta<\kappa_\zeta} I_\eta) = \emptyset\}$$

still has cardinal $\kappa_{\zeta+1}$.

If $\zeta < \lambda$ and $\xi \in \Delta'_\zeta$, $I_\xi \cap J$ is included in the finite set J_ζ; so $\{P_J u_\xi : \xi \in \Delta'_\zeta\}$ is countable, and there is a v_ζ such that $\Delta''_\zeta = \{\xi : \xi \in \Delta'_\zeta,\, P_J u_\xi = v_\zeta\}$ has cardinal $\kappa_{\zeta+1}$. Note that

$$\int v_\zeta = \int u_\xi \ge \delta$$

whenever $\zeta < \lambda$ and $\xi \in \Delta''_\zeta$.

Because λ is regular, we can apply (i) or (ii) above to find an $A \in [\lambda]^\lambda$ such that $\int \prod_{i=0}^{k} v_{\zeta_i} \ge \gamma$ whenever $\zeta_0, \dots , \zeta_k \in A$. Set $\Gamma = \bigcup_{\zeta\in A} \Delta''_\zeta$; because A must be cofinal with λ, $\#(\Gamma) = \kappa$.

If $\xi,\, \eta \in \Gamma$ are distinct, then $I_\xi \cap I_\eta \subseteq J$. So $\langle \mathfrak{C}_{I_\xi} \rangle_{\xi\in\Gamma}$ is relatively independent over $\mathfrak{C}_J$. Take any $\xi_0, \dots , \xi_k \in \Gamma$; for each $i \le k$, let $\zeta_i \in A$ be such that $\xi_i \in \Delta''_{\zeta_i}$. By 458Lh again,

$$\int \textstyle\prod_{i=0}^{n} u_{\xi_i} \ge \int \prod_{i=0}^{n} P_J u_{\xi_i} = \int \prod_{i=0}^{n} v_{\zeta_i} \ge \gamma,$$

so we are done (provided (α)-(β) are true).

(d) Now let us unwind these conditions from the bottom.

(i) If (α) is true, but (β) is not, take $\epsilon \in \,]0, \delta[$ such that $(\delta - \epsilon)^{k+1} > \gamma + (k + 1)\epsilon$. For each $\xi < \kappa$, let $u'_\xi \in W$ be such that $u'_\xi \le \chi 1$ and $\int |u_\xi - u'_\xi| \le \epsilon$. (Such a u'_ξ exists because $\bigcup\{\mathfrak{C}_K : K \in [I]^{<\omega}\}$ is topologically dense in $\mathfrak{B}_I$ and $u \wedge \chi 1 \in W$ for every $u \in W$.) Then $\int u'_\xi \ge \delta - \epsilon$ for each ξ, so we can apply (c) to $\langle u'_\xi \rangle_{\xi<\kappa}$ to see that there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=0}^{k} u'_{\xi_i} \ge \gamma + (k + 1)\epsilon$ for all $\xi_0, \dots , \xi_k \in \Gamma$. Now (because u_ξ and u'_ξ all lie between 0 and $\chi 1$) we have

$$|\textstyle\prod_{i=0}^{k} u_{\xi_i} - \prod_{i=0}^{k} u'_{\xi_i}| \le \sum_{i=0}^{k} |u_{\xi_i} - u'_{\xi_i}|$$

(see 285O), so that

$$\int \textstyle\prod_{i=0}^{k} u_{\xi_i} \ge \int \prod_{i=0}^{k} u'_{\xi_i} - \sum_{i=0}^{k} \int |u_{\xi_i} - u'_{\xi_i}| \ge \gamma$$

whenever $\xi_0, \dots , \xi_n \in \Gamma$, and the theorem is still true.

(ii) Finally, for the general case, set $M = 1 + \sup_{\xi<\kappa} \|u_\xi\|_\infty$, and $u'_\xi = \frac{1}{M} u_\xi$ for $\xi < \kappa$. Then every u'_ξ belongs to $[0, \chi 1]$ and $\int u'_\xi \ge \frac{\delta}{M}$. By (i) there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=0}^k u'_{\xi_i} \ge \frac{\gamma}{M^{k+1}}$ for all $\xi_0, \dots, \xi_k \in \Gamma$; in which case $\int \prod_{i=0}^k u_{\xi_i} \ge \gamma$ for all $\xi_0, \dots, \xi_k \in \Gamma$.

This completes the proof.

525T Corollary (Argyros & Kalamidas 82) (a) If κ is an infinite cardinal and $k \in \mathbb{N}$, (κ, κ, k) is a measure-precaliber triple of every probability algebra.

(b) If κ is a cardinal of uncountable cofinality and $k \in \mathbb{N}$, (κ, κ, k) is a precaliber triple of every measurable algebra. In particular, every measurable algebra satisfies Knaster's condition.

(c) If κ is a cardinal of uncountable cofinality, $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $k \ge 1$ and $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A} \setminus \{0\}$, then there are a $\delta > 0$ and a $\Gamma \in [\kappa]^\kappa$ such that $\bar{\mu}(\inf_{\xi \in I} a_\xi) \ge \delta$ for every $I \in [\Gamma]^k$.

(d) For any measurable algebra $\mathfrak{A}$, $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{K}}$; and if $\mathfrak{m}(\mathfrak{A}) > \omega_1$, then $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. So if $\omega \le \kappa < \mathfrak{m}_K$, κ is a measure-precaliber of every probability algebra.

proof Really this is just the special case of 525S in which every u_ξ belongs to $\{\chi a : a \in \mathfrak{A}\}$.

(a) If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar{\mu} a_\xi = \delta > 0$, take any $\gamma \in \left]0, \delta^k\right[$. Setting $u_\xi = \chi a_\xi$ for each ξ, $\int u_\xi \ge \delta$ for each ξ, so there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=1}^k u_{\xi_i} \ge \gamma$ for every $\xi_1, \dots, \xi_k \in \Gamma$; in which case $\inf_{\xi \in J} a_\xi \ne 0$ for every $J \in [\Gamma]^k$. As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, (κ, κ, k) is a measure-precaliber triple of $(\mathfrak{A}, \bar{\mu})$.

(b) This now follows at once from 525Db, since any non-zero measurable algebra can be given a probability measure. Taking $\kappa = \omega_1$ and $k = 2$, we have Knaster's condition.

(c) For the quantitative version, we have only to note that there must be some $\alpha > 0$ such that $\#(\{\xi : \bar{\mu} a_\xi \ge \alpha\}$ has cardinal κ, and take $\delta < \alpha^k$.

(d) By (b), $\mathfrak{A}$ satisfies Knaster's condition; it follows at once that $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{K}}$, while $\operatorname{sat}(\mathfrak{A}) \le \omega_1$. If $\mathfrak{m}(\mathfrak{A}) > \omega_1$, then ω_1 is a precaliber of $\mathfrak{A}$ (517Ig) so $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. By 525Fb, every infinite cardinal less than $\mathfrak{m}_{\mathrm{K}}$ is a measure-precaliber of every probability algebra.

525X Basic exercises (a) Let (X, Σ, μ) be any measure space and $\mathfrak{A}$ its measure algebra. (i) Show that $(\mathfrak{A}^+, \supseteq) \preccurlyeq_{\mathrm{T}} (\Sigma \setminus \mathcal{N}(\mu), \supseteq)$. (ii) Show that a pair (κ, λ) is a downwards precaliber pair of $\Sigma \setminus \mathcal{N}(\mu)$ iff it is a precaliber pair of $\mathfrak{A}$.

>(b) Let $\mathfrak{A}$ be a measurable algebra. Show that ω_1 is a precaliber of $\mathfrak{A}$ iff *either* $\mathfrak{A}$ is purely atomic *or* $\tau(\mathfrak{A}) \le \omega$ and $\operatorname{cov} \mathcal{N}_\omega > \omega_1$ *or* $\operatorname{cov} \mathcal{N}_{\omega_1} > \omega_1$. (*Hint*: 525G, 523F.)

>(c) (i) Suppose that $\operatorname{add} \mathcal{N}_\omega = \operatorname{cov} \mathcal{N}_\omega = \kappa$. Show that κ is not a precaliber of $\mathfrak{B}_\omega$. (ii) Suppose that $\operatorname{non} \mathcal{N}_\omega = \mathfrak{c}$. Show that $\mathfrak{c}$ is not a precaliber of $\mathfrak{B}_\omega$.

(d) Let (X, Σ, μ) be a complete strictly localizable measure space and $\mathfrak{A}$ its measure algebra. Show that the supported relation $(\Sigma \setminus \mathcal{N}(\mu), \ni, X)$ has the same precaliber pairs as the Boolean algebra $\mathfrak{A}$.

(e) Suppose that (κ, λ) is a precaliber pair of every measurable algebra, that I is a set, and that $X \subseteq \mathbb{R}^I$ is a compact set such that $\#(\{i : x(i) \ne 0\}) < \lambda$ for every $x \in X$. Show that $\#(\{i : x(i) \ne 0\}) < \kappa$ for every x belonging to the closed convex hull of X in $\mathbb{R}^I$. (*Hint*: 461I.)

(f) Suppose that $\lambda \le \kappa$ are infinite cardinals, $(\mathfrak{A}, \bar{\mu})$ is a homogeneous probability algebra, and that $\gamma < 1$ is such that whenever $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ and $\bar{\mu} a_\xi \ge \gamma$ for every $\xi < \kappa$, there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in \Gamma\}$ is centered. Show that (κ, λ) is a measure-precaliber pair of $(\mathfrak{A}, \bar{\mu})$. (*Hint*: given that $\inf_{\xi<\kappa} \bar{\mu} a_\xi > 0$, take $(\mathfrak{C}, \bar{\lambda}) = \widehat{\bigotimes}_m (\mathfrak{A}, \bar{\mu}) \cong (\mathfrak{A}, \bar{\mu})$ to be the probability algebra free product of a large finite number of copies of $(\mathfrak{A}, \bar{\mu})$, and consider $c_\xi = \sup_{j<m} \varepsilon_j a_\xi$ for $\xi < \kappa$.)

(g) Let $\mathfrak{A}$ be a Boolean algebra, and λ, κ cardinals such that (κ, λ) is a measure-precaliber pair of every probability algebra. Suppose that $A \subseteq \mathfrak{A} \setminus \{0\}$ has positive intersection number. Show that if $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in A, then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in \Gamma\}$ is centered.

(h) Let κ be a cardinal such that (α) $\lambda^\omega < \kappa$ for every $\lambda < \kappa$ (β) $\lambda^\omega < \operatorname{cf}\kappa$ for every $\lambda < \operatorname{cf}\kappa$. Show that κ is a measure-precaliber of every probability algebra.

(i) Show that if $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon measure space and $\mu X > 0$, then $\operatorname{cov}\mathcal{N}(\mu) \ge \mathfrak{m}_{\mathrm{K}}$. (*Hint*: assemble 524Md, 524Pc, 525Tb.)

525Z Problem Can we, in ZFC, find an infinite cardinal κ which is not a measure-precaliber of all probability algebras? From 525N we see that a negative answer will require a model of set theory in which $2^\kappa > \kappa^+$ for all strong limit cardinals κ of countable cofinality; for such models see Foreman & Woodin 91, Cummings 92.

525 Notes and comments There seem to be three methods of proving that a cardinal is a precaliber of a measure algebra. First, we have the counting arguments of 516L; since we know something about the centering numbers of measure algebras (524Me), this gives us a start (see the proof of 525K). Next, we can try to use Martin numbers, as in 517Ig and 525F; since we can relate the Martin number of a measure algebra to the cardinals of §523 (524Md), we get the formulation 525J. In third place, we have arguments based on the special structure of measure algebras, using 525H to apply Δ-system theorems from infinitary combinatorics. Subject to the generalized continuum hypothesis, these ideas are enough to answer the most natural questions (525O). Without this simplification, they leave conspicuous gaps. The most important seems to be 525Z. Even if we know all the cardinals $\operatorname{add}\mathcal{N}_\kappa$, $\operatorname{cov}\mathcal{N}_\kappa$, $\operatorname{non}\mathcal{N}_\kappa$ and $\operatorname{cf}\mathcal{N}_\kappa$ of §523, we may still not be able to determine which cardinals are precalibers; 525Xb is an exceptional special case.

I have presented this section with a bias towards measure-precalibers rather than precalibers. When there is a difference, the former search deeper. 'Cofinality ω_1' has a rather special position in this theory (525Ib), deriving from the combinatorial arguments of 5A1I.

Version of 24.1.14

526 Asymptotic density zero

In §491, I devoted some paragraphs to the ideal $\mathcal{Z}$ of subsets of $\mathbb{N}$ with asymptotic density zero, as part of an investigation into equidistributed sequences in topological measure spaces. Here I return to $\mathcal{Z}$ to examine its place in the Tukey ordering of partially ordered sets. We find that it lies strictly between $\mathbb{N}^{\mathbb{N}}$ and ℓ^1 (526B, 526J, 526L) but in some sense is closer to ℓ^1 (526Ga). On the way, I mention the ideal $\mathcal{N}\mathit{wd}$ of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$ (526H-526L) and ideals of sets with negligible closures (526I-526M).

526A Proposition For $I \subseteq \mathbb{N}$, set $\nu I = \sup_{n\ge 1} \frac{1}{n}\#(I\cap n)$.

(a) ν is a strictly positive submeasure (definition: 392A) on $\mathcal{P}\mathbb{N}$. We have a metric ρ on $\mathcal{P}\mathbb{N}$ defined by setting $\rho(I,J) = \nu(I\triangle J)$ for all I, $J \subseteq \mathbb{N}$, under which the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and upper asymptotic density $d^* : \mathcal{P}\mathbb{N} \to [0,1]$ are uniformly continuous and $\mathcal{P}\mathbb{N}$ is complete.

(b) $\mathcal{Z}$ is a separable closed subset of $\mathcal{P}\mathbb{N}$.

(c) If $\mathcal{I} \subseteq \mathcal{Z}$ is such that $\sum_{I\in\mathcal{I}} \nu I$ is finite, then $\bigcup\mathcal{I} \in \mathcal{Z}$.

(d) With the subspace topology, $(\mathcal{Z},\subseteq)$ is a metrizably compactly based directed set (definition: 513K).

proof (a) It is elementary to check that ν is a strictly positive submeasure. By 392H, ρ is a metric under which the Boolean operations are uniformly continuous. Since

$$|d^*(I) - d^*(J)| \le d^*(I\triangle J) \le \nu(I\triangle J)$$

for all I, $J \subseteq \mathbb{N}$, d^* is uniformly continuous. Let $\langle I_j\rangle_{j\in\mathbb{N}}$ be a sequence in $\mathcal{P}\mathbb{N}$ such that $\rho(I_j,I_{j+1}) \le 2^{-j}$ for every $j \in \mathbb{N}$. Set $I = \bigcap_{m\in\mathbb{N}}\bigcup_{j\ge m} I_j$. For any $m \in \mathbb{N}$ and $n \ge 1$,

$$\frac{1}{n}\#(n\cap(I_m\triangle I)) \le \frac{1}{n}\sum_{j=m}^\infty \#(n\cap(I_j\triangle I_{j+1})) \le \sum_{j=m}^\infty \rho(I_j,I_{j+1}) \le 2^{-m+1},$$

so $\rho(I_m,I) \le 2^{-m+1}$. Thus $\langle I_j\rangle_{j\in\mathbb{N}}$ is convergent to I; as $\langle I_j\rangle_{j\in\mathbb{N}}$ is arbitrary, $\mathcal{P}\mathbb{N}$ is complete (cf. 2A4E).

(b) $\mathcal{Z}$ is a closed subset of $\mathcal{P}\mathbb{N}$. **P** If I belongs to the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$, and $\epsilon > 0$, let $J \in \mathcal{Z}$ be such that $\rho(I, J) \le \frac{1}{2}\epsilon$, and let $m \ge 1$ be such that $\frac{1}{n}\#(J \cap n) \le \frac{1}{2}\epsilon$ for every $n \ge m$; then $\frac{1}{n}\#(I \cap n) \le \epsilon$ for every $n \ge m$. As ϵ is arbitrary, $I \in \mathcal{Z}$; as I is arbitrary, $\mathcal{Z}$ is closed. **Q**

$\mathcal{Z}$ is separable because $[\mathbb{N}]^{<\omega}$ is a countable dense set. (If $I \in \mathcal{Z}$ and $n \in \mathbb{N}$, $\rho(a, a \cap n) \le \sup_{m>n} \frac{1}{m}\#(m \cap I)$.)

(c) Let $\epsilon > 0$. Then there is a finite $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\sum_{I \in \mathcal{I}\setminus\mathcal{I}_0} \nu I \le \epsilon$. Set $J = \bigcup \mathcal{I}$, $J_0 = \bigcup \mathcal{I}_0$; then $J_0 \in \mathcal{Z}$ so there is an $n_0 \in \mathbb{N}$ such that $\#(J_0 \cap n) \le n\epsilon$ for every $n \ge n_0$. If $n \ge n_0$, then

$$\#(J \cap n) \le \#(J_0 \cap n) + \sum_{I \in \mathcal{I}\setminus\mathcal{I}_0} \#(I \cap n) \le n\epsilon + \sum_{I \in \mathcal{I}\setminus\mathcal{I}_0} n\nu I \le 2n\epsilon.$$

As ϵ is arbitrary, $J \in \mathcal{Z}$.

(d) $\mathcal{Z}$ is closed under $\cup$, so is a directed set under $\subseteq$, and $\cup : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ is continuous. If $a \in \mathcal{Z}$, then on $\{b : b \subseteq a\}$ the topology $\mathfrak{T}_\rho$ induced by ρ agrees with the usual compact Hausdorff topology $\mathfrak{S}$ of $\mathcal{P}\mathbb{N} \cong \{0,1\}^{\mathbb{N}}$. **P** If $n \in \mathbb{N}$ and $\rho(b, c) < \frac{1}{n+1}$, then $b \cap n = c \cap n$; so $\mathfrak{T}_\rho$ is finer than $\mathfrak{S}$ on $\mathcal{P}\mathbb{N}$. If $\epsilon > 0$, there is an $m \in \mathbb{N}$ such that $\#(a \cap n) \le n\epsilon$ whenever $n \ge m$; now $\rho(b, c) \le \epsilon$ whenever b, $c \subseteq a$ and $b \cap m = c \cap m$. So $\mathfrak{S}$ is finer than $\mathfrak{T}_\rho$ on $\{b : b \subseteq a\}$. **Q** Since $\{b : b \subseteq a\}$ is $\mathfrak{S}$-compact, it is $\mathfrak{T}_\rho$-compact.

Now suppose that $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{Z}$ with $\mathfrak{T}_\rho$-limit a. Then it has a subsequence $\langle a_{n_k} \rangle_{k\in\mathbb{N}}$ such that $\rho(a, a_{n_k}) \le 2^{-k}$ for every k. Set $b = \bigcup_{k\in\mathbb{N}} a_{n_k}$. Then, given $\epsilon > 0$, let r, $m \in \mathbb{N}$ be such that $2^{-r} \le \epsilon$ and $\#(n \cap (a \cup \bigcup_{k\le r} a_{n_k})) \le n\epsilon$ for every $n \ge m$; then

$$\begin{aligned}\#(n \cap b) &\le \#(n \cap (a \cup \bigcup_{k\le r} a_{n_k})) + \sum_{k=r+1}^{\infty} \#(n \cap a_{n_k} \setminus a)\\ &\le n\epsilon + \sum_{k=r+1}^{\infty} 2^{-k} n \le 2n\epsilon\end{aligned}$$

for every $n \ge m$. So $b \in \mathcal{Z}$ and $\{a_{n_k} : k \in \mathbb{N}\}$ is bounded above in $\mathcal{Z}$.

526B Proposition (Fremlin 91) $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{Z} \preccurlyeq_{\mathrm{T}} \ell^1$.

proof (a) For $\alpha \in \mathbb{N}^{\mathbb{N}}$, set

$$\phi(\alpha) = \{2^n i : n \in \mathbb{N},\, i \le \alpha(n)\}.$$

Then $\phi(\alpha) \in \mathcal{Z}$, because if $k \in \mathbb{N}$ then

$$\#(m \cap \phi(\alpha)) \le \sum_{n=0}^{k} \alpha(n) + \lceil 2^{-k} m \rceil$$

for every m. Also $\phi : \mathbb{N}^{\mathbb{N}} \to \mathcal{Z}$ is a Tukey function, because if $\phi(\alpha) \subseteq a \in \mathcal{Z}$ then $\alpha(n) \le \min\{i : 2^n i \notin a\}$ for every $n \in \mathbb{N}$. So $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

(b) Give $\mathcal{Z}$ the metric ρ of 526A. Then $\mathcal{Z}$ is complete and separable and the lattice operation $\cup$ is uniformly continuous (526Aa). By 524C, $(\mathcal{Z}, \subseteq', [\mathcal{Z}]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\omega), \le, \ell^1(\omega))$. Since $\mathcal{Z}$ is upwards-directed, $(\mathcal{Z}, \subseteq, \mathcal{Z}) \equiv_{\mathrm{GT}} (\mathcal{Z}, \subseteq', [\mathcal{Z}]^{<\omega})$ (513Id) and $(\mathcal{Z}, \subseteq, \mathcal{Z}) \preccurlyeq_{\mathrm{GT}} (\ell^1, \le, \ell^1)$, that is, $\mathcal{Z} \preccurlyeq_{\mathrm{T}} \ell^1$.

526C The next three lemmas are steps on the way to Theorem 526F. I give them in much more generality than is required by that theorem because a couple of them will be useful later, and I think they are interesting in themselves. But if you are reading this primarily for the sake of 526F, you might save time by looking ahead to the proof there and working backwards, extracting arguments adequate for the special case of 526E which is actually required.

Lemma Let $\langle (\mathfrak{A}_n, \bar\mu_n) \rangle_{n\in\mathbb{N}}$ be a sequence of purely atomic probability algebras, and $\mathfrak{A} = \prod_{n\in\mathbb{N}} \mathfrak{A}_n$ the simple product algebra. Then there is an order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathcal{P}\mathbb{N}$ such that $\limsup_{n\to\infty} \bar\mu_n a(n)$ is the upper asymptotic density $d^*(\pi a)$ for every $a \in \mathfrak{A}$; consequently, $\lim_{n\to\infty} \bar\mu_n a(n)$ is the asymptotic density $d(\pi a)$ of πa if either is defined.

proof (a) For each $n \in \mathbb{N}$, let C_n be the set of atoms of $\mathfrak{A}_n$, and choose rational numbers $\alpha_n(c)$ such that $\alpha_n(c) \le \bar\mu_n c$ for each $c \in C_n$, $\sum_{c\in C_n} \alpha_n(c) > 1 - 2^{-n}$, and $\{c : c \in C_n,\, \alpha_n(c) > 0\}$ is finite. Express $\alpha_n(c)$

as $r_n(c)/s_n$ for each $c \in C_n$, where $r_n(c) \in \mathbb{N}$ and $s_n \in \mathbb{N} \setminus \{0\}$; let $\langle I_n(c)\rangle_{c\in C_n}$ be a disjoint family of subsets of $\mathbb{N}$ with $\#(I_n(c)) = r_n(c)$ for each c, and set $J_n = \bigcup_{c\in C_n} I_n(c)$; let $\pi_n : \mathfrak{A}_n \to \mathcal{P}J_n$ be the Boolean homomorphism such that $\pi_n(c) = I_n(c)$ for each $c \in C_n$. Then

$$(1-2^{-n})s_n < s_n\sum_{c\in C_n}\alpha_n(c) = \sum_{c\in C_n} r_n(c) = \#(J_n) \le s_n.$$

Note that $\#(J_n) > 0$. Also

$$\begin{aligned}\#(J_n)(\bar\mu_n d - 2^{-n}) &\le \#(J_n)\sum_{c\in C_n, c\subseteq d}\alpha_n(c)\\ &\le s_n\sum_{c\in C_n, c\subseteq d}\alpha_n(c) = \sum_{c\in C_n, c\subseteq d} r_n(c) = \#(\pi_n d)\end{aligned}$$

and

$$(1-2^{-n})\#(\pi_n d) = (1-2^{-n})s_n\sum_{c\in C_n, c\subseteq d}\alpha_n(c) \le \#(J_n)\cdot\bar\mu_n d$$

for every $d \in \mathfrak{A}_n$. So, for $a \in \mathfrak{A}$,

$$\limsup_{n\to\infty}\bar\mu_n a(n) = \limsup_{n\to\infty}\frac{\#(\pi_n a(n))}{\#(J_n)}.$$

(b) Let $\langle m_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that

$$m_n\#(J_n) \ge 2^n\max(\#(J_{n+1}), \sum_{i<n} m_i\#(J_i))$$

for every n. Set $n_k = n$ if $\sum_{i<n} m_i \le k < \sum_{i\le n} m_i$ where $n \in \mathbb{N}$; then $\lim_{k\to\infty} n_k = \infty$. Set $l_k = \sum_{i<k}\#(J_{n_i})$, so that $l_{k+1} - l_k = \#(J_{n_k})$ for each k, and let $\phi_k : \mathcal{P}J_{n_k} \to \mathcal{P}(l_{k+1}\setminus l_k)$ be a Boolean isomorphism; set

$$\pi a = \bigcup_{k\in\mathbb{N}}\phi_k\pi_{n_k}a(n_k)$$

for $a \in \mathfrak{A}$, so that $\pi : \mathfrak{A} \to \mathcal{P}\mathbb{N}$ is an order-continuous Boolean isomorphism.

(c) Let $a \in \mathfrak{A}$, and set

$$\gamma = \limsup_{n\to\infty}\bar\mu_n a(n) = \limsup_{n\to\infty}\frac{\#(\pi_n a(n))}{\#(J_n)},$$

$$\gamma' = \limsup_{l\to\infty}\frac{1}{l}\#(l\cap\pi a).$$

Then $\gamma \le \gamma'$. **P** Setting $l'_n = \sum_{i<n} m_i\#(J_i)$, we have $\#(l'_{n+1}\cap\pi a) \ge m_n\#(\pi_n a(n))$, while $l'_{n+1} = l'_n + m_n\#(J_n) \le (1+2^{-n})m_n\#(J_n)$ for each n; but this means that

$$\gamma' \ge \limsup_{n\to\infty}\frac{1}{l'_{n+1}}\#(l'_{n+1}\cap\pi a) \ge \limsup_{n\to\infty}\frac{m_n\#(\pi_n a(n))}{(1+2^{-n})m_n\#(J_n)} = \gamma. \textbf{ Q}$$

Also $\gamma' \le \gamma$. **P** Let $\epsilon > 0$. Let $n^* \ge 1$ be such that $2^{-n^*} \le \epsilon$ and $\#(\pi_n a(n)) \le (\gamma+\epsilon)\#(J_n)$ for every $n \ge n^*$. Suppose that $l \ge l'_{n^*+1}$. Then l is of the form $l'_{n+1} + j\#(J_{n+1}) + i$ where $n \ge n^*$, $j < m_{n+1}$ and $i < \#(J_{n+1})$. Now $l'_{n+1} = l'_n + m_n\#(J_n)$, so

$$\begin{aligned}\#(l'_{n+1}\cap\pi a) &\le l'_n + m_n\#(\pi_n a(n)) \le l'_n + m_n\#(J_n)(\gamma+\epsilon)\\ &\le m_n\#(J_n)(\gamma+\epsilon+2^{-n}) \le m_n\#(J_n)(\gamma+2\epsilon)\end{aligned}$$

by the choice of m_n. Accordingly

$$\begin{aligned}\#(l\cap\pi a) &\le m_n\#(J_n)(\gamma+2\epsilon) + (j+1)\#(\pi_{n+1}a(n+1))\\ &\le m_n\#(J_n)(\gamma+2\epsilon) + j(\gamma+\epsilon)\#(J_{n+1}) + \#(J_{n+1})\\ &\le (\gamma+2\epsilon)l + \#(J_{n+1}) \le (\gamma+2\epsilon)l + 2^{-n}m_n\#(J_n)\end{aligned}$$

(by the choice of m_n)

$$\leq (\gamma+3\epsilon)l.$$

As this is true for any $l \geq l'_{n^*+1}$, $\gamma' \leq \gamma+3\epsilon$; as ϵ is arbitrary, $\gamma' \leq \gamma$. **Q**

(d) Thus

$$\limsup_{n\to\infty} \bar{\mu}_n a(n) = \limsup_{n\to\infty} \frac{1}{n}\#(n \cap \pi a)$$

for every $a \in \mathfrak{A}$. But as π is a Boolean homomorphism, it follows at once that

$$\liminf_{n\to\infty} \bar{\mu}_n a(n) = \liminf_{n\to\infty} \frac{1}{n}\#(n \cap \pi a)$$

for every a, so that the limits are equal if either is defined.

526D Lemma Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and $\kappa \geq \max(\omega, c(\mathfrak{A}), \tau(\mathfrak{A}))$ a cardinal. Let $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$ be the measure algebra of the usual measure on $\{0,1\}^\kappa$, and $\gamma > 0$. Then there is a function $\theta : \mathfrak{A} \to \mathfrak{B}_\kappa$ such that

(i) $\theta(\sup A) = \sup \theta[A]$ for every non-empty $A \subseteq \mathfrak{A}$ such that $\sup A$ is defined in $\mathfrak{A}$;

(ii) $\bar{\nu}_\kappa\theta(a) = 1 - e^{-\gamma\bar{\mu}a}$ for every $a \in \mathfrak{A}$, interpreting $e^{-\infty}$ as 0;

(iii) if $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$, and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{\theta(a) : a \subseteq a_i\}$ for each i, then $\langle\mathfrak{C}_i\rangle_{i\in I}$ is stochastically independent.

proof (a) By 495M[5], we have exactly this result for some probability algebra $(\mathfrak{B},\bar{\lambda})$ in place of $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$. Set $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$, and give $\mathfrak{A}^f$ its measure metric ρ (323Ad). Then $\theta\restriction\mathfrak{A}^f$ is uniformly continuous for ρ and the measure metric σ of $\mathfrak{B}$. **P** If $\epsilon > 0$, there is a $\delta > 0$ such that $|e^{-\gamma s} - e^{-\gamma t}| \leq \frac{1}{2}\epsilon$ whenever s, $t \in [0,\infty[$ and $|s-t| \leq \delta$. Now if a, $a' \in \mathfrak{A}$ and $\bar{\mu}(a \triangle a') \leq \delta$, set $b = a \cap a'$; then $\theta(b) \subseteq \theta(a)$ and $\bar{\mu}a - \bar{\mu}b \leq \delta$, so

$$\sigma(\theta(a),\theta(b)) = \bar{\lambda}(\theta(a)\setminus\theta(b)) = \bar{\lambda}\theta(a) - \bar{\lambda}\theta(b) = e^{-\gamma\bar{\mu}b} - e^{-\gamma\bar{\mu}a} \leq \frac{1}{2}\epsilon.$$

Similarly, $\sigma(\theta(a'),\theta(b)) \leq \frac{1}{2}\epsilon$ so $\sigma(\theta(a),\theta(a')) \leq \epsilon$. As ϵ is arbitrary, this gives the result. **Q**

(b) By 521Eb, there is a set $B \subseteq \mathfrak{A}^f$, of cardinal at most κ, which is dense for ρ. Accordingly $\theta[B]$ is dense in $f[\mathfrak{A}^f]$ for σ. Taking $\mathfrak{D}$ to be the closed subalgebra of $\mathfrak{B}$ generated by $\theta[B]$, $\tau(\mathfrak{D}) \leq \kappa$ and $\theta[\mathfrak{A}^f] \subseteq \mathfrak{D}$. But if $a \in \mathfrak{A}\setminus\mathfrak{A}^f$ then $\theta(a) = 1$, so $\theta[\mathfrak{A}] \subseteq \mathfrak{D}$. Now there is a measure-preserving Boolean homomorphism $\phi : \mathfrak{D} \to \mathfrak{B}_\kappa$ (332N), and $\phi\theta : \mathfrak{A} \to \mathfrak{B}_\kappa$ has the properties we need.

526E Lemma Let $\langle(\mathfrak{A}_n,\bar{\mu}_n)\rangle_{n\in\mathbb{N}}$ be a sequence of finite probability algebras and $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ a sequence in $]0,\infty[$. Write P for the set

$$\{p : p \in \prod_{n\in\mathbb{N}}\mathfrak{A}_n,\ \lim_{n\to\infty}\gamma_n\bar{\mu}_n p(n) = 0\},$$

with the ordering inherited from the product partial order on $\prod_{n\in\mathbb{N}}\mathfrak{A}_n$. Then $P \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

proof (a) By 526D, we can find for each n a probability algebra $(\mathfrak{B}_n,\bar{\nu}_n)$ and a function $\theta_n : \mathfrak{A}_n \to \mathfrak{B}_n$ such that, for all a, $a' \in \mathfrak{A}_n$,

$$\theta_n(a \cup a') = \theta_n(a) \cup \theta_n(a'),$$

$$\bar{\nu}_n\theta_n(a) = 1 - \exp(-\gamma_n\bar{\mu}a).$$

We may suppose that $\mathfrak{B}_n$ is generated by $\theta_n[\mathfrak{A}_n]$, so is itself finite. Set $\mathfrak{A} = \prod_{n\in\mathbb{N}}\mathfrak{A}_n$, $\mathfrak{B} = \prod_{n\in\mathbb{N}}\mathfrak{B}_n$, $\theta(p) = \langle\theta_n(p(n))\rangle_{n\in\mathbb{N}}$ for $p \in \mathfrak{A}$; then $\theta(\sup A) = \sup\theta[A]$ for any non-empty subset A of $\mathfrak{A}$. Set

$$Q = \{q : q \in \prod_{n\in\mathbb{N}}\mathfrak{B}_n,\ \lim_{n\to\infty}\bar{\nu}_n q(n) = 0\}.$$

Then $\theta\restriction P$ is a Tukey function from P to Q. **P** $P = f^{-1}[Q]$, because $\lim_{n\to\infty}\gamma_n\xi_n = 0$ iff $\lim_{n\to\infty} 1 - e^{-\gamma_n\xi_n} = 0$. So $\theta\restriction P$ is a function from P to Q. If $q \in Q$, $A = \{p : p \in \mathfrak{A},\ \theta(p) \subseteq q\}$ has a supremum $p_0 \in \mathfrak{A}$; now $\theta(p_0) = \sup\theta[A] \subseteq q$, so $\theta(p_0) \in Q$ and $p_0 \in P$ is an upper bound for A in P. **Q**

[5] Formerly 495J.

(b) By 526C, we have an order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathcal{P}\mathbb{N}$ such that $\pi(q) \in \mathcal{Z}$ iff $q \in Q$. Now $\pi\restriction Q$ is a Tukey function from Q to $\mathcal{Z}$. **P** If $d \in \mathcal{Z}$, set $B = \{q : q \in \mathfrak{B},\ \pi(q) \subseteq d\}$. Because π is an order-continuous Boolean homomorphism, B contains its supremum, and B is bounded above in Q. **Q**

(c) Thus $\pi\theta\restriction P : P \to \mathcal{Z}$ is a Tukey function and $P \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

526F Theorem $(\ell^1, \le, \ell^1) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathcal{Z}, \subseteq, \mathcal{Z})$.

proof (a) Let $Q \subseteq \mathbb{N}^{\mathbb{N}}$ be the set of strictly increasing sequences α such that $\alpha(0) > 0$. For $\alpha \in Q$, set

$$P_\alpha = \{x : x \in \ell^1,\ \|x\|_\infty \le \alpha(0),\ \lim_{n\to\infty} 2^n \sum_{i=\alpha(n)}^{\infty} x(i)^+ = 0\}$$

$$= \{x : x \in \ell^\infty,\ \|x\|_\infty \le \alpha(0),\ \lim_{n\to\infty} 2^n \sum_{i=\alpha(n)}^{\alpha(n+1)-1} x(i)^+ = 0\}$$

because

$$2^n \sum_{i=\alpha(n)}^{\infty} x(i)^+ = \sum_{m=n}^{\infty} 2^{n-m} 2^m \sum_{i=\alpha(m)}^{\alpha(m+1)-1} x(i)^+ \le 2 \sup_{m\ge n} 2^m \sum_{i=\alpha(m)}^{\alpha(m+1)-1} x(i)^+$$

for every n and x.

The point is that $P_\alpha \preccurlyeq_{\mathrm{T}} \mathcal{Z}$. **P** For each $n \in \mathbb{N}$ set $k_n = 2^{2n}(\alpha(n+1) - \alpha(n))$,

$$V_n = (\alpha(n+1) \setminus \alpha(n)) \times k_n\alpha(0) \subseteq \mathbb{N} \times \mathbb{N}, \quad \mathfrak{A}_n = \mathcal{P}V_n,$$

and let $\bar\mu_n$ be the uniform probability measure on $\mathfrak{A}_n$, so that $\bar\mu_n d = \#(d)/\#(V_n)$ for $d \subseteq V_n$. For $n \in \mathbb{N}$ and $x \in \ell^\infty$ set

$$f_n(x) = \{(i,j) : \alpha(n) \le i < \alpha(n+1),\ j < k_n \min(\alpha(0), x(i))\} \subseteq V_n.$$

Then

$$|\#(f_n(x)) - k_n \textstyle\sum_{\alpha(n)\le i<\alpha(n+1)} \min(\alpha(0), x(i)^+)| \le \alpha(n+1) - \alpha(n),$$

so if $\|x\|_\infty \le \alpha(0)$ then

$$\Bigl|2^n\alpha(0)(\alpha(n+1) - \alpha(n))\bar\mu f_n(x) - 2^n \sum_{i=\alpha(n)}^{\alpha(n+1)-1} x(i)^+\Bigr| \le 2^n(\alpha(n+1) - \alpha(n))/k_n$$

$$= 2^{-n}.$$

Accordingly

$$P_\alpha = \{x : x \in \ell^\infty,\ \|x\|_\infty \le \alpha(0),\ \lim_{n\to\infty} 2^n(\alpha(n+1) - \alpha(n))\bar\mu_n f_n(x) = 0\}.$$

Let $\mathfrak{A} = \prod_{n\in\mathbb{N}} \mathfrak{A}_n$ be the simple product of the Boolean algebras $\mathfrak{A}_n$, and I the ideal

$$\{a : a \in \mathfrak{A},\ \lim_{n\to\infty} 2^n(\alpha(n+1) - \alpha(n))\bar\mu_n a(n) = 0\}$$

of $\mathfrak{A}$. For $x \in \ell^\infty$, set $f(x) = \langle f_n(x)\rangle_{n\in\mathbb{N}}$. Observe that $f : \ell^\infty \to \mathfrak{A}$ is supremum-preserving in the sense that $f(\sup A) = \sup f[A]$ for any non-empty bounded subset A of ℓ^∞.

The last formula for P_α shows that $f(x) \in I$ for every $x \in P_\alpha$. But if $a \in I$, $A = \{x : x \in P_\alpha,\ f(x) \subseteq a\}$ is upwards-directed and has a supremum $x_0 \in \ell^\infty$, with $\|x_0\|_\infty \le \alpha(0)$. Now $f(x_0) = \sup_{x\in A} f(x) \subseteq a$, so $x_0 \in P_\alpha$ and is an upper bound for A in P_α. Thus $f\restriction P_\alpha$ is a Tukey function from P_α to I, and $P_\alpha \preccurlyeq_{\mathrm{T}} I$. By 526E, $I \preccurlyeq_{\mathrm{T}} \mathcal{Z}$, so $P_\alpha \preccurlyeq_{\mathrm{T}} \mathcal{Z}$. **Q**

Thus $(P_\alpha, \le, P_\alpha) \preccurlyeq_{\mathrm{GT}} (\mathcal{Z}, \subseteq, \mathcal{Z})$; it follows at once that $(P_\alpha, \le, \ell^1) \preccurlyeq_{\mathrm{GT}} (\mathcal{Z}, \subseteq, \mathcal{Z})$.

(b) Now, for $\alpha \in \mathbb{N}^{\mathbb{N}}$, take $\tilde P_\alpha = P_\beta$ where $\beta(n) = 1 + n + \max_{i\le n} \alpha(i)$ for $n \in \mathbb{N}$. Then $\tilde P_\alpha \subseteq \tilde P_{\alpha'}$ whenever $\alpha \le \alpha'$ in $\mathbb{N}^{\mathbb{N}}$, $\bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \tilde P_\alpha = \ell^1$ and $(\tilde P_\alpha, \le, \ell^1) \preccurlyeq_{\mathrm{GT}} (\mathcal{Z}, \subseteq, \mathcal{Z})$ for every α; so $(\ell^1, \le, \ell^1) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathcal{Z}, \subseteq, \mathcal{Z})$, by 512K.

526G Corollary Let $\mathcal{N}$ be the ideal of Lebesgue negligible subsets of $\mathbb{R}$.

(a) $\operatorname{add}_\omega \mathcal{Z} = \operatorname{add}\mathcal{N} = \operatorname{add}_\omega \ell^1$ and $\operatorname{cf}\mathcal{Z} = \operatorname{cf}\mathcal{N} = \operatorname{cf}\ell^1$.

(b) If $\mathcal{A} \subseteq \mathcal{Z}$ and $\#(\mathcal{A}) < \operatorname{add}\mathcal{N}$, there is a $J \in \mathcal{Z}$ such that $I \setminus J$ is finite for every $I \in \mathcal{A}$.

proof (a)(i) Putting 526B and 513Ie together, we see that

$$\operatorname{add}_\omega \mathbb{N}^\mathbb{N} \geq \operatorname{add}_\omega \mathcal{Z} \geq \operatorname{add}_\omega \ell^1,$$

that is,

$$\mathfrak{b} \geq \operatorname{add}_\omega \mathcal{Z} \geq \operatorname{add}\mathcal{N}$$

(522A, 524I). Next, we can deduce from 526F that $\operatorname{add}_\omega \ell^1 \geq \min(\operatorname{add}_\omega \mathbb{N}^\mathbb{N}, \operatorname{add}_\omega \mathcal{Z})$. **P** Let (ϕ,ψ) be a Galois-Tukey connection from $(\ell^1, \leq, \ell^1)$ to

$$(\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N}) \ltimes (\mathcal{Z}, \subseteq, \mathcal{Z}) = (\mathbb{N}^\mathbb{N} \times \mathcal{Z}^{\mathbb{N}^\mathbb{N}}, T, \mathbb{N}^\mathbb{N} \times \mathcal{Z}),$$

where

$$T = \{((p,f),(q,a)) : p \leq q \text{ in } \mathbb{N}^\mathbb{N},\ f(q) \subseteq a \in \mathcal{Z}\}.$$

We can interpret ϕ as a pair (ϕ_1,ϕ_2) where ϕ_1 is a function from ℓ^1 to $\mathbb{N}^\mathbb{N}$ and ϕ_2 is a function from $\ell^1 \times \mathbb{N}^\mathbb{N}$ to $\mathcal{Z}$, and saying that (ϕ,ψ) is a Galois-Tukey connection means just that

$$\text{if } \phi_1(x) \leq q \text{ and } \phi_2(x,q) \subseteq a \text{ then } x \leq \psi(q,a).$$

Now suppose that $A \subseteq \ell^1$ and $\#(A) < \min(\operatorname{add}_\omega \mathbb{N}^\mathbb{N}, \operatorname{add}_\omega \mathcal{Z})$. Then there is a sequence $\langle q_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^\mathbb{N}$ such that for every $x \in A$ there is an $n \in \mathbb{N}$ such that $\phi_1(x) \leq q_n$. Next, for each $n \in \mathbb{N}$ there is a sequence $\langle a_{nm}\rangle_{m\in\mathbb{N}}$ in $\mathcal{Z}$ such that for every $x \in A$ there is an $m \in \mathbb{N}$ such that $\phi_2(x,q_n) \subseteq a_{nm}$. In this case, $B = \{\psi(q_n, a_{nm}) : m,\ n \in \mathbb{N}\}$ is a countable subset of $\mathcal{Z}$, and for every $x \in A$ there are m, $n \in \mathbb{N}$ such that $\phi_1(x) \leq q_n$ and $\phi_2(x,q_n) \subseteq a_{nm}$, so that $x \leq \psi(q_n,a_{nm}) \in B$. As A is arbitrary, $\operatorname{add}_\omega \ell^1 \geq \min(\operatorname{add}_\omega \mathbb{N}^\mathbb{N}, \operatorname{add}_\omega \mathcal{Z})$. **Q**

Thus we have $\operatorname{add}\mathcal{N} \geq \min(\mathfrak{b}, \operatorname{add}_\omega \mathcal{Z}) = \operatorname{add}_\omega \mathcal{Z}$, and $\operatorname{add}_\omega \mathcal{Z} = \operatorname{add}\mathcal{N}$. And we know from 524I, with $\kappa = \omega$ there, that $\operatorname{add}\mathcal{N} = \operatorname{add}_\omega \ell^1$

(ii) On the other hand, 524I, 526F, 512Da and 512Jb tell us that

$$\begin{aligned}\operatorname{cf}\mathcal{N} = \operatorname{cf}\ell^1 = \operatorname{cov}(\ell^1, \subseteq, \ell^1) &\leq \operatorname{cov}((\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N}) \ltimes (\mathcal{Z}, \subseteq, \mathcal{Z}))\\ &= \max(\operatorname{cov}(\mathbb{N}^\mathbb{N}, \leq, \mathbb{N}^\mathbb{N}), \operatorname{cov}(\mathcal{Z}, \subseteq, \mathcal{Z})) = \max(\mathfrak{d}, \operatorname{cf}\mathcal{Z}).\end{aligned}$$

But from 526B we see that $\mathfrak{d} \leq \operatorname{cf}\mathcal{Z} \leq \operatorname{cf}\ell^1$, so $\operatorname{cf}\mathcal{Z} = \operatorname{cf}\mathcal{N}$, while 524I tells us that $\operatorname{cf}\mathcal{N} = \operatorname{cf}\ell^1$.

(b) By (a), there is a countable set $\mathcal{D} \subseteq \mathcal{Z}$ such that every member of $\mathcal{A}$ is included in a member of $\mathcal{D}$. By 491Ae, there is a $J \in \mathcal{Z}$ such that $I \setminus J$ is finite for every $I \in \mathcal{D}$; this J serves.

526H I turn now to ideals of nowhere dense sets.

Proposition Let $\mathcal{N}\mathit{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^\mathbb{N}$ and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^\mathbb{N}$.

(a) $\mathcal{N}\mathit{wd}$ is isomorphic, as partially ordered set, to $(\mathcal{N}\mathit{wd})^\mathbb{N}$.

(b) $(\mathcal{N}\mathit{wd}, \subseteq', [\mathcal{N}\mathit{wd}]^{\leq\omega}) \equiv_{\mathrm{GT}} (\mathcal{M}, \subseteq, \mathcal{M})$.

(c) $\mathcal{N}\mathit{wd} \preccurlyeq_{\mathrm{T}} \ell^1$.

(d) Let X be a set and $\mathcal{V}$ a countable family of subsets of X. Set

$$\mathcal{D} = \{D : D \subseteq X, \text{ for every } V \in \mathcal{V} \text{ there is a } V' \in \mathcal{V} \text{ such that } V' \subseteq V \setminus D\}.$$

Then $\mathcal{D} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathit{wd}$.

(e) If X is any non-empty Polish space without isolated points, and $\mathcal{N}\mathit{wd}(X)$ is the ideal of nowhere dense subsets of X, then $\mathcal{N}\mathit{wd} \equiv_{\mathrm{T}} \mathcal{N}\mathit{wd}(X)$.

(f) If X is a compact metrizable space and $\mathcal{C}_{\mathrm{nwd}}$ is the family of closed nowhere dense subsets of X with the Fell (or Vietoris) topology (4A2T), then $(\mathcal{C}_{\mathrm{nwd}}, \subseteq)$ is a metrizably compactly based directed set.

Remark Recall that if R is any relation then R' is the relation $\{(x,B) : (x,y) \in R \text{ for some } y \in B\}$; see 512F-512G.

proof Enumerate $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ as $\langle\sigma_n\rangle_{n\in\mathbb{N}}$. For $\sigma\in S$ write $I_\sigma=\{\alpha:\sigma\subseteq\alpha\in\mathbb{N}^{\mathbb{N}}\}$.

(a) Define $\phi:\mathcal{N}\mathrm{wd}\to\mathcal{N}\mathrm{wd}^{\mathbb{N}}$ by setting $\phi(F)(n)=\{\alpha:\text{<}n\text{>}^\frown\alpha\in F\}$ (notation: 5A1C). Then ϕ is an isomorphism between $\mathcal{N}\mathrm{wd}$ and $\mathcal{N}\mathrm{wd}^{\mathbb{N}}$.

(b)(i) Choose $\phi:\mathcal{M}\to\mathcal{N}\mathrm{wd}^{\mathbb{N}}$ such that $M\subseteq\bigcup_{n\in\mathbb{N}}\phi(M)(n)$ for every $M\in\mathcal{M}$. Then ϕ is a Tukey function so $\mathcal{M}\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathrm{wd}^{\mathbb{N}}\cong\mathcal{N}\mathrm{wd}$, that is, $(\mathcal{M},\subseteq,\mathcal{M})\preccurlyeq_{\mathrm{GT}}(\mathcal{N}\mathrm{wd},\subseteq,\mathcal{N}\mathrm{wd})$. By 513Id and 512Gb,

$$(\mathcal{M},\subseteq,\mathcal{M})\equiv_{\mathrm{GT}}(\mathcal{M},\subseteq',[\mathcal{M}]^{\le\omega})\preccurlyeq_{\mathrm{GT}}(\mathcal{N}\mathrm{wd},\subseteq',[\mathcal{N}\mathrm{wd}]^{\le\omega}).$$

(ii) For $n\in\mathbb{N}$ and $\tau\in\mathbb{N}^n$, define $g_\tau:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ by saying that $g_\tau(\alpha)=\tau^\frown\alpha$. Note that g_τ is a homeomorphism between $\mathbb{N}^{\mathbb{N}}$ and I_τ, so that $g_\tau[F]$ and $g_\tau^{-1}[F]$ are nowhere dense whenever F is.

Now for any $F\in\mathcal{N}\mathrm{wd}$ we can find a $\phi(F)\in\mathcal{N}\mathrm{wd}$ such that $F\subseteq\phi(F)$ and for every $\sigma\in S$ either $I_\sigma\cap\phi(F)=\emptyset$ or there is a $\tau\in S$, extending σ, such that $g_\tau[F]\subseteq\phi(F)$. **P** Choose $\langle\tau_n\rangle_{n\in\mathbb{N}}$, $\langle\upsilon_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Given that $I_{\upsilon_i}\cap(F\cup g_{\tau_j}[F])=\emptyset$ for all i, $j<n$, set $E=I_{\sigma_n}\cap(F\cup\bigcup_{j<n}g_{\tau_j}[F])$. If $E=\emptyset$ set $\upsilon_n=\sigma_n$ and $\tau_n=\emptyset$, so that $g_{\tau_n}[F]=F$. If E is not empty, it is still nowhere dense, so we can find $\upsilon_n\supseteq\sigma_n$ such that $I_{\upsilon_n}\cap E=\emptyset$. In this case, $\bigcup_{i\le n}I_{\upsilon_i}$ is a closed set not including I_{σ_n}, so we can find a $\tau_n\supseteq\sigma_n$ such that $I_{\tau_n}\cap\bigcup_{i\le n}I_{\upsilon_i}=\emptyset$, and $I_{\upsilon_i}\cap g_{\tau_n}[F]=\emptyset$ for $i\le n$. Thus in both cases we shall have $\bigcup_{i\le n}I_{\upsilon_i}\cap(F\cup\bigcup_{j\le n}g_{\tau_j}[F])=\emptyset$, and the induction proceeds.

Set $\phi(F)=\overline{F\cup\bigcup_{j\in\mathbb{N}}g_{\tau_j}[F]}$. Because $\upsilon_i\supseteq\sigma_i$ and $\phi(F)\cap I_{\upsilon_i}$ is empty for every $i\in\mathbb{N}$, $\phi(F)\in\mathcal{N}\mathrm{wd}$. If $\sigma\in S$ is such that $\phi(F)$ meets I_σ, there is an $n\in\mathbb{N}$ such that $\sigma=\sigma_n$; now we cannot have $\upsilon_n=\sigma_n$ so we must have $\tau_n\supseteq\sigma_n$ and $g_{\tau_n}[F]\subseteq\phi(F)$. Thus we have found an suitable set $\phi(F)$. **Q**

For each $M\in\mathcal{M}$ let $\mathcal{E}_M$ be a non-empty countable family of closed nowhere dense sets covering M, and set $\psi(M)=\{g_\tau^{-1}[E]:E\in\mathcal{E}_M,\,\tau\in S\}$. Then (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{N}\mathrm{wd},\subseteq',[\mathcal{N}\mathrm{wd}]^{\le\omega})$ to $(\mathcal{M},\subseteq,\mathcal{M})$. **P** Suppose that $F\in\mathcal{N}\mathrm{wd}$ and $M\in\mathcal{M}$ are such that $\phi(F)\subseteq M$. If $F=\emptyset$ then certainly there is an $F'\in\psi(M)$ covering F. Otherwise, $\phi(F)$ is a non-empty closed set included in the union of the countable set $\mathcal{E}_M$ of closed sets. By Baire's theorem, there must be a $\sigma\in S$ and an $E\in\mathcal{E}_M$ such that $\emptyset\ne\phi(F)\cap I_\sigma\subseteq E$. In this case, there is a $\tau\supseteq\sigma$ such that $g_\tau[F]\subseteq\phi(F)$, so that $g_\tau[F]\subseteq E$ and $F\subseteq g_\tau^{-1}[E]\in\psi(M)$ and $F\subseteq'\psi(M)$. As F and M are arbitrary, (ϕ,ψ) is a Galois-Tukey connection. **Q**

(iii) Thus we have

$$(\mathcal{M},\subseteq,\mathcal{M})\preccurlyeq_{\mathrm{GT}}(\mathcal{N}\mathrm{wd},\subseteq',[\mathcal{N}\mathrm{wd}]^{\le\omega})\preccurlyeq_{\mathrm{GT}}(\mathcal{M},\subseteq,\mathcal{M})$$

and $(\mathcal{M},\subseteq,\mathcal{M})\equiv_{\mathrm{GT}}(\mathcal{N}\mathrm{wd},\subseteq',[\mathcal{N}\mathrm{wd}]^{\le\omega})$.

(c) We can use the idea of 522O. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ enumerate a base for the topology of $\mathbb{N}^{\mathbb{N}}$ not containing $\emptyset$. By 522N, there is for each $n\in\mathbb{N}$ a countable family $\mathcal{V}_n$ of open subsets of U_n such that $\bigcap\mathcal{V}\ne\emptyset$ for every $\mathcal{V}\in[\mathcal{V}_n]^{\le 2^n}$ and every dense open subset of U_n includes some member of $\mathcal{V}_n$. Enumerate $\mathcal{V}_n$ as $\langle U_{nm}\rangle_{m\in\mathbb{N}}$.

For each $F\in\mathcal{N}\mathrm{wd}$ let $f_F:\mathbb{N}\to\mathbb{N}$ be such that $F\cap U_{n,f_F(n)}=\emptyset$ for every $n\in\mathbb{N}$, and for n, $i\in\mathbb{N}$ set

$$\begin{aligned}\phi(F)(2^n(2i+1)-1)&=2^{-n}\text{ if }f_F(n)=i,\\&=0\text{ otherwise.}\end{aligned}$$

Then $\sum_{i=0}^{\infty}\phi(F)(i)=2$ for each F, so we have a function $\phi:\mathcal{N}\mathrm{wd}\to\ell^1$.

Suppose that $x\in\ell^1$. Set $\mathcal{A}=\{F:F\in\mathcal{N}\mathrm{wd},\,\phi(F)\le x\}$ and $E=\bigcup\mathcal{A}$. The set

$$K=\{n:\#(\{i:x(2^n(2i+1)-1)\ge 2^{-n}\})\ge 2^n\}$$

is finite; set $k=\sup(\{0\}\cup K)$. If $n>k$, then $\#(\{f_F(n):F\in\mathcal{A}\})<2^n$, so $\bigcap_{F\in\mathcal{A}}U_{n,f_F(n)}$ is a non-empty open subset of U_n disjoint from $\bigcup_{F\in\mathcal{A}}F=E$. Thus $\{n:U_n\subseteq\overline{E}\}\subseteq\{0,\ldots,k\}$ is finite, and therefore in fact is empty, that is, $E\in\mathcal{N}\mathrm{wd}$.

As x is arbitrary, $\phi:\mathcal{N}\mathrm{wd}\to\ell^1$ is a Tukey function, and witnesses that $\mathcal{N}\mathrm{wd}\preccurlyeq_{\mathrm{T}}\ell^1$.

(d) If $\mathcal{V}=\emptyset$ then $\mathcal{D}=\mathcal{P}X$ has a greatest element and the result is trivial (any function from $\mathcal{D}$ to $\mathcal{N}\mathrm{wd}$ will be a Tukey function). Otherwise, choose a function $h:S\to\mathcal{V}\cup\{X\}$ such that $h(\emptyset)=X$ and $\langle h(\sigma^\frown\text{<}i\text{>})\rangle_{i\in\mathbb{N}}$ runs over $\{V:h(\sigma)\supseteq V\in\mathcal{V}\}$ for every $\sigma\in S$. Note that $h(\tau)\subseteq h(\sigma)$ whenever $\tau\supseteq\sigma$, and that $\{h(\tau):\sigma\subseteq\tau\in\mathbb{N}^n\}=\{V:V\in\mathcal{V},\,V\subseteq h(\sigma)\}$ whenever $m\in\mathbb{N}$, $\sigma\in\mathbb{N}^m$ and $n>m$. For each $D\in\mathcal{D}$ we can choose a sequence $\langle\tau_{Dn}\rangle_{n\in\mathbb{N}}$ in S such that $\tau_{Dn}\supseteq\sigma_n$ and $D\cap h(\tau_{Dn})$ is empty and $\#(\tau_{Dn})\ge n$ for every $n\in\mathbb{N}$. Set $\phi(D)=\mathbb{N}^{\mathbb{N}}\setminus\bigcup_{n\in\mathbb{N}}I_{\tau_{Dn}}$, so that $\phi(D)\in\mathcal{N}\mathrm{wd}$.

Take any $F\in\mathcal{N}\!wd$, and set $D_0=\bigcup\{D:D\in\mathcal{D},\,\phi(D)\subseteq F\}$. Then $D_0\in\mathcal{D}$. **P** Let $V\in\mathcal{V}$. Let $\upsilon\in\mathbb{N}^1$ be such that $h(\upsilon)=V$. Take $\tau\supseteq\upsilon$ such that $F\cap I_\tau=\emptyset$. **?** If $D_0\cap h(\tau)\neq\emptyset$, then there is a $D\in\mathcal{D}$ such that $\phi(D)\subseteq F$ and $D\cap h(\tau)\neq\emptyset$. $I_\tau\cap\phi(D)$ is empty, that is, $I_\tau\subseteq\bigcup_{n\in\mathbb{N}}I_{\tau_{Dn}}$; because $\#(\tau_{Dn})\ge n$ for every n, this can happen only because there is some $n\in\mathbb{N}$ such that $\tau_{Dn}\subseteq\tau$. But this means that $D\cap h(\tau)\subseteq D\cap h(\tau_{Dn})=\emptyset$, which is impossible. **X** Thus $D_0\cap h(\tau)$ is empty, and $h(\tau)$ is a member of $\mathcal{V}$ included in $V\setminus D_0$. As V is arbitrary, $D_0\in\mathcal{V}$. **Q**

As F is arbitrary, ϕ is a Tukey function and $\mathcal{D}\preccurlyeq_{\mathrm{T}}\mathcal{N}\!wd$, as claimed.

(e)(i) Taking $\mathcal{V}$ to be a countable base for the topology of X not containing $\emptyset$, we have

$$\mathcal{N}\!wd(X)=\{F:F\subseteq X,\text{ for every }V\in\mathcal{V}\text{ there is a }V'\in\mathcal{V}\text{ such that }V'\subseteq V\setminus F\},$$

so (d) tells us that $\mathcal{N}\!wd(X)\preccurlyeq_{\mathrm{T}}\mathcal{N}\!wd$.

(ii) X has a dense subset Y which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ (5A4Le). Let $\mathcal{N}\!wd(Y)$ be the family of nowhere dense subsets of Y. For $F\in\mathcal{N}\!wd(Y)$ let $\phi(F)$ be its closure in X. Then ϕ is a Tukey function from $\mathcal{N}\!wd(Y)$ to $\mathcal{N}\!wd(X)$, so $\mathcal{N}\!wd\cong\mathcal{N}\!wd(Y)\preccurlyeq_{\mathrm{T}}\mathcal{N}\!wd(X)$.

(f) By 4A2Tg, the Fell topology on the family $\mathcal{C}$ of all closed subsets of X is compact and metrizable. $E\cup F\in\mathcal{C}_{\mathrm{nwd}}$ for all E, $F\in\mathcal{C}_{\mathrm{nwd}}$, and $\cup:\mathcal{C}_{\mathrm{nwd}}\times\mathcal{C}_{\mathrm{nwd}}\to\mathcal{C}_{\mathrm{nwd}}$ is continuous (4A2T(b-ii)). If $F\in\mathcal{C}_{\mathrm{nwd}}$, the set $\{E:E\in\mathcal{C}_{\mathrm{nwd}},\,E\subseteq F\}=\{E:E\in\mathcal{C},\,E\cup F=F\}$ is closed in $\mathcal{C}$, therefore compact. Now suppose that $\langle E_k\rangle_{k\in\mathbb{N}}$ is a sequence in $\mathcal{C}_{\mathrm{nwd}}$ converging to $E\in\mathcal{C}_{\mathrm{nwd}}$. If $X=\emptyset$ then of course $\{E_k:k\in\mathbb{N}\}$ is bounded above in $\mathcal{C}_{\mathrm{nwd}}$. Otherwise, let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of X not containing $\emptyset$, and choose $\langle k_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Given $k_i\in\mathbb{N}$ for $i<n$, let $V_n\subseteq U_n$ be a non-empty open set such that $\overline{V}_n\cap(E\cup\bigcup_{i<n}E_{k_i})=\emptyset$; given that $E\cap\overline{V}_i=\emptyset$ for $i\le n$, choose $k_n\ge n$ such that $E_{k_n}\cap\bigcup_{i\le n}\overline{V}_i$ is empty. (This is possible because $\bigcup_{i\le n}\overline{V}_i$ is compact, so the family of sets disjoint from it is open in the Fell topology.) Continue. At the end of the induction, $G=\bigcup_{n\in\mathbb{N}}V_n$ is a dense open set disjoint from $\bigcup_{n\in\mathbb{N}}E_{k_n}$, so $X\setminus G$ is an upper bound for $\{E_{k_n}:n\in\mathbb{N}\}$ in $\mathcal{C}_{\mathrm{nwd}}$. Thus all the conditions of 513K are satisfied, and $\mathcal{C}_{\mathrm{nwd}}$ is metrizably compactly based.

526I A related type of ideal is the following. I express the result in more general form because it has some measure theory in it.

Proposition (Fremlin 91) Let X be a second-countable topological space and μ a σ-finite topological measure on X. Let $\mathcal{E}$ be the ideal of subsets of X with negligible closures. Then, writing $\mathcal{N}\!wd$ for the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$, $\mathcal{E}\preccurlyeq_{\mathrm{T}}\mathcal{N}\!wd$ and $\mathcal{E}\preccurlyeq_{\mathrm{T}}\mathcal{Z}$.

proof (a) If $\mu X=0$ then $\mathcal{E}$ has a greatest element and the result is trivial. Otherwise, there is a probability measure on X with the same measurable sets and the same negligible sets as μ (215B(vii)); so we may suppose that μ itself is a probability measure. Let $\mathcal{U}$ be a countable base for the topology of X, containing X and closed under finite unions.

(b) For $k\in\mathbb{N}$ let $\mathcal{V}_k$ be the countable set $\{V:V\in\mathcal{U},\,\mu V>1-2^{-k}\}$. Set

$$\mathcal{E}_k=\{E:E\subseteq X,\text{ for every }V\in\mathcal{V}_k\text{ there is a }U\in\mathcal{V}_k\text{ such that }U\subseteq V\setminus E\}.$$

Then $\mathcal{E}=\bigcap_{k\in\mathbb{N}}\mathcal{E}_k$. **P** Because $X\in\mathcal{V}_k$, $\mu\overline{E}\le 2^{-k}$ for every $E\in\mathcal{E}_k$, so $\bigcap_{k\in\mathbb{N}}\mathcal{E}_k\subseteq\mathcal{E}$. On the other hand, if $E\in\mathcal{E}$ and $k\in\mathbb{N}$ and $V\in\mathcal{V}_k$, then $\mu(V\setminus\overline{E})>1-2^{-k}$ and $\mathcal{U}'=\{U:U\in\mathcal{U},\,U\subseteq V\setminus E\}$ has union $V\setminus\overline{E}$. As $\mathcal{U}'$ is countable, there is a finite $\mathcal{U}'_1\subseteq\mathcal{U}'$ such that $U=\bigcup\mathcal{U}'_1$ has measure greater than $1-2^{-k}$, so that $U\in\mathcal{V}_k$ and $U\subseteq V\setminus E$. As V is arbitrary, $E\in\mathcal{V}_k$; as E and k are arbitrary, $\mathcal{E}\subseteq\bigcap_{k\in\mathbb{N}}\mathcal{E}_k$. **Q**

This means that the map $E\mapsto(E,E,E,\dots)$ is a Tukey function from $\mathcal{E}$ to $\prod_{k\in\mathbb{N}}\mathcal{E}_k$, so that $\mathcal{E}\preccurlyeq_{\mathrm{T}}\prod_{k\in\mathbb{N}}\mathcal{E}_k$. At the same time, $\mathcal{E}_k\preccurlyeq_{\mathrm{T}}\mathcal{N}\!wd$ for every k, by 526Hd. So $\mathcal{E}\preccurlyeq_{\mathrm{T}}\mathcal{N}\!wd^{\mathbb{N}}\cong\mathcal{N}\!wd$ (513Eg, 526Ha).

(c) Let $\mathfrak{A}$ be the countable subalgebra of $\mathcal{P}X$ generated by $\mathcal{U}$. Then there is a Boolean homomorphism $\pi:\mathfrak{A}\to\mathcal{P}\mathbb{N}$ such that $d(\pi E)$ is defined and equal to μE for every $E\in\mathfrak{A}$. **P** This is easy to prove directly (see 491Xu), but we can also argue as follows. Let $\langle\mathfrak{A}_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of finite subalgebras of $\mathfrak{A}$ with union $\mathfrak{A}$. By 526C, we have a Boolean homomorphism $\pi':\prod_{n\in\mathbb{N}}\mathfrak{A}_n\to\mathcal{P}\mathbb{N}$ such that $d(\pi'\langle E_n\rangle_{n\in\mathbb{N}})=\lim_{n\to\infty}\mu E_n$ whenever $E_n\in\mathfrak{A}_n$ for every n and the limit on the right is defined. For each $n\in\mathbb{N}$ let $\pi_n:\mathfrak{A}\to\mathfrak{A}_n$ be a Boolean homomorphism extending the identity homomorphism on $\mathfrak{A}_n$ (314K, or otherwise); set $\pi E=\pi'\langle\pi_nE\rangle_{n\in\mathbb{N}}$ for $E\in\mathfrak{A}$; this works. **Q**

Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence running over the closed sets belonging to $\mathfrak{A}$. Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}\setminus\{0\}$ such that $\frac{1}{k_n}\#(k_n\cap\pi E)\ge\mu E-2^{-n}$ whenever E belongs to the subalgebra $\mathfrak{B}_n$ of $\mathfrak{A}$ generated by $\{V_i:i\le n\}$. Define $\phi:\mathcal{E}\to\mathcal{P}\mathbb{N}$ by setting

$$\phi(E)=\bigcap\{k_i\cup\pi V_i:i\in\mathbb{N},\,E\subseteq V_i\}.$$

Then ϕ is a Tukey function from $\mathcal{E}$ to $\mathcal{Z}$.

P (i) If $E\in\mathcal{E}$ and $\epsilon>0$ there is a $U\in\mathcal{U}$ such that $U\subseteq X\setminus E$ and $\mu U\ge 1-\epsilon$. Let $i\in\mathbb{N}$ be such that $X\setminus U=V_i$; then $\phi(E)\subseteq k_i\cup\pi V_i$, so

$$d^*(\phi(E))\le d^*(\pi V_i)=\mu V_i\le\epsilon.$$

As ϵ is arbitrary, $\phi(E)\in\mathcal{Z}$. Thus ϕ is a function from $\mathcal{E}$ to $\mathcal{Z}$.

(ii) Take any $\mathcal{A}\subseteq\mathcal{E}$, and set $F=\overline{\bigcup\mathcal{A}}$, $a=\bigcup_{E\in\mathcal{E}}\phi(E)$. If $n\in\mathbb{N}$ and $i\in k_n\setminus a$, then $i\notin\phi(E)$ for every $E\in\mathcal{A}$, so for every $E\in\mathcal{A}$ there is a $j<n$ such that $E\subseteq V_j$ and $i\notin\pi V_j$. Set $F_{ni}=\bigcup\{V_j:j<n,\ i\notin\pi V_j\}$, so that $i\notin\pi F_{ni}$, while $\bigcup\mathcal{A}\subseteq F_{ni}$ and $F\subseteq F_{ni}$. Set $F_n=\mathbb{N}\cap\bigcap_{i\in k_n\setminus a}F_{ni}$, so that $F\subseteq F_n$ and no member of $k_n\setminus a$ belongs fo πF_n, that is, $k_n\cap\pi F_n\subseteq a$. Note that $F_n\in\mathfrak{B}_n$. So we have

$$\frac{1}{k_n}\#(k_n\cap a)\ge\frac{1}{k_n}\#(k_n\cap\pi F_n)\ge\mu F_n-2^{-n}\ge\mu F-2^{-n}.$$

This means that $d^*(a)\ge\mu F$. So if $\{\phi(E):E\in\mathcal{A}\}$ is bounded above in $\mathcal{Z}$, $\mathcal{A}$ must be bounded above in $\mathcal{E}$; that is, ϕ is a Tukey function. **Q**

Thus $\mathcal{E}\preccurlyeq_{\mathrm{T}}\mathcal{Z}$ also.

526J Proposition Let $\mathcal{E}_{\mathrm{Leb}}$ be the ideal of subsets of $\mathbb{R}$ whose closures are Lebesgue negligible. Then $\mathbb{N}^{\mathbb{N}}\preccurlyeq_{\mathrm{T}}\mathcal{E}_{\mathrm{Leb}}$ but $\mathcal{E}_{\mathrm{Leb}}\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$; consequently $\mathcal{Z}\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$, $\mathcal{N}\mathrm{wd}\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$ and $\ell^1\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$.

proof (a) Enumerate $\mathbb{Q}\cap[0,1]$ as $\langle q_i\rangle_{i\in\mathbb{N}}$. Define $\phi:\mathbb{N}^{\mathbb{N}}\to\mathcal{E}_{\mathrm{Leb}}$ by setting $\phi(f)(n)=\{n+q_i:n\in\mathbb{N},\ i\le f(n)\}$. Then it is easy to see that ϕ is a Tukey function, because if $F\subseteq\mathbb{N}^{\mathbb{N}}$ and $\{f(n):f\in F\}$ is unbounded, then $\bigcup_{f\in F}\phi(f)$ is dense in $[n,n+1]$ so does not belong to $\mathcal{E}_{\mathrm{Leb}}$.

(b) Let $\psi:\mathcal{E}_{\mathrm{Leb}}\to\mathbb{N}^{\mathbb{N}}$ be any function. Let μ be Lebesgue measure on $\mathbb{R}$, and choose $\langle f(n)\rangle_{n\in\mathbb{N}}$ inductively in $\mathbb{N}$ such that $\mu^*\{t:t\in[0,1],\ \psi(\{t\})(i)\le f(i)\text{ for every }i\le n\}>\frac{1}{2}$ for every n. Set

$$A_n=\{t:t\in[0,1],\ \psi(\{t\})(i)\le f(i)\text{ for every }i\le n\},\qquad F=\textstyle\bigcap_{n\in\mathbb{N}}\overline{A_n}$$

so that $\mu F\ge\frac{1}{2}$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ enumerate the set of open intervals of $\mathbb{R}$, meeting F, with rational endpoints, and for each $n\in\mathbb{N}$ choose $t_n\in A_n\cap U_n$. Then $\psi(\{t_n\})(i)\le f(i)$ whenever $n\ge i$, so $\{\psi(\{t_n\}):n\in\mathbb{N}\}$ is bounded above in $\mathbb{N}^{\mathbb{N}}$; but $\overline{\{t_n:n\in\mathbb{N}\}}$ includes F, so $\{\{t_n\}:n\in\mathbb{N}\}$ is not bounded above in $\mathcal{E}_{\mathrm{Leb}}$. Thus ψ cannot be a Tukey function.

(c) Accordingly $\mathcal{E}_{\mathrm{Leb}}\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$; since $\mathcal{E}_{\mathrm{Leb}}\preccurlyeq_{\mathrm{T}}\mathcal{Z}\preccurlyeq_{\mathrm{T}}\ell^1$ and $\mathcal{E}_{\mathrm{Leb}}\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathrm{wd}$ (526I, 526B), $\mathcal{Z}\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$, $\mathcal{N}\mathrm{wd}\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$ and $\ell^1\not\preccurlyeq_{\mathrm{T}}\mathbb{N}^{\mathbb{N}}$.

526K Proposition Let $\mathcal{N}\mathrm{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$. Then $\mathcal{Z}\not\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathrm{wd}$, so $\mathcal{Z}\not\preccurlyeq_{\mathrm{T}}\mathcal{E}_{\mathrm{Leb}}$ and $\ell^1\not\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathrm{wd}$.

proof Let $\phi:\mathcal{Z}\to\mathcal{N}\mathrm{wd}$ be any function. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ enumerate a base for the topology of $\mathbb{N}^{\mathbb{N}}$ which contains $\emptyset$ and is closed under finite unions. For each $n\in\mathbb{N}$, set

$$a_n=\{i:i\in\mathbb{N},\ \phi(a)\cap U_n\ne\emptyset\text{ whenever }i\in a\in\mathcal{Z}\}.$$

Set

$$a=\{\min(a_n\setminus n^2):n\in\mathbb{N},\ a_n\not\subseteq n^2\}$$

(interpreting n^2 in the formula above as a member of $\mathbb{N}$ rather than as a subset of $\mathbb{N}^2$). Then $a\in\mathcal{Z}$ and $a\cap a_n\ne\emptyset$ whenever a_n is infinite. Set $K=\{n:n\in\mathbb{N},\ \phi(a)\cap U_n=\emptyset\}$, so that K is infinite and $\bigcup_{n\in K}U_n=\mathbb{N}^{\mathbb{N}}\setminus\overline{\phi(a)}$ is dense, while a_n is finite for every $n\in K$ (since otherwise there is an $i\in a\cap a_n$, and $\phi(a)\cap U_n$ will not be empty). For $n\in\mathbb{N}$, $\bigcup_{m\in K\cap n}U_m$ belongs to $\mathcal{U}$; let $r(n)\in\mathbb{N}$ be such that

$U_{r(n)}=\bigcup_{m\in K\cap n}U_m$. Then $r(n)\in K$ for every n, so $a_{r(n)}$ is always finite. Take a strictly increasing sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ in K such that $a_{r(n)}\subseteq k_n$ for every n. For $i<k_0$, set $b_i=\{i\}$; for $k_n\le i<k_{n+1}$, choose $b_i\in\mathcal{Z}$ such that $i\in b_i$ and $\phi(b_i)\cap U_{r(n)}$ is empty (such exists because $i\notin a_{r(n)}$).

Examine $E=\bigcup_{i\in\mathbb{N}}\phi(b_i)\subseteq\mathbb{N}^{\mathbb{N}}$. If $m\in K$ then $U_m\subseteq U_{r(n)}$ for every $n>m$ so $U_m\cap\phi(b_i)=\emptyset$ for every $i\ge k_{m+1}$ and $U_m\cap E=\bigcup_{i<k_{m+1}}U_m\cap\phi(b_i)$ is nowhere dense. As $\bigcup_{m\in K}U_m$ is a dense open set, E is nowhere dense. On the other hand, $\bigcup_{i\in\mathbb{N}}b_i=\mathbb{N}$. So $\{b:b\in\mathcal{Z},\,\phi(b)\subseteq E\}$ is not bounded above in $\mathcal{Z}$, and ϕ cannot be a Tukey function. As ϕ is arbitrary, $\mathcal{Z}\not\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathit{wd}$.

Because $\mathcal{E}_{\mathrm{Leb}}\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathit{wd}$ (526I) and $\mathcal{Z}\preccurlyeq_{\mathrm{T}}\ell^1$ (526B), it follows that $\mathcal{Z}\not\preccurlyeq_{\mathrm{T}}\mathcal{E}_{\mathrm{Leb}}$ and $\ell^1\not\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathit{wd}$.

526L Proposition (Mátrai p09) $\mathcal{N}\mathit{wd}\not\preccurlyeq_{\mathrm{T}}\mathcal{Z}$, so $\mathcal{N}\mathit{wd}\not\preccurlyeq_{\mathrm{T}}\mathcal{E}_{\mathrm{Leb}}$ and $\ell^1\not\preccurlyeq_{\mathrm{T}}\mathcal{Z}$.

proof (a)(i) Fix a non-empty zero-dimensional compact metrizable space X without isolated points, and write $\mathcal{N}\mathit{wd}(X)$ for the ideal of nowhere dense subsets of X; the bulk of the argument here will be a proof that $\mathcal{N}\mathit{wd}(X)\not\preccurlyeq_{\mathrm{T}}\mathcal{Z}$. Let $\mathcal{V}$ be the family of non-empty open-and-closed subsets of X. For $V\in\mathcal{V}$ write $\mathcal{N}\mathit{wd}(V)=\mathcal{N}\mathit{wd}(X)\cap\mathcal{P}V$ for the family of nowhere dense subsets of V. As in 526A, set $\nu I=\sup_{n\ge1}\frac{1}{n}\#(I\cap n)$ for $I\subseteq\mathbb{N}$. Take any function $f:\mathcal{N}\mathit{wd}(X)\to\mathcal{Z}$.

(ii) Let Q be the set of pairs $\sigma=(m_\sigma,I_\sigma)$ where $I_\sigma\subseteq m_\sigma\in\mathbb{N}$; for σ, $\tau\in Q$, say that $\sigma\le\tau$ if either $\sigma=\tau$ or $2m_\sigma\le m_\tau$ and $I_\sigma=m_\sigma\cap I_\tau$. Then $(Q,\le)$ is a partially ordered set. For $\sigma\in Q$ and $\epsilon>0$, let $D(\sigma,\epsilon)$ be the set of those $E\subseteq X$ for which there is an $F\in\mathcal{N}\mathit{wd}(X)$, including E, such that $f(F)\cap m_\sigma\subseteq I_\sigma$ and $\nu(f(F)\setminus I_\sigma)\le\epsilon$.

(iii) If $\sigma\in Q$, $\epsilon>0$ and $k\ge 2m_\sigma$, then

$$D(\sigma,\epsilon)\subseteq\bigcup\{D(\tau,\epsilon):\sigma\le\tau\in Q,\ m_\tau=k,\ \nu(I_\tau\setminus I_\sigma)\le\epsilon\}.$$

P If $E\in D(\sigma,\epsilon)$, let $F\in\mathcal{N}\mathit{wd}(X)$ be such that $E\subseteq F$, $f(F)\cap m_\sigma\subseteq I_\sigma$ and $\nu(f(F)\setminus I_\sigma)\le\epsilon$. Set $\tau=(k,I_\sigma\cup(k\cap f(F))$; then $\sigma\le\tau$ and F witnesses that $E\in D(\tau,\epsilon)$, while $\nu(I_\tau\setminus I_\sigma)\le\epsilon$. **Q**

(iv) If $\sigma\in Q$ and ϵ, $\delta>0$, then

$$D(\sigma,\epsilon)\subseteq\bigcup\{D(\tau,\delta):\sigma\le\tau\in Q,\ \nu(I_\tau\setminus I_\sigma)\le\epsilon\}.$$

P If $E\in D(\sigma,\epsilon)$, let $F\in\mathcal{N}\mathit{wd}(X)$ be such that $E\subseteq F$, $F\cap m_\sigma\subseteq I_\sigma$ and $\nu(f(F)\setminus I_\sigma)\le\epsilon$. As $f(F)\in\mathcal{Z}$, there is a $k\ge 2m_\sigma$ such that $\nu(f(F)\setminus k)\le\delta$. Set $\tau=(k,I_\sigma\cup(k\cap f(F))$; then F witnesses that $E\in D(\tau,\delta)$, while $\nu(I_\tau\setminus I_\sigma)\le\epsilon$. **Q**

(v) Suppose that $n\ge1$ and that $\langle\sigma_j\rangle_{j\le n}$, $\langle\tau_j\rangle_{j\le n}$ are finite sequences in Q such that $m_{\tau_j}\le m_{\sigma_j}$ for $j\le n$ and $\sigma_j\le\tau_{j+1}$ for $j<n$. Then $\nu(\bigcup_{j<n}I_{\tau_{j+1}}\setminus I_{\sigma_j})\le 3\max_{j<n}\nu(I_{\tau_{j+1}}\setminus I_{\sigma_j})$. **P** Note first that we certainly have $m_{\sigma_j}\le m_{\tau_{j+1}}\le m_{\sigma_{j+1}}$ for every $j<n$. Set $K=\bigcup_{j<n}I_{\tau_{j+1}}\setminus I_{\sigma_j}$ and $\epsilon=\max_{j<n}\nu(I_{\tau_{j+1}}\setminus I_{\sigma_j})$. If $m\in\mathbb{N}$, set $J=\{j:j<n,\ \sigma_j\ne\tau_{j+1},\ m_{\sigma_j}\le m\}$, $J'=\{j:j\in J,\ m_{\tau_{j+1}}\le m\}$. Then

$$\#(m\cap K)\le\sum_{j\in J}\#(m\cap I_{\tau_{j+1}}\setminus I_{\sigma_j})$$

(because if $j<n$ and $m\le m_{\sigma_j}$, then $m\cap I_{\tau_{j+1}}\setminus I_{\sigma_j}=\emptyset$)

$$\le\epsilon m+\sum_{j\in J'}\#(I_{\tau_{j+1}}\setminus I_{\sigma_j})$$

(because $\#(J\setminus J')\le1$)

$$\le\epsilon(m+\sum_{j\in J'}m_{\tau_{j+1}})\le\epsilon(m+2m)$$

(because $2m_{\tau_{j+1}}\le 2m_{\sigma_{j+1}}\le 2m_{\sigma_{j'}}\le m_{\tau_{j'+1}}\le m$ whenever j, j' are successive members of J')

$$=3\epsilon m.$$

As m is arbitrary, $\nu K\le3\epsilon$. **Q**

(b)(i) Suppose that $V\in\mathcal{V}$ and that $\mathcal{C}_0,\dots,\mathcal{C}_n\subseteq\mathcal{N}\mathit{wd}(X)$ are such that every nowhere dense subset of V is included in some member of $\bigcup_{i\le n}\mathcal{C}_i$. Then there is an $i\le n$ such that every nowhere dense subset of

V is included in some member of $\mathcal{C}_i$. **P?** Otherwise, for each $i \le n$ we can find a nowhere dense subset E_i of V not included in any member of $\mathcal{C}_i$; now $E = \bigcup_{i \le n} E_i$ is a nowhere dense subset of V not included in any member of $\bigcup_{i \le n} \mathcal{C}_i$. **XQ**

(ii) Suppose that $V \in \mathcal{V}$ and that $\langle \mathcal{C}_n \rangle_{n \in \mathbb{N}}$ is a sequence of subsets of $\mathcal{N}\mathrm{wd}(X)$ such that every nowhere dense subset of V is included in some member of $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n$. Then for any $E \in \mathcal{N}\mathrm{wd}(V)$ there are a $U \in \mathcal{V}$ and an $n \in \mathbb{N}$ such that $E \subseteq U$ and every nowhere dense subset of U is included in some member of $\mathcal{C}_n$. **P** As $V \ne \emptyset$ we can suppose that $E \ne \emptyset$. Let $\langle U_n \rangle_{n \in \mathbb{N}}$ be a non-increasing sequence in $\mathcal{V}$ such that $U_0 = V$ and $\bigcap_{n \in \mathbb{N}} U_n = \overline{E}$. **?** If, for every $n \in \mathbb{N}$, there is an $E_n \in \mathcal{N}\mathrm{wd}(U_n)$ not included in any member of $\mathcal{C}_n$, consider $F = \bigcup_{n \in \mathbb{N}} E_n$; then $F \in \mathcal{N}\mathrm{wd}(V)$ but F is not included in any member of any $\mathcal{C}_n$. **X** So some U_n will serve. **Q**

(c) (The key.) Suppose that $V \in \mathcal{V}$, $\sigma \in Q$ and $\epsilon > 0$ are such that $\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma, \epsilon)$. Let $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ be a sequence in $]0, \infty[$. Then there are an $n \in \mathbb{N}$, $U_0, \dots, U_n \in \mathcal{V}$ and $\tau \in Q$ such that

$$\sigma \le \tau, \quad \nu(I_\tau \setminus I_\sigma) \le 8\epsilon,$$

$$V \subseteq \textstyle\bigcup_{j \le n} U_j, \quad \mathcal{N}\mathrm{wd}(U_j) \subseteq D(\tau, \epsilon_j) \text{ for every } j \le n.$$

P It is enough to consider the case in which $\sum_{n=0}^\infty \epsilon_n \le \epsilon$. Let $\langle x_n \rangle_{n \in \mathbb{N}}$ run over a dense subset of V. Choose $\langle \sigma_n \rangle_{n \in \mathbb{N}}$, $\langle k_n \rangle_{n \in \mathbb{N}}$, $\langle U_n \rangle_{n \ge 1}$ and $\langle \tau_n \rangle_{n \ge 1}$ inductively, as follows. Start with $\sigma_0 = \sigma$, $k_0 = m_{\sigma_0}$. Given that $\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_n, \epsilon)$, we know from (a-iv) that

$$\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_n, \epsilon) \subseteq \bigcup \{D(\tau, \epsilon_{n+1}) : \sigma_n \le \tau \in Q, \, \nu(I_\tau \setminus I_{\sigma_n}) \le \epsilon\},$$

so by (b-ii) we can find a $U_{n+1} \in \mathcal{V}$ and a $\tau_{n+1} \ge \sigma_n$ such that $x_n \in U_{n+1}$, $\nu(I_{\tau_{n+1}} \setminus I_{\sigma_n}) \le \epsilon$ and $\mathcal{N}\mathrm{wd}(U_{n+1}) \subseteq D(\tau_{n+1}, \epsilon_{n+1})$. Next, taking $k_{n+1} = \max(m_{\tau_{n+1}}, 2m_{\sigma_n})$, (a-iii) tells us that

$$\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_n, \epsilon) \subseteq \bigcup \{D(\tau, \epsilon) : \sigma_n \le \tau \in Q, \, m_\tau = k_{n+1}, \, \nu(I_\tau \setminus I_{\sigma_n}) \le \epsilon\},$$

so from (b-i) we see that there is a $\sigma_{n+1} \in Q$ such that $\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_{n+1}, \epsilon)$, $m_{\sigma_{n+1}} = k_{n+1}$, $\sigma_n \le \sigma_{n+1}$ and $\nu(I_{\sigma_{n+1}} \setminus I_{\sigma_n}) \le \epsilon$. Continue.

At the end of the induction, set $E = V \setminus \bigcup_{n \in \mathbb{N}} U_{n+1}$. Because $\{x_n : n \in \mathbb{N}\}$ is dense in V, so is $\bigcup_{n \in \mathbb{N}} U_{n+1}$, and $E \in \mathcal{N}\mathrm{wd}(V)$. By (a-iv) and (b-ii) again, there are a $U_0 \in \mathcal{V}$ and a $\tau_0 \ge \sigma$ such that $E \subseteq U_0$, $\mathcal{N}\mathrm{wd}(U_0) \subseteq D(\tau_0, \epsilon_0)$ and $\nu(I_{\tau_0} \setminus I_\sigma) \le \epsilon$. Now $V \subseteq \bigcup_{n \in \mathbb{N}} U_n$; since V is compact, there is an $n \in \mathbb{N}$ such that $V \subseteq \bigcup_{j \le n} U_j$.

I have still to define τ. Set $k = 2\max(k_n, m_{\tau_0})$. For each $j \le n$, (a-iii) and (b-i), as before, show us that there is an $\upsilon_j \in Q$ such that $\tau_j \le \upsilon_j$, $m_{\upsilon_j} = k$, $\nu(I_{\upsilon_j} \setminus I_{\tau_j}) \le \epsilon_j$ and $\mathcal{N}\mathrm{wd}(U_j) \subseteq D(\upsilon_j, \epsilon_j)$. Try setting $\tau = (k, \bigcup_{j \le n} I_{\upsilon_j})$. Then surely $\mathcal{N}\mathrm{wd}(U_j) \subseteq D(\tau, \epsilon_j)$ for each j. To estimate $\nu(I_\tau \setminus I_\sigma)$, set $K = \bigcup_{j<n} I_{\tau_{j+1}} \setminus I_{\sigma_j}$, $K' = \bigcup_{j<n} I_{\sigma_{j+1}} \setminus I_{\sigma_j}$. By (a-v), νK and $\nu K'$ are both at most 3ϵ. Now

$$\begin{aligned} I_\tau \setminus I_\sigma &\subseteq \bigcup_{j \le n} (I_{\upsilon_j} \setminus I_{\tau_j}) \cup (I_{\tau_0} \setminus I_\sigma) \\ &\qquad \cup \bigcup_{j<n} (I_{\tau_{j+1}} \setminus I_{\sigma_j}) \cup \bigcup_{j \le n} (I_{\sigma_j} \setminus I_\sigma) \\ &= \bigcup_{j \le n} (I_{\upsilon_j} \setminus I_{\tau_j}) \cup (I_{\tau_0} \setminus I_\sigma) \cup K \cup K', \end{aligned}$$

and

$$\begin{aligned} \nu(I_\tau \setminus I_\sigma) &\le \sum_{j=0}^n \nu(I_{\upsilon_j} \setminus I_{\tau_j}) + \nu(I_{\tau_0} \setminus I_\sigma) + \nu K + \nu K' \\ &\le 7\epsilon + \sum_{j=0}^n \epsilon_j \le 8\epsilon, \end{aligned}$$

as required. **Q**

(d) Now we can find $T \subseteq S = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, $\langle \delta_t \rangle_{t \in T}$, $\langle \sigma_t \rangle_{t \in T}$, $\langle \tau_t \rangle_{t \in T}$ and $\langle V_t \rangle_{t \in T}$ such that

T is a tree (that is, $t{\restriction}k \in T$ whenever $t \in T$ and $k \in \mathbb{N}$),
$\delta_t > 0$ for every $t \in T$, $\sum_{t\in T}\delta_t < \infty$,
$\sigma_t \in Q$, $\tau_t \in Q$, $\sigma_t \le \tau_t$ for every $t \in T$,
$\sigma_t = \tau_{t\restriction n}$ whenever $n \in \mathbb{N}$ and $t \in T \cap \mathbb{N}^{n+1}$,
$\nu(I_{\tau_t} \setminus I_{\sigma_t}) \le \delta_t$ for every $t \in T$,
$V_t \in \mathcal{V}$, $\mathcal{N}\mathrm{wd}(V_t) \subseteq D(\sigma_t, \delta_t)$ for every $t \in T$,
$\bigcup\{V_t : t \in T \cap \mathbb{N}^n\} = X$ for every $n \in \mathbb{N}$.

P Begin by choosing strictly positive δ_t, for $t \in S$, such that $\delta_\emptyset = 1$ and $\sum_{t\in S}\delta_t$ is finite. Now choose $T_n \subseteq \mathbb{N}^n$ and $\langle\sigma_t\rangle_{t\in T_n}$, $\langle V_t\rangle_{t\in T_n}$ inductively, as follows. Start with $T_0 = \{\emptyset\}$, $\sigma_\emptyset = (0, \emptyset)$ and $V_\emptyset = X$. Then

$$\mathcal{N}\mathrm{wd}(V_\emptyset) = \mathcal{N}\mathrm{wd}(X) = D((0,\emptyset),1) = D(\sigma_\emptyset, \delta_\emptyset),$$

so the process starts. Given that T_n, $\langle\sigma_t\rangle_{t\in T_n}$ and $\langle V_t\rangle_{t\in T_n}$ have been defined, then for each $t \in T_n$ use (c) to find $n_t \in \mathbb{N}$, $\langle V_{t^\frown<i>}\rangle_{i\le n_t} \in \mathcal{V}^{n_t+1}$ and $\tau_t \in Q$ such that $\sigma_t \le \tau_t$, $\nu(I_{\tau_t} \setminus I_{\sigma_t}) \le \delta_t$, $V_t \subseteq \bigcup_{i\le n_t} V_{t^\frown<i>}$ and $\mathcal{N}\mathrm{wd}(V_{t^\frown<i>}) \subseteq D(\tau_t, \delta_{t^\frown<i>})$ for every $i \le n_t$. Set $T_{n+1} = \{t^\frown<i> : t \in T_n,\ i \le n_t\}$ and $\sigma_t = \tau_{t\restriction n}$ for every $t \in T_{n+1}$, and continue. At the end of the construction, set $T = \bigcup_{n\in\mathbb{N}} T_n$. **Q**

(e) Let $\langle y_n\rangle_{n\in\mathbb{N}}$ run over a dense subset of X. For $n \in \mathbb{N}$, take $t_n \in T \cap \mathbb{N}^n$ such that $y_n \in V_{t_n}$. Since $\mathcal{N}\mathrm{wd}(V_{t_n}) \subseteq D(\sigma_{t_n}, \delta_{t_n})$, we can choose an $F_n \in \mathcal{N}\mathrm{wd}(X)$, containing y_n, such that $\nu(f(F_n) \setminus I_{\sigma_{t_n}}) \le \delta_{t_n}$. Now $\{f(F_n) : n \in \mathbb{N}\}$ is bounded above in $\mathcal{Z}$. **P** Set $K = \bigcup_{t\in T} I_{\sigma_t}$. As $I_{\sigma_\emptyset} = \emptyset$,

$$K = \bigcup_{n\in\mathbb{N}}\bigcup_{t\in T\cap\mathbb{N}^{n+1}} I_{\sigma_t} \setminus I_{\sigma_{t\restriction n}} = \bigcup_{t\in T} I_{\tau_t} \setminus I_{\sigma_t};$$

as $\sum_{t\in T}\nu(I_{\tau_t} \setminus I_{\sigma_t})$ is finite, $K \in \mathcal{Z}$ (526Ac). Next,

$$\bigcup_{n\in\mathbb{N}} f(F_n) \setminus K \subseteq \bigcup_{n\in\mathbb{N}} f(F_n) \setminus I_{\sigma_{t_n}};$$

as

$$\sum_{n=0}^\infty \nu(f(F_n) \setminus I_{\sigma_{t_n}}) \le \sum_{n=0}^\infty \delta_{t_n}$$

is finite, $\bigcup_{n\in\mathbb{N}} f(F_n) \setminus K \in \mathcal{Z}$, so $\bigcup_{n\in\mathbb{N}} f(F_n)$ also belongs to $\mathcal{Z}$, and is an upper bound for $\{f(F_n) : n \in \mathbb{N}\}$. **Q**

(f) On the other hand, $\{F_n : n \in \mathbb{N}\}$ is certainly not bounded above in $\mathcal{N}\mathrm{wd}(X)$, since $\bigcup_{n\in\mathbb{N}} F_n$ includes the dense set $\{y_n : n \in \mathbb{N}\}$. So f cannot be a Tukey function. Since f is arbitrary, $\mathcal{N}\mathrm{wd}(X) \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

(g) Since $\mathcal{N}\mathrm{wd}(X) \equiv_{\mathrm{T}} \mathcal{N}\mathrm{wd}$ (526He), it follows that $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$. Since $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}} \preccurlyeq_{\mathrm{T}} \mathcal{Z}$ (526I, 526J) and $\mathcal{N}\mathrm{wd} \preccurlyeq_{\mathrm{T}} \ell^1$ (526Hc), we see that $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

Remark A somewhat stronger result is in Solecki & Todorčević 10.

526M Having introduced ideals of sets with negligible closures, I add a simple result which will be useful later.

Proposition Let X be a second-countable topological space and μ a σ-finite topological measure on X. Let $\mathcal{E}$ be the ideal of subsets of X with negligible closures, $\mathcal{N}(\mu)$ the null ideal of μ, and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$. Then

$$(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}});$$

consequently $\mathrm{add}(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \ge \mathfrak{m}_{\mathrm{countable}}$.

proof (a) Suppose first that μ is a probability measure. Let $\mathcal{U}$ be a countable base for the topology of X, containing $\emptyset$ and closed under finite unions. For each $n \in \mathbb{N}$, let $\langle U_{ni}\rangle_{i\in\mathbb{N}}$ run over $\{U : U \in \mathcal{U},\ \mu U \ge 1-2^{-n}\}$. For $f \in \mathbb{N}^{\mathbb{N}}$, set

$$\psi(f) = \bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n} X \setminus U_{i,f(i)} \in \mathcal{N}(\mu).$$

For $E \in \mathcal{E}$, set

$$\phi(E) = \{f : f \in \mathbb{N}^{\mathbb{N}},\ E \not\subseteq \psi(f)\}.$$

Then $\phi(E) \in \mathcal{M}$. **P** Since $X \setminus \overline{E}$ is a conegligible open set, we can find for each $i \in \mathbb{N}$ a $g(i) \in \mathbb{N}$ such that $E \cap U_{i,g(i)} = \emptyset$. Now

$$M=\bigcup_{n\in\mathbb{N}}\bigcap_{i\ge n}\{f:f\in\mathbb{N}^{\mathbb{N}},\,f(i)\ne g(i)\}$$

belongs to $\mathcal{M}$. If $f\in\phi(E)$, there is an $x\in E\setminus\psi(f)$, so that $x\in\bigcap_{i\ge n}U_{i,f(i)}$ for some n; now if $i\ge n$, we have $x\in U_{i,f(i)}\setminus U_{i,g(i)}$, so $f(i)\ne g(i)$; thus $f\in M$. Accordingly $\phi(E)\subseteq M\in\mathcal{M}$. **Q**

Now (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{E},\subseteq,\mathcal{N}(\mu))$ to $(\mathcal{M},\not\ni,\mathbb{N}^{\mathbb{N}})$, and $(\mathcal{E},\subseteq,\mathcal{N}(\mu))\preccurlyeq_{\mathrm{GT}}(\mathcal{M},\not\ni,\mathbb{N}^{\mathbb{N}})$.

(b) If $\mu X=0$, then of course $(\mathcal{E},\subseteq,\mathcal{N}(\mu))\preccurlyeq_{\mathrm{GT}}(\mathcal{M},\not\ni,\mathbb{N}^{\mathbb{N}})$ (take $\phi(E)=\emptyset$ for every $E\in\mathcal{E}$, $\psi(f)=X$ for every $f\in\mathbb{N}^{\mathbb{N}}$). Otherwise, there is a probability measure ν on X with the same domain and the same null ideal as μ, so (a) tells us that $(\mathcal{E},\subseteq,\mathcal{N}(\mu))\preccurlyeq_{\mathrm{GT}}(\mathcal{M},\not\ni,\mathbb{N}^{\mathbb{N}})$.

(c) Accordingly

$$\operatorname{add}(\mathcal{E},\subseteq,\mathcal{N}(\mu))\ge\operatorname{add}(\mathcal{M},\not\ni,\mathbb{N}^{\mathbb{N}})=\operatorname{cov}\mathcal{M}$$

(512Db). But, writing $\mathcal{M}(\mathbb{R})$ for the ideal of meager subsets of $\mathbb{R}$, $\operatorname{cov}\mathcal{M}=\operatorname{cov}\mathcal{M}(\mathbb{R})=\mathfrak{m}_{\mathrm{countable}}$, by 522Wb and 522Sa.

Remark If $X=\mathbb{R}$ and μ is Lebesgue measure, then $\operatorname{add}(\mathcal{E},\subseteq,\mathcal{N}(\mu))=\mathfrak{m}_{\mathrm{countable}}$ and $\operatorname{cov}(\mathcal{E},\subseteq,\mathcal{N}(\mu))=\operatorname{non}\mathcal{M}$; see Bartoszyński & Shelah 92 or Bartoszyński & Judah 95, 2.6.14.

526X Basic exercises (a) Let $\nu:\mathcal{P}\mathbb{N}\to[0,1]$ be the submeasure described in 526A. Show that $d^*(I)=\lim_{n\to\infty}\nu(I\setminus n)$ for every $I\subseteq\mathbb{N}$.

(b) For I, $J\subseteq\mathbb{N}$ say that $I\subseteq^* J$ if $I\setminus J$ is finite. Show that $(\mathcal{Z},\subseteq^*,\mathcal{Z})\equiv_{\mathrm{GT}}(\mathcal{Z},\subseteq',[\mathcal{Z}]^{\le\omega})$.

(c) Let $\mathcal{N}\mathit{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$. Show that $\operatorname{add}_\omega\mathcal{N}\mathit{wd}=\operatorname{add}\mathcal{M}$, $\operatorname{non}\mathcal{N}\mathit{wd}=\omega$, $\operatorname{cov}\mathcal{N}\mathit{wd}=\mathfrak{m}_{\mathrm{countable}}$ and $\operatorname{cf}\mathcal{N}\mathit{wd}\le\operatorname{cf}\mathcal{M}$.

(d) Let X be a topological space with a countable π-base, and $\mathcal{N}\mathit{wd}(X)$ the ideal of nowhere dense subsets of X. Show that $\mathcal{N}\mathit{wd}(X)\preccurlyeq_{\mathrm{T}}\mathcal{N}\mathit{wd}$, where $\mathcal{N}\mathit{wd}$ is the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$.

(e) In 526Hf, show that $\mathcal{C}_{\mathrm{nwd}}$ is a G_δ subset of the family $\mathcal{C}$ of all closed subsets of X with its Fell topology, so is a Polish space in the subspace topology.

(f) Let $\mathcal{C}_{\mathrm{Leb}}$ be the family of closed Lebesgue negligible subsets of $[0,1]$. Show that $\mathcal{C}_{\mathrm{Leb}}$ with its Fell topology is a Polish space and a metrizably compactly based directed set.

(g) Let $\mathcal{E}_{\mathrm{Leb}}$ be the ideal of subsets of $\mathbb{R}$ with negligible closures. (i) Show that it is Tukey equivalent to the partially ordered set $\mathcal{C}_{\mathrm{Leb}}$ of 526Xf. (ii) Show that it is isomorphic to $\mathcal{E}_{\mathrm{Leb}}^{\mathbb{N}}$. (iii) Show that if we write $\mathcal{E}_\sigma$ for the σ-ideal of subsets of $\mathbb{R}$ generated by $\mathcal{E}_{\mathrm{Leb}}$, then $(\mathcal{E}_{\mathrm{Leb}},\subseteq',[\mathcal{E}_{\mathrm{Leb}}]^{\le\omega})\equiv_{\mathrm{GT}}(\mathcal{E}_\sigma,\subseteq,\mathcal{E}_\sigma)$. (iv) Show that $\operatorname{add}_\omega\mathcal{E}_{\mathrm{Leb}}=\operatorname{add}\mathcal{E}_\sigma$ and $\operatorname{cf}\mathcal{E}_{\mathrm{Leb}}=\operatorname{cf}\mathcal{E}_\sigma$.

526Y Further exercises (a) Let X be a locally compact separable metrizable space. Let $\mathcal{C}_{\mathrm{nwd}}$ be the family of closed nowhere dense sets in X with its Fell topology. Show that $\mathcal{C}_{\mathrm{nwd}}$ is a metrizably compactly based directed set.

(b) Let $\mathfrak{Z}$ be the asymptotic density algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$ and define $\bar d^*:\mathfrak{Z}\to[0,1]$ by setting $\bar d^*(I^\bullet)=d^*(I)$ for every $I\subseteq\mathbb{N}$, as in 491I. Show that if $A\subseteq\mathfrak{Z}$ is non-empty, downwards-directed and has infimum 0, and $\#(A)<\mathfrak{p}$, then $\inf_{a\in A}\bar d^*(a)=0$. (Compare 491Id.)

(c) Show that $\operatorname{wdistr}(\mathfrak{Z})=\omega_1$.

(d) Show that $\mathfrak{m}(\mathfrak{Z})\ge\mathfrak{m}_{\sigma\text{-linked}}$.

(e) Show that $\mathrm{FN}(\mathcal{P}\mathbb{N})\le\mathrm{FN}(\mathfrak{Z})\le\max(\mathrm{FN}^*(\mathcal{P}\mathbb{N}),(\operatorname{cf}\mathcal{N})^+)$.

(f) Show that $\tau(\mathfrak{Z})\ge\mathfrak{p}$.

526 Notes and comments The 'positive' results of this section are straightforward enough, except perhaps for 526F. As elsewhere in this chapter, I am attempting to describe a framework which will accommodate the many arguments which have been found effective in discussing the cardinal functions of these partially ordered sets. I note that in this section I use the symbol $\mathcal{M}$ to represent the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$, rather than the ideal of meager subsets of $\mathbb{R}$, as elsewhere in the chapter. If you miss this point, however, none of the formulae here are dangerous, because the two ideals are Tukey equivalent, and indeed isomorphic (522Wb).

When we come to 'negative' results, we have problems of a new kind. The special character of Tukey functions is that they need not be of any particular type. They are not asked to be order-preserving, and even if we have partially ordered sets with natural Polish topologies (as in 526A, 526Xe and 526Xf, for instance), Tukey functions between them are not required to be Borel measurable. This means that in order to show that there is *no* Tukey function between a given pair of partially ordered sets, we have had to consider arbitrary functions, or seek to calculate suitable invariants which we know to be related to the Tukey ordering, like precaliber triples (516C), and show that they are incompatible with the existence of a Tukey function. For a discussion of a class of invariants giving very sharp distinctions, see MÁTRAI P09, §3.

Putting 526B and 526H-526L together, we find that we have a complete description of the Tukey ordering on the set $\{\mathbb{N}^{\mathbb{N}}, \mathcal{E}_{\mathrm{Leb}}, \mathcal{N}\mathrm{wd}, \mathcal{Z}, \ell^1\}$, given by the diagram

$$\begin{array}{ccccc} & & \mathcal{N}\mathrm{wd} & — & \ell^1 \\ & & | & & | \\ \mathbb{N}^{\mathbb{N}} & — & \mathcal{E}_{\mathrm{Leb}} & — & \mathcal{Z} \end{array}$$

if we interpret this in the same way as for Cichoń's diagram (522B). Moreover, this is exact, in that no two of the five are Tukey equivalent, and $\mathcal{Z}$ and $\mathcal{N}\mathrm{wd}$ are Tukey incomparable. Note that all five of these partially ordered sets are either themselves metrizably compactly based directed sets (526A, 513Xj, 513Xl) or are Tukey equivalent to metrizably compactly based directed sets (526He-526Hf, 526Xf-526Xg).

In 526Yb-526Yf I list miscellaneous facts about the asymptotic density algebra. A remarkable description of its Dedekind completion is in 556S below.

Version of 22.9.21

527 Skew products of ideals

The methods of this chapter can be applied to a large proportion of the partially ordered sets which arise in analysis. In this section I look at skew products of ideals, constructed by a method suggested by Fubini's theorem and the Kuratowski-Ulam theorem (527E). At the end of the section I introduce 'harmless' algebras (527M-527O).

527A Notation If (X, Σ, μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ; $\mathcal{N}$ will be the null ideal of Lebesgue measure on $\mathbb{R}$. If X is a topological space, $\mathcal{B}(X)$ will be the Borel σ-algebra of X and $\mathcal{M}(X)$ the σ-ideal of meager subsets of X; $\mathcal{M}$ will be the ideal $\mathcal{M}(\mathbb{R})$ of meager subsets of $\mathbb{R}$.

527B Skew products of ideals Suppose that $\mathcal{I} \triangleleft \mathcal{P}X$ and $\mathcal{J} \triangleleft \mathcal{P}Y$ are ideals of subsets of sets X, Y respectively.

(a) I will write $\mathcal{I} \ltimes \mathcal{J}$ for their **skew product** $\{W : W \subseteq X \times Y,\ \{x : W[\{x\}] \notin \mathcal{J}\} \in \mathcal{I}\}$. (This use of the symbol $\ltimes$ is unconnected with the usage in §512 except by the vaguest of analogies.) It is easy to check that $\mathcal{I} \ltimes \mathcal{J} \triangleleft \mathcal{P}(X \times Y)$.

Similarly, $\mathcal{I} \rtimes \mathcal{J}$ will be $\{W : W \subseteq X \times Y,\ \{y : W^{-1}[\{y\}] \notin \mathcal{I}\} \in \mathcal{J}\}$.

(b) Suppose that X and Y are not empty and that $\mathcal{I}$ and $\mathcal{J}$ are proper ideals. Then

$$\mathrm{add}(\mathcal{I} \ltimes \mathcal{J}) = \min(\mathrm{add}\,\mathcal{I}, \mathrm{add}\,\mathcal{J}), \quad \mathrm{cf}(\mathcal{I} \ltimes \mathcal{J}) \ge \max(\mathrm{cf}\,\mathcal{I}, \mathrm{cf}\,\mathcal{J}),$$

$$\mathrm{non}(\mathcal{I} \ltimes \mathcal{J}) = \max(\mathrm{non}\,\mathcal{I}, \mathrm{non}\,\mathcal{J}), \quad \mathrm{cov}(\mathcal{I} \ltimes \mathcal{J}) = \min(\mathrm{cov}\,\mathcal{I}, \mathrm{cov}\,\mathcal{J}).$$

P (i) If $\langle W_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{I} \ltimes \mathcal{J}$ with $\kappa < \min(\mathrm{add}\,\mathcal{I}, \mathrm{add}\,\mathcal{J})$, set $W = \bigcup_{\xi<\kappa} W_\xi$. For each $\xi < \kappa$, $H_\xi = \{x : W_\xi[\{x\}] \notin \mathcal{J}\}$ belongs to $\mathcal{I}$; as $\kappa < \mathrm{add}\,\mathcal{I}$, $H = \bigcup_{\xi<\kappa} H_\xi \in \mathcal{I}$. For any $x \notin H$, $W[\{x\}] = \bigcup_{\xi<\kappa} W_\xi[\{x\}] \in \mathcal{J}$ because $\kappa < \mathrm{add}\,\mathcal{J}$; so $W \in \mathcal{I} \ltimes \mathcal{J}$. As $\langle W_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{add}(\mathcal{I} \ltimes \mathcal{J}) \ge \min(\mathrm{add}\,\mathcal{I}, \mathrm{add}\,\mathcal{J})$.

In the other direction, as $X \notin \mathcal{I}$, $\mathcal{J} = \{F : F \subseteq Y,\ X \times F \in \mathcal{I} \ltimes \mathcal{J}\}$, so $F \mapsto X \times F$ is a Tukey function from $\mathcal{J}$ to $\mathcal{I} \ltimes \mathcal{J}$ and $\mathrm{add}\,\mathcal{J} \ge \mathrm{add}(\mathcal{I} \ltimes \mathcal{J})$, $\mathrm{cf}\,\mathcal{J} \le \mathrm{cf}(\mathcal{I} \ltimes \mathcal{J})$. Similarly, $E \mapsto E \times Y$ is a Tukey function from $\mathcal{I}$ to $\mathcal{I} \ltimes \mathcal{J}$ and $\mathrm{add}\,\mathcal{I} \ge \mathrm{add}(\mathcal{I} \ltimes \mathcal{J})$, $\mathrm{cf}\,\mathcal{I} \le \mathrm{cf}(\mathcal{I} \ltimes \mathcal{J})$.

(ii) Let $A \subseteq X$ and $B \subseteq Y$ be such that $A \notin \mathcal{I}$, $B \notin \mathcal{J}$, $\#(A) = \mathrm{non}\,\mathcal{I}$ and $\#(B) = \mathrm{non}\,\mathcal{J}$. Then $A \times B \notin \mathcal{I} \ltimes \mathcal{J}$, so $\mathrm{non}(\mathcal{I} \ltimes \mathcal{J}) \le \#(A \times B)$. But note that as $\mathcal{I}$ and $\mathcal{J}$ are ideals, A and B are either singletons or infinite; so $\#(A \times B) = \max(\#(A), \#(B))$ and $\mathrm{non}(\mathcal{I} \ltimes \mathcal{J}) \le \max(\mathrm{non}\,\mathcal{I}, \mathrm{non}\,\mathcal{J})$.

In the other direction, take any $W \in \mathcal{P}(X \times Y) \setminus (\mathcal{I} \ltimes \mathcal{J})$. Set $E = \{x : W[\{x\}] \notin \mathcal{J}\}$. Then $\#(E) \ge \mathrm{non}\,\mathcal{I}$ and $\#(W[\{x\}]) \ge \mathrm{non}\,\mathcal{J}$ for every $x \in E$, so $\#(W) \ge \max(\mathrm{non}\,\mathcal{I}, \mathrm{non}\,\mathcal{J})$; as W is arbitrary, $\mathrm{non}(\mathcal{I} \ltimes \mathcal{J}) \ge \max(\mathrm{non}\,\mathcal{I}, \mathrm{non}\,\mathcal{J})$.

(iii) If $\mathcal{A} \subseteq \mathcal{I}$ covers X, then $\{A \times Y : A \in \mathcal{A}\} \subseteq \mathcal{I} \ltimes \mathcal{J}$ covers $X \times Y$; so $\mathrm{cov}(\mathcal{I} \ltimes \mathcal{J}) \le \mathrm{cov}\,\mathcal{I}$. Similarly, $\mathrm{cov}(\mathcal{I} \ltimes \mathcal{J}) \le \mathrm{cov}\,\mathcal{J}$.

Now suppose that $\mathcal{W} \subseteq \mathcal{I} \ltimes \mathcal{J}$ and that $\#(\mathcal{W}) < \min(\mathrm{cov}\,\mathcal{I}, \mathrm{cov}\,\mathcal{J})$. For each $W \in \mathcal{W}$ set $E_W = \{x : W[\{x\}] \notin \mathcal{J}\}$; then $E_W \in \mathcal{I}$ for every W, so there is an $x \in X \setminus \bigcup_{W\in\mathcal{W}} E_W$, because $\#(\mathcal{W}) < \mathrm{cov}\,\mathcal{I}$. Now $W[\{x\}] \in \mathcal{J}$ for every W, so there is a $y \in Y \setminus \bigcup_{W\in\mathcal{W}} W[\{x\}]$, because $\#(\mathcal{W}) < \mathrm{cov}\,\mathcal{J}$. In this case $(x, y) \in (X \times Y) \setminus \bigcup \mathcal{W}$. As $\mathcal{W}$ is arbitrary, $\mathrm{cov}(\mathcal{I} \ltimes \mathcal{J}) \ge \min(\mathrm{cov}\,\mathcal{I}, \mathrm{cov}\,\mathcal{J})$ and we have equality. **Q**

(c) The idea of the operation $\ltimes$ here is that we iterate notions of 'negligible set' in a way indicated by Fubini's theorem: a measurable subset of $\mathbb{R}^2$ is negligible iff almost every vertical section is negligible, that is, iff it belongs to $\mathcal{N} \ltimes \mathcal{N}$. However it is immediately apparent that $\mathcal{N} \ltimes \mathcal{N}$ contains many non-measurable sets, and indeed many sets of full outer measure (527Xa). We are therefore led to the following idea. If Λ is a family of subsets of $X \times Y$, write $\mathcal{I} \ltimes_\Lambda \mathcal{J} \subseteq \mathcal{I} \ltimes \mathcal{J}$ for the ideal generated by $(\mathcal{I} \ltimes \mathcal{J}) \cap \Lambda$. Note that if $\kappa \le \min(\mathrm{add}\,\mathcal{I}, \mathrm{add}\,\mathcal{J})$ and $\bigcup \mathcal{W} \in \Lambda$ for every $\mathcal{W} \in [\Lambda]^{<\kappa}$, then $\mathrm{add}(\mathcal{I} \ltimes_\Lambda \mathcal{J}) \ge \kappa$; in particular, $\mathcal{I} \ltimes_\Lambda \mathcal{J}$ will be a σ-ideal whenever $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals and Λ is a σ-algebra of subsets of $X \times Y$. Typical applications will be with Λ a Borel σ-algebra or an algebra of the form $\Sigma\widehat{\otimes}\mathrm{T}$. Thus 252F tells us that

> if (X, Σ, μ) and (Y, T, ν) are measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$ then $\mathcal{N}(\mu) \ltimes_\Lambda \mathcal{N}(\nu) \subseteq \mathcal{N}(\lambda)$.

If μ and ν are σ-finite then we get

$$\mathcal{N}(\lambda) = \mathcal{N}(\mu) \ltimes_{\Sigma\widehat{\otimes}\mathrm{T}} \mathcal{N}(\nu)$$

(252C). If we take $\mathcal{B} = \mathcal{B}(\mathbb{R}^2)$ to be the Borel σ-algebra of $\mathbb{R}^2$, then all four ideals $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$, $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$, $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$ and $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$ become interesting. In the next few paragraphs I will sketch some of the ideas needed to deal with ideals of these kinds.

527C We are already familiar with $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$; I begin by repeating a result from §417 in this language.

Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be σ-finite effectively locally finite τ-additive topological measure spaces, both measures being inner regular with respect to the Borel sets. Let $\tilde\lambda$ be the τ-additive product measure on $X \times Y$ (417C, 417F[6]). Then $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{N}(\nu) = \mathcal{N}(\tilde\lambda)$.

proof (a) To begin with, suppose that μ and ν are complete. Then 417C(b-vi) and 417G[7] tell us that $\tilde\lambda$ is inner regular with respect to the Borel sets, and that a Borel subset of $X \times Y$ is $\tilde\lambda$-negligible iff it belongs to $\mathcal{N}(\mu) \ltimes \mathcal{N}(\nu)$. On the other hand, because μ and ν are σ-finite, so is $\tilde\lambda$ (251K), and every $\tilde\lambda$-negligible set is included in a $\tilde\lambda$-negligible Borel set. **P** Suppose that $W \in \mathcal{N}(\tilde\lambda)$. Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a cover of $X \times Y$ by sets of finite measure. Because $\tilde\lambda$ is inner regular with respect to the Borel sets, we can find $V_n \in \mathcal{B}(X \times Y)$ such that $V_n \subseteq W_n \setminus W$ and $\tilde\lambda V_n = \tilde\lambda W_n$ for each n. Now

$$W \subseteq (X \times Y) \setminus \bigcup_{n\in\mathbb{N}} V_n \in \mathcal{N}(\tilde\lambda) \cap \mathcal{B}(X \times Y). \ \mathbf{Q}$$

[6] Formerly 417G.

[7] Formerly 417H.

So $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{N}(\nu) = \mathcal{N}(\tilde{\lambda})$.

(b) For the general case, let $\hat{\mu}$ and $\hat{\nu}$ be the completions of μ and ν. Just because they extend μ and ν, they are τ-additive topological measures; they are inner regular with respect to the Borel sets by 412Ha; and similarly they are inner regular with respect to subsets of open sets of finite measure, that is, they are effectively locally finite. Write $\tilde{\lambda}\hat{}$ for the τ-additive product of $\hat{\mu}$ and $\hat{\nu}$; this is a complete locally determined effectively locally finite topological measure on $X \times Y$. By 417C(b-iv), it is inner regular with respect to $(\Sigma\widehat{\otimes}\mathrm{T}) \vee \mathcal{B}(X \times Y)$, and of course

$$\tilde{\lambda}\hat{}(E \times F) = \hat{\mu}E \cdot \hat{\nu}F = \mu E \cdot \nu F$$

whenever $E \in \Sigma$ and $F \in \mathrm{T}$. By the uniqueness assertion in 417Ca, $\tilde{\lambda}\hat{} = \tilde{\lambda}$. So

$$\mathcal{N}(\tilde{\lambda}) = \mathcal{N}(\tilde{\lambda}\hat{}) = \mathcal{N}(\hat{\mu}) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{N}(\hat{\nu})$$

(by (a) above)

$$= \mathcal{N}(\mu) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{N}(\nu)$$

by 212Eb.

527D The case $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$ is also well known.

Theorem Let X and Y be topological spaces, with product $X \times Y$. Write $\mathcal{M}^* = \mathcal{M}(X) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{M}(Y)$ and $\mathcal{M}_1^* = \mathcal{M}(X) \ltimes_{\widehat{\mathcal{B}}(X\times Y)} \mathcal{M}(Y)$, writing $\widehat{\mathcal{B}}(X \times Y)$ for the Baire-property algebra of $X \times Y$.

(a) If $\mathcal{M}(X \times Y) \subseteq \mathcal{M}_1^*$, then $\mathcal{M}^* = \mathcal{M}_1^* = \mathcal{M}(X \times Y)$.

(b) Let $\mathfrak{G}$ be the category algebra of Y (514I). If $\pi(\mathfrak{G}) < \operatorname{add}\mathcal{M}(X)$ then $\mathcal{M}^* = \mathcal{M}(X \times Y)$.

proof (a)(i) ? If $\mathcal{M}_1^* \ne \mathcal{M}(X \times Y)$, there is a set $W \in \mathcal{M}_1^* \setminus \mathcal{M}(X \times Y)$; take $W_1 \in \widehat{\mathcal{B}}(X \times Y) \cap (\mathcal{M}(X) \ltimes \mathcal{M}(Y))$ such that $W_1 \supseteq W$. By 4A3Sa[8], there is an open set $V \subseteq X \times Y$ such that $W_1 \triangle V$ is meager and $V \cap V'$ is empty whenever $V' \subseteq X \times Y$ is open and $V' \cap W_1$ is meager. As $W_1 \notin \mathcal{M}(X \times Y)$, V cannot be empty; let $G \subseteq X$, $H \subseteq Y$ be non-empty open sets such that $G \times H \subseteq V$. In this case, $G \times H$ cannot be meager, so neither G nor H can be meager. (If $F \subseteq X$ is nowhere dense, then $F \times Y$ is nowhere dense in $X \times Y$; so $M \times Y \in \mathcal{M}(X \times Y)$ whenever $M \in \mathcal{M}(X)$; as $G \times Y \notin \mathcal{M}(X \times Y)$, $G \notin \mathcal{M}(X)$.) But now we see that

$$\{x : (G \times H)[\{x\}] \notin \mathcal{M}(Y)\} = G \notin \mathcal{M}(X),$$

so that $G \times H \notin \mathcal{M}_1^*$; but $(G \times H) \setminus W_1$ is meager, so belongs to $\mathcal{M}_1^*$, and W_1 is also supposed to belong to $\mathcal{M}_1^*$. **X**

(ii) So $\mathcal{M}_1^* = \mathcal{M}(X \times Y)$. Of course $\mathcal{M}^* \subseteq \mathcal{M}_1^*$ just because $\mathcal{B}(X \times Y) \subseteq \widehat{\mathcal{B}}(X \times Y)$. In the other direction, if $W \in \mathcal{M}(X \times Y)$ there is a meager F_σ set $W' \supseteq W$. Now W' is a Borel set in $\mathcal{M}(X \times Y) = \mathcal{M}_1^*$, so $W' \in \mathcal{M}(X) \ltimes \mathcal{M}(Y)$ witnesses that $W \in \mathcal{M}^*$. Thus $\mathcal{M}(X \times Y) \subseteq \mathcal{M}^*$ and the three classes are equal.

(b) By (a), I have only to show that $W \in \mathcal{M}^*$ whenever $W \subseteq X \times Y$ is meager. Let $D \subseteq \mathfrak{G} \setminus \{0\}$ be an order-dense subset with cardinal $\pi(\mathfrak{G})$. Let H be the smallest comeager regular open subset of Y, so that an open subset of Y is meager iff it is disjoint from H (4A3Sa again). For each $d \in D$ let $V_d \subseteq Y$ be an open set such that $V_d^\bullet = d$ in $\mathfrak{G}$; since $H^\bullet = 1$, we may suppose that $V_d \subseteq H$. Observe that if $F \subseteq Y$ is a non-meager closed set, then there is a $d \in D$ such that $0 \ne d \subseteq F^\bullet$ in $\mathfrak{G}$, in which case $V_d \setminus F$ is meager; as $V_d \subseteq H$, $V_d \subseteq F$.

If $W \subseteq X \times Y$ is a nowhere dense closed set, it belongs to $\mathcal{M}^*$. **P** Set $E = \{x : W[\{x\}] \text{ is not meager}\}$. For each $d \in D$, the set

$$E_d = \{x : V_d \subseteq W[\{x\}]\} = \{x : (x, y) \in W \text{ for every } y \in V_d\}$$

is a closed set in X and $E_d \times V_d \subseteq W$; so $\operatorname{int} E_d \times V_d$ is an open subset of W. As W is nowhere dense, and $V_d \ne \emptyset$, $\operatorname{int} E_d$ must be empty, and $E_d \in \mathcal{M}(X)$. Next, $E = \bigcup_{d\in D} E_d$ and $\#(D) = \pi(\mathfrak{G}) < \operatorname{add}\mathcal{M}(X)$, so $E \in \mathcal{M}(X)$ and $W \in \mathcal{M}^*$. **Q**

Since $\mathcal{M}^*$ is a σ-ideal, it follows that every meager subset of $X \times Y$ belongs to $\mathcal{M}^*$, as required.

[8]Formerly 4A3Ra.

527E Corollary If X and Y are separable metrizable spaces, then $\mathcal{M}(X\times Y)=\mathcal{M}(X)\ltimes_{\mathcal{B}(X\times Y)}\mathcal{M}(Y)$.

proof $\pi(\mathfrak{C})\le\pi(Y)\le w(Y)\le\omega<\operatorname{add}\mathcal{M}(X)$ (514Ja, 5A4Ba, 4A2P(a-i)).

Remark The case $X=Y=\mathbb{R}$ is the **Kuratowski-Ulam theorem**.

527F If we mix measure and category, as in $\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{N}$ and $\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}$, we encounter some new phenomena. To deal with the first we need the following, which is important for other reasons.

Lemma (see Cichoń & Pawlikowski 86) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X generated by $\Sigma\cap\mathcal{I}$; suppose that the quotient algebra $\Sigma/\Sigma\cap\mathcal{I}$ is non-zero, atomless and has countable π-weight. Let Y be a set, T a σ-algebra of subsets of Y, and $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ a sequence of finite covers of Y by members of T. Set

$$\mathcal{H}^*_n=\{\bigcup_{m\ge n}H_m:H_m\in\mathcal{H}_m\cup\{\emptyset\}\text{ for every }m\ge n\}$$

for each $n\in\mathbb{N}$. Then there is a sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}^{\mathbb{N}}\times X\times Y$ such that

(i) for every $n\in\mathbb{N}$, W_n is expressible as the union of a sequence of sets of the form $I\times E\times F$ where $I\subseteq\mathbb{N}^{\mathbb{N}}$ is open-and-closed, $E\in\Sigma$ and $F\in\mathrm{T}$;

(ii) whenever $n\in\mathbb{N}$, $\alpha\in\mathbb{N}^{\mathbb{N}}$ and $x\in X$ then $\{y:(\alpha,x,y)\in W_n\}\in\mathcal{H}^*_n$;

(iii) setting $W=\bigcap_{n\in\mathbb{N}}W_n$, the set $\{(\alpha,x):\alpha\in\mathbb{N}^{\mathbb{N}},\,x\in X,\,(\alpha,x,f(x))\notin W\}$ belongs to $[\mathbb{N}^{\mathbb{N}}]^{\le\omega}\ltimes\mathcal{I}$ for every (Σ,T)-measurable function $f:X\to Y$.

proof If $X\in\mathcal{I}$ or Y is empty, we can take every W_n to be $\emptyset$; suppose otherwise.

(a) Set $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$. There is a family $\langle U_\sigma\rangle_{\sigma\in S}$ such that

every U_σ belongs to $\Sigma\setminus\mathcal{I}$,

for every $\sigma\in S$, $\langle U_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ is a disjoint sequence of subsets of U_σ and $U_\sigma\setminus\bigcup_{i\in\mathbb{N}}U_{\sigma^\frown<i>}\in\mathcal{I}$,

(see 5A1C for the notation here),

for every $E\in\Sigma\setminus\mathcal{I}$ there is a $\sigma\in S$ such that $U_\sigma\setminus E\in\mathcal{I}$.

P Let D be a countable order-dense set in $\mathfrak{A}=\Sigma/\Sigma\cap\mathcal{I}$. Then the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by D is countable and atomless and non-trivial. Let $\mathcal{E}$ be the subalgebra of $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ generated by the sets $I_\sigma=\{\alpha:\sigma\subseteq\alpha\in\mathbb{N}^{\mathbb{N}}\}$ for $\sigma\in S$. This is also an atomless countable Boolean algebra, and must therefore be isomorphic to $\mathfrak{B}$ (316M). Let $\pi:\mathcal{E}\to\mathfrak{B}$ be an isomorphism, and set $b_\sigma=\pi(I_\sigma)$ for each $\sigma\in S$. Set $U_\emptyset=X$ and for $n\in\mathbb{N}$, $\sigma\in\mathbb{N}^n$ choose a disjoint sequence $\langle U_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ of subsets of U_σ such that $U^\bullet_{\sigma^\frown<i>}=b_{\sigma^\frown<i>}$ for every i. This construction ensures that $U_\sigma\in\Sigma\setminus\mathcal{I}$ for every σ. If $E\in\Sigma\setminus\mathcal{I}$, there must be a non-zero $d\in D$ such that $d\subseteq E^\bullet$; now $\pi^{-1}(d)\in\mathcal{E}\setminus\{\emptyset\}$, so there is a $\tau\in S$ such that $I_\tau\subseteq\pi^{-1}(d)$, $b_\tau\subseteq E^\bullet$ and $U_\tau\setminus E\in\mathcal{I}$. Finally, if $\sigma\in S$, set $E=U_\sigma\setminus\bigcup_{i\in\mathbb{N}}U_{\sigma^\frown<i>}$; then for every $\tau\in S$ either $\tau\subseteq\sigma$ and $U_\tau\setminus E\supseteq U_{\sigma^\frown<0>}\notin\mathcal{I}$, or $U_\tau\cap U_\sigma=\emptyset$ and $U_\tau\setminus E=U_\tau\notin\mathcal{I}$, or there is an $i\in\mathbb{N}$ such that $\tau\supseteq\sigma^\frown<i>$ and again $U_\tau\setminus E=U_\tau\notin\mathcal{I}$. This means that E must belong to $\mathcal{I}$, so that $\langle U_\sigma\rangle_{\sigma\in S}$ has all the required properties. **Q**

(b) Enumerate S as $\langle\tau_k\rangle_{k\in\mathbb{N}}$. Let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n$. For $n\in\mathbb{N}$, set

$$K_n=\{(\sigma,k):\sigma\in\mathbb{N}^{n+2},\,k<\#(\tau_{\sigma(n)}),\,\sigma(n+1)=\#(\tau_k),\,H_{\tau_{\sigma(n)}(k)}\in\mathcal{H}_n\},$$

$$V_n=\bigcup_{(\sigma,k)\in K_n}\{(\alpha,x,y):\tau_k\subseteq\alpha\in\mathbb{N}^{\mathbb{N}},\,x\in U_\sigma,\,y\in H_{\tau_{\sigma(n)}(k)}\}.$$

If $\alpha\in\mathbb{N}^{\mathbb{N}}$ and $x\in X$ the section $\{y:(\alpha,x,y)\in V_n\}$ is either empty or $H_{\tau_{\sigma(n)}(k)}$ where $\sigma\in\mathbb{N}^{n+2}$, $x\in U_\sigma$ and $\tau_k=\alpha\restriction\sigma(n+1)$; in either case it belongs to $\mathcal{H}_n\cup\{\emptyset\}$.

So if we now set $W_n=\bigcup_{m\ge n}V_m$, W_n satisfies (i) and (ii) for every n.

(c) Set $W=\bigcap_{n\in\mathbb{N}}W_n$. **?** Suppose, if possible, that $f:X\to Y$ is a (Σ,T)-measurable function such that $\{(\alpha,x):(\alpha,x,f(x))\notin W\}\notin[\mathbb{N}^{\mathbb{N}}]^{\le\omega}\ltimes\mathcal{I}$. Note that

$$\{V:V\subseteq\mathbb{N}^{\mathbb{N}}\times X\times Y,\,\{x:(\alpha,x,f(x))\in V\}\in\Sigma\text{ for every }\alpha\in\mathbb{N}^{\mathbb{N}}\}$$

is a σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}}\times X\times Y$ containing $I\times E\times F$ whenever I is open-and-closed, $E\in\Sigma$ and $F\in\mathrm{T}$, so contains every V_n and every W_n.

Set

$$A_0=\{\alpha:\alpha\in\mathbb{N}^{\mathbb{N}},\,\{x:(\alpha,x,f(x))\notin W\}\notin\mathcal{I}\},$$

so that A_0 is uncountable. For each $\alpha\in A_0$,

$$\bigcup_{n\in\mathbb{N}}\{x:(\alpha,x,f(x))\notin W_n\}=\{x:(\alpha,x,f(x))\notin W\}$$

does not belong to $\mathcal{I}$. So there is an $n\in\mathbb{N}$ such that

$$A_1=\{\alpha:\alpha\in A_0,\,\{x:(\alpha,x,f(x))\notin W_n\}\notin\mathcal{I}\}$$

is uncountable. For each $\alpha\in A_1$, set $G_\alpha=\{x:(\alpha,x,f(x))\notin W_n\}$; then $G_\alpha\in\Sigma\setminus\mathcal{I}$, so there is a $\sigma\in S$ such that $U_\sigma\setminus G_\alpha\in\mathcal{I}$. Let $\sigma\in S$ be such that

$$A_2=\{\alpha:\alpha\in A_1,\,U_\sigma\setminus G_\alpha\in\mathcal{I}\}$$

is infinite. Set $m=\max(n,\#(\sigma))$; then $U_\sigma\cap\{x:(\alpha,x,f(x))\in V_m\}\in\mathcal{I}$ for every $\alpha\in A_2$. Set $M=\#(\mathcal{H}_m)$.

Take $k\in\mathbb{N}$ such that $\#(\{\alpha\restriction k:\alpha\in A_2\})\ge M$. Let $\langle\alpha_i\rangle_{i<M}$ be a family in A_2 such that $\alpha_i\restriction k\ne\alpha_j\restriction k$ for distinct i, $j<M$; let $\langle r_i\rangle_{i<M}$, $\langle l_i\rangle_{i<M}$ be such that $\alpha_i\restriction k=\tau_{r_i}$ for each i and $\mathcal{H}_m=\{H_{l_i}:i<M\}$. Let $s\in\mathbb{N}$ be such that $\tau_s(r_i)$ is defined and equal to l_i for $i<M$. Let $\sigma'\in\mathbb{N}^{m+2}$ be such that $\sigma'\supseteq\sigma$, $\sigma'(m)=s$ and $\sigma'(m+1)=k$. Then $U_{\sigma'}\notin\mathcal{I}$ and $U_{\sigma'}\setminus G_\alpha\in\mathcal{I}$ for every $\alpha\in A_2$.

Suppose that $i<M$ and $x\in U_{\sigma'}$. Then

$$\{y:(\alpha_i,x,y)\in V_m\}=H_{\tau_{\sigma'(m)}}(j)=H_{\tau_s(j)}$$

where $(\sigma',j)\in K_m$, that is, j is such that $\tau_j\subseteq\alpha_i$ and $\#(\tau(j))=\sigma'(m+1)=k$. Thus $j=r_i$, $\tau_s(j)=l_i$ and $\{y:(\alpha_i,x,y)\in V_m\}=H_{l_i}$. But this means that, for any $x\in U_{\sigma'}$,

$$\bigcup_{i<M}\{y:(\alpha_i,x,y)\in V_m\}=\bigcup_{i<M}H_{l_i}=Y$$

contains $f(x)$; that is, $U_{\sigma'}\subseteq\bigcup_{i<M}\{x:(\alpha_i,x,f(x))\in V_m\}$. On the other hand,

$$U_{\sigma'}\cap\{x:(\alpha_i,x,f(x))\in V_m\}\subseteq U_\sigma\cap\{x:(\alpha_i,x,f(x))\in V_m\}\in\mathcal{I}$$

for each $i<M$, while $U_{\sigma'}$ itself does not belong to $\mathcal{I}$. So this is impossible. **X**

Thus $\langle W_n\rangle_{n\in\mathbb{N}}$ satisfies (iii).

527G Theorem Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X which is generated by $\Sigma\cap\mathcal{I}$; suppose that the quotient algebra $\Sigma/\Sigma\cap\mathcal{I}$ is non-zero, atomless and has countable π-weight. Let (Y,T,ν) be an atomless perfect semi-finite measure space such that $\nu Y>0$. Set $\mathcal{K}=\mathcal{I}\ltimes_{\Sigma\widehat{\otimes}\mathrm{T}}\mathcal{N}(\nu)$. Then $[\mathfrak{c}]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{K}$, so $\operatorname{add}\mathcal{K}=\omega_1$ and $\operatorname{cf}\mathcal{K}\ge\mathfrak{c}$.

proof (a) To begin with (down to the end of (d)) suppose that ν is totally finite. Because ν is atomless, we can for each $n\in\mathbb{N}$ find a finite cover $\mathcal{H}_n$ of Y by measurable sets with measures at most 2^{-n}. Let T_0 be the σ-algebra generated by $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n$, so that T_0 is a σ-subalgebra of T. Construct $\langle\mathcal{H}^*_n\rangle_{n\in\mathbb{N}}$, $\langle W_n\rangle_{n\in\mathbb{N}}$ and W from $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ as in 527F. Then if $f:X\to Y$ is (Σ,T_0)-measurable, $\{(\alpha,x):(\alpha,x,f(x))\notin W\}\in[\mathbb{N}^{\mathbb{N}}]^{\le\omega}\ltimes\mathcal{I}$. Note that $\nu H\le 2^{-n+1}$ for every $H\in\mathcal{H}^*_n$, so $\nu\{y:(\alpha,x,y)\in W_n\}\le 2^{-n+1}$ for every $\alpha\in\mathbb{N}^{\mathbb{N}}$, $x\in X$ and $n\in\mathbb{N}$. For each $\alpha\in\mathbb{N}^{\mathbb{N}}$ set $K_\alpha=\{(x,y):(\alpha,x,y)\in W\}$. Observe that $K_\alpha\in\Sigma\widehat{\otimes}\mathrm{T}$ because $W\in\mathcal{B}(\mathbb{N}^{\mathbb{N}})\widehat{\otimes}\Sigma\widehat{\otimes}\mathrm{T}$, and that

$$\nu K_\alpha[\{x\}]\le\inf_{n\in\mathbb{N}}\nu\{y:(\alpha,x,y)\in W_n\}=0$$

for every $x\in X$, so $K_\alpha\in\mathcal{K}$ for every $\alpha\in\mathbb{N}^{\mathbb{N}}$.

(b) Set $\hat\Sigma=\{E\triangle M:E\in\Sigma,\,M\in\mathcal{I}\}$. Then $\hat\Sigma$ is a σ-algebra of subsets of X (cf. 212Ca) and $\mathcal{I}$ is a σ-ideal in $\hat\Sigma$; also the identity embedding of Σ in $\hat\Sigma$ induces an isomorphism between $\Sigma/\Sigma\cap\mathcal{I}$ and $\hat\Sigma/\mathcal{I}$ (cf. 322Da). Consequently $\hat\Sigma/\mathcal{I}$ has countable π-weight, therefore is ccc, and $\hat\Sigma$ is closed under Souslin's operation (431G).

(c) Let $A\subseteq\mathbb{N}^{\mathbb{N}}$ be an uncountable set, and $V\in\Sigma\widehat{\otimes}\mathrm{T}$ a set disjoint from $\bigcup_{\alpha\in A}K_\alpha$. (I aim to show that $(X\times Y)\setminus V\notin\mathcal{I}\ltimes\mathcal{N}(\nu)$.) There must be sequences $\langle C_n\rangle_{n\in\mathbb{N}}$ in Σ, $\langle F_n\rangle_{n\in\mathbb{N}}$ in T such that V belongs to the σ-algebra generated by $\{C_n\times F_n:n\in\mathbb{N}\}$; we can of course arrange that $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n\subseteq\{F_n:n\in\mathbb{N}\}$. Let T_1 be the σ-subalgebra of T generated by $\{F_n:n\in\mathbb{N}\}$, so that $\mathrm{T}_0\subseteq\mathrm{T}_1$ and $V\in\Sigma\widehat{\otimes}\mathrm{T}_1$. Let $g:Y\to\mathbb{R}$

be the Marczewski functional defined by setting $g=\sum_{n=0}^{\infty}3^{-n}\chi F_n$. Because ν is perfect, there is a Borel set $H\subseteq g[Y]$ such that $g^{-1}[H]$ is conegligible. Let $h:H\to Y$ be any function such that $g(h(t))=t$ for every $t\in H$; note that h is $(\mathcal{B}(H),\mathrm{T}_1)$-measurable, where $\mathcal{B}(H)$ is the Borel σ-algebra of H, just because $\overline{g[F_n]}\cap\overline{g[Y\setminus F_n]}$ is empty for every n. Set $V_0=\{(x,t):x\in X,\,t\in H,\,(x,h(t))\in V\}$; then $V_0\in\Sigma\widehat{\otimes}\mathcal{B}(H)$. It follows that V_0 belongs to the class of sets obtainable by Souslin's operation from sets of the form $E\times F$ where $E\in\Sigma$ and $F\subseteq H$ is relatively closed in H. (Use 421F.)

Set $\tilde{E}=V_0^{-1}[H]$. Because H is analytic and $\hat{\Sigma}$ is closed under Souslin's operation, $\tilde{E}\in\hat{\Sigma}$ and there is a $(\hat{\Sigma},\mathcal{B}(H))$-measurable function $f_1:\tilde{E}\to H$ such that $(x,f_1(x))\in V_0$ for every $x\in\tilde{E}$ (423N). Now $f_2=hf_1:\tilde{E}\to Y$ is $(\hat{\Sigma},\mathrm{T}_1)$-measurable and $(x,f_2(x))\in V$ for every $x\in\tilde{E}$.

For every $n\in\mathbb{N}$, $E_n=f_1^{-1}[F_n]$ belongs to $\hat{\Sigma}$, so there is an $E'_n\in\Sigma$ such that $E_n\triangle E'_n\in\mathcal{I}$. Similarly, there is an $\tilde{E}'\in\Sigma$ such that $\tilde{E}\triangle\tilde{E}'\in\mathcal{I}$. Because $\mathcal{I}$ is generated by $\Sigma\cap\mathcal{I}$, there is an $M_0\in\Sigma\cap\mathcal{I}$ including $(\tilde{E}\triangle\tilde{E}')\cup\bigcup_{n\in\mathbb{N}}(E_n\triangle E'_n)$. Now $\tilde{E}\setminus M_0=\tilde{E}'\setminus M_0$ belongs to Σ. Set $f_3=f_2\restriction\tilde{E}\setminus M_0$; then $f_3^{-1}[F_n]=E'_n\setminus M_0\in\Sigma$ for every n, so f_3 is (Σ,T_1)-measurable. Take any $y_0\in Y$, and set $f(x)=f_3(x)$ if $x\in\tilde{E}\setminus M_0$, y_0 for other $x\in X$; then f is (Σ,T_1)-measurable, therefore (Σ,T_0)-measurable.

The set $\{(\alpha,x):(\alpha,x,f(x))\notin W\}$ belongs to $[\mathbb{N}^{\mathbb{N}}]^{\le\omega}\ltimes\mathcal{I}$, so there must be an $\alpha\in A$ such that $M_1=\{x:(\alpha,x,f(x))\notin W\}$ belongs to $\mathcal{I}$. **?** Suppose, if possible, that $(X\times Y)\setminus V\in\mathcal{I}\ltimes\mathcal{N}(\nu)$. Then there must be an $x\in X\setminus(M_0\cup M_1)$ such that $V[\{x\}]$ is conegligible. In this case, $V[\{x\}]\cap g^{-1}[H]$ is conegligible, so is not empty, and there is a $y\in V[\{x\}]\cap g^{-1}[H]$. Consider $y'=h(g(y))$; then $g(y')=g(y)$, so $\{n:y'\in F_n\}=\{n:y\in F_n\}$, and $\{F:y\in F\iff y'\in F\}$ is a σ-algebra of subsets of Y containing every F_n and therefore containing $V[\{x\}]$. So $y'\in V[\{x\}]$ and $(x,g(y))\in V_0$. This means that $x\in\tilde{E}$; as $x\notin M_0$, $f(x)=f_3(x)=f_2(x)$ and $(x,f(x))\in V$. On the other hand, $x\notin M_1$, so $(\alpha,x,f(x))\in W$ and $(x,f(x))\in K_\alpha$; contradicting the choice of V as a set disjoint from K_α. **X**

This shows that $(X\times Y)\setminus V\notin\mathcal{I}\ltimes\mathcal{N}(\nu)$. As V is arbitrary, $\bigcup_{\alpha\in A}K_\alpha\notin\mathcal{K}$.

(d) This is true for every uncountable $A\subseteq\mathbb{N}^{\mathbb{N}}$. But this means that $A\mapsto\bigcup_{\alpha\in A}K_\alpha$ is a Tukey function from $[\mathbb{N}^{\mathbb{N}}]^{\le\omega}$ to $\mathcal{K}$, and $[\mathfrak{c}]^{\le\omega}\cong[\mathbb{N}^{\mathbb{N}}]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{K}$.

(e) Thus the theorem is true if νY is finite. For the general case, let $Y_0\in\mathrm{T}$ be such that $0<\nu Y_0<\infty$. Then the subspace measure ν_{Y_0} is still atomless and perfect (214Ka, 451Dc), so $[\mathfrak{c}]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{K}_0$, where $\mathcal{K}_0=\mathcal{I}\ltimes_{\Sigma\widehat{\otimes}(\mathrm{T}\cap\mathcal{P}Y_0)}\mathcal{N}(\nu_{Y_0})$. But $\mathcal{K}_0=\mathcal{K}\cap\mathcal{P}(X\times Y_0)$, so the identity map from $\mathcal{K}_0$ to $\mathcal{K}$ is a Tukey function, and

$$[\mathfrak{c}]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{K}_0\preccurlyeq_{\mathrm{T}}\mathcal{K}$$

in this case also. It follows at once that $\operatorname{add}\mathcal{K}\le\operatorname{add}[\mathfrak{c}]^{\le\omega}=\omega_1$, so that $\operatorname{add}\mathcal{K}=\omega_1$ and $\operatorname{cf}\mathcal{K}\ge\operatorname{cf}[\mathfrak{c}]^{\le\omega}=\mathfrak{c}$.

527H Corollary $\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{N}\equiv_{\mathrm{T}}[\mathfrak{c}]^{\le\omega}$.

proof By 527G, $[\mathfrak{c}]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{N}$. In the other direction, all we need to observe is that $\#(\mathcal{B}(\mathbb{R}^2))=\mathfrak{c}$. Let $\langle W_\xi\rangle_{\xi<\mathfrak{c}}$ run over $\mathcal{B}(\mathbb{R}^2)\cap(\mathcal{M}\ltimes\mathcal{N})$, and for $V\in\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{N}$ choose $\xi_V<\mathfrak{c}$ such that $V\subseteq W_{\xi_V}$; then $V\mapsto\{\xi_V\}:\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{N}\to[\mathfrak{c}]^{\le\omega}$ is a Tukey function, so $\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\preccurlyeq_{\mathrm{T}}[\mathfrak{c}]^{\le\omega}$.

527I I now turn to the ideal $\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}$.

Lemma Let X be a set, Σ a σ-algebra of subsets of X, and Y a topological space with a countable π-base $\mathcal{H}$. Let $\mathcal{W}$ be the family of subsets of $X\times Y$ of the form $\bigcup_{H\in\mathcal{H}}E_H\times H$, where $E_H\in\Sigma$ for every $H\in\mathcal{H}$, and $\mathcal{D}_0$ the family of sets $D\subseteq X\times Y$ such that $(X\times Y)\setminus D\in\mathcal{W}$ and $D[\{x\}]$ is nowhere dense for every $x\in X$; let $\mathcal{L}_0$ be the σ-ideal of subsets of $X\times Y$ generated by $\mathcal{D}_0$. Then $\Sigma\widehat{\otimes}\mathcal{B}(Y)\subseteq\{W\triangle L:W\in\mathcal{W},\,L\in\mathcal{L}_0\}$.

proof Write $\mathcal{V}$ for $\{W\triangle L:W\in\mathcal{W},\,L\in\mathcal{L}_0\}$. Then $\mathcal{W}$ and $\mathcal{V}$ are closed under countable unions. Next, $(X\times Y)\setminus W\in\mathcal{V}$ for every $W\in\mathcal{W}$. **P** Express W as $\bigcup_{H\in\mathcal{H}}E_H\times H$ where $E_H\in\Sigma$ for every $H\in\mathcal{H}$. For $H\in\mathcal{H}$, set

$$F_H=X\setminus\bigcup\{E_{H'}:H'\in\mathcal{H},\,H'\cap H\ne\emptyset\}\in\Sigma,$$

and set $W'=\bigcup_{H\in\mathcal{H}}F_H\times H$. Then W' and $W\cup W'$ belong to $\mathcal{W}$. Set $D=(X\times Y)\setminus(W\cup W')$. If $x\in X$ and $G\subseteq Y$ is a non-empty open set, let $H\subseteq G$ be a non-empty member of $\mathcal{H}$. Then either $x\in F_H$ and

H is a non-empty open subset of $G \setminus D[\{x\}]$, or there is an $H' \in \mathcal{H}$ such that $H \cap H' \neq \emptyset$ and $x \in E_{H'}$, in which case $H \cap H'$ is a non-empty open subset of $G \setminus D[\{x\}]$. As G is arbitrary, $D[\{x\}]$ is nowhere dense; as x is arbitrary, $D \in \mathcal{D}_0$. But now observe that $(X \times Y) \setminus W = W' \triangle D$ belongs to $\mathcal{V}$. **Q**

It follows that the complement of any member of $\mathcal{V}$ belongs to $\mathcal{V}$, so $\mathcal{V}$ is a σ-algebra. Now $E \times G \in \mathcal{V}$ for every $E \in \Sigma$ and open $G \subseteq Y$. **P** For $H \in \mathcal{H}$, set $E_H = E$ if $H \subseteq G$, $\emptyset$ otherwise; set $W = \bigcup_{H \in \mathcal{H}} E_H \times H \in \mathcal{W}$. Then $W \subseteq E \times G$. But, defining W' from W as just above, we see that W' is disjoint from $E \times G$. So

$$(E \times G) \triangle W \subseteq (X \times Y) \setminus (W \cup W') \in \mathcal{D}_0$$

and $E \times G \in \mathcal{V}$. **Q**

Accordingly $\mathcal{V}$ includes the σ-algebra generated by $\{E \times G : E \in \Sigma,\, G \subseteq Y \text{ is open}\}$, which is $\Sigma \widehat{\otimes} \mathcal{B}(Y)$.

527J Theorem (see Fremlin 91) Let X be a topological space and μ a σ-finite quasi-Radon measure on X with countable Maharam type; let Y be a topological space of countable π-weight. Then $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{M}(Y) \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

proof Write $\mathcal{L}$ for $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{M}(Y)$, and fix a countable π-base $\mathcal{H}$, not containing $\emptyset$, for the topology of Y.

(a) We need to know that for every Borel set $V \subseteq X \times Y$ there are sets V', $V'' \in \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$ such that $V' \subseteq V \subseteq V''$ and $V'' \setminus V' \in \mathcal{L}$. **P** Let $\mathcal{V}^*$ be the family of all subsets of $X \times Y$ with this property. Because $\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$ is a σ-algebra and $\mathcal{L}$ is a σ-ideal of sets, $\mathcal{V}^*$ is a σ-algebra. If $W \subseteq X \times Y$ is open, set

$$U_H = \bigcup \{G : G \subseteq X \text{ is open, } G \times H \subseteq W\}, \quad U'_H = \{x : H \cap W[\{x\}] \neq \emptyset\}$$

for $H \in \mathcal{H}$, so all the U_H and U'_H are open (U'_H is just the projection of the open set $W \cap (X \times H)$). Set $V_1 = \bigcup_{H \in \mathcal{H}} U_H \times H$ and $V_2 = \bigcup_{H \in \mathcal{H}} ((X \setminus U'_H) \times H)$. Then V_1 and V_2 both belong to $\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$, $V_1 \subseteq W$ and $W \cap V_2 = \emptyset$.

Let $x \in X$. **?** If the open set $V_1[\{x\}] \cup V_2[\{x\}]$ is not dense, there is an $H \in \mathcal{H}$ disjoint from both $V_1[\{x\}]$ and $V_2[\{x\}]$. In this case x must belong to U'_H, and there is a point $y \in H \cap W[\{x\}]$. (x, y) belongs to the open set $(X \times H) \cap W$, so there are open sets $G \subseteq X$, $\tilde{H} \subseteq Y$ such that $(x, y) \in G \times \tilde{H} \subseteq (X \times H) \cap W$. Now there is an $H' \in \mathcal{H}$ such that $H' \subseteq \tilde{H}$, in which case $x \in G \subseteq U_{H'}$. But this will mean that $H' \subseteq V_1[\{x\}]$ and H' is a non-empty subset of $H \cap V_1[\{x\}]$, which is impossible. **X**

Thus $V_1[\{x\}] \cup V_2[\{x\}]$ is dense for every x, and if we set $V_3 = (X \times Y) \setminus V_2$ we shall have $V_3 \setminus V_1 \in \mathcal{L}$, while both V_1 and V_3 belong to $\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$, and $V_1 \subseteq W \subseteq V_3$. So $W \in \mathcal{V}^*$. This is true for every open set $W \subseteq X \times Y$, so the σ-algebra $\mathcal{V}^*$ must contain every Borel set, as required. **Q**

It follows that every member of $\mathcal{L}$ is included in a member of $\mathcal{L} \cap (\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y))$. **P** If $V \in \mathcal{L}$ there is a Borel set $V' \supseteq V$ which belongs to $\mathcal{L}$, and now there is a set $V'' \in \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$ such that $V'' \supseteq V'$ and $V'' \setminus V' \in \mathcal{L}$, in which case $V'' \supseteq V$ also must belong to $\mathcal{L}$. **Q**

Thus $\mathcal{L} = \mathcal{N}(\mu) \ltimes_{\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)} \mathcal{M}(Y)$.

(b) To begin with let us suppose that X is compact and metrizable, μ is totally finite and Y is a Baire space.

(i) Taking $\Sigma = \mathcal{B}(X)$, define $\mathcal{W}$, $\mathcal{D}_0$ and $\mathcal{L}_0$ as in 527I. Now let $\mathcal{D}$ be the family of closed subsets belonging to $\mathcal{D}_0$, and $\mathcal{L}_1$ the σ-ideal of subsets of $X \times Y$ generated by $\{E \times Y : E \in \mathcal{N}(\mu)\} \cup \mathcal{D}$.

(ii) $\mathcal{D}_0 \subseteq \mathcal{L}_1$. **P** If $D \in \mathcal{D}_0$, express $(X \times Y) \setminus D$ as $\bigcup_{H \in \mathcal{H}} E_H \times H$ where $E_H \in \mathcal{B}(X)$ for every $H \in \mathcal{H}$. Because μ is totally finite, μ is outer regular with respect to the open sets (412Wb). So for each $n \in \mathbb{N}$ we can find a family $\langle G_{nH} \rangle_{H \in \mathcal{H}}$ of open sets in X such that $E_H \subseteq G_{nH}$ for every H and $\sum_{H \in \mathcal{H}} \mu(G_{nH} \setminus E_H) \leq 2^{-n}$. Set $D_n = (X \times Y) \setminus \bigcup_{H \in \mathcal{H}} (G_{nH} \times H)$. Then D_n is closed and $D_n \subseteq D \in \mathcal{D}_0$ so $D_n \in \mathcal{D}$. Set $E = \bigcap_{n \in \mathbb{N}} \bigcup_{H \in \mathcal{H}} (G_{nH} \setminus E_H)$; then $E \in \mathcal{N}(\mu)$ and

$$D \subseteq (E \times Y) \cup \bigcup_{n \in \mathbb{N}} D_n \in \mathcal{L}_1. \textbf{ Q}$$

(iii) Of course every member of $\mathcal{D}$ belongs to $\mathcal{L}$, so $\mathcal{L}_1 \subseteq \mathcal{L}$. But in fact $\mathcal{L} = \mathcal{L}_1$. **P** If $V \in \mathcal{L}$, there is a $V' \in (\mathcal{N}(\mu) \ltimes \mathcal{M}(Y)) \cap (\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y))$ such that $V \subseteq V'$, by (a). By 527I, we can express V' as $W \triangle L$ where $W \in \mathcal{W}$ and $L \in \mathcal{L}_0$. By (ii), $\mathcal{L}_0 \subseteq \mathcal{L}_1$, so $W \in \mathcal{L}$. There is therefore a negligible set $E \subseteq X$

such that $W[\{x\}]$ is meager for every $x \in X \setminus E$. But $W[\{x\}]$ is always open, and Y is a Baire space, so $W \subseteq E \times Y \in \mathcal{L}_1$. Accordingly V' and V belong to $\mathcal{L}_1$. As V is arbitrary, $\mathcal{L} \subseteq \mathcal{L}_1$. **Q**

(iv) Let $\mathcal{G}$ be a countable base for the topology of X containing X. Let $\mathcal{U}_0$ be the family of those sets $U \subseteq X \times Y$ such that U is expressible as a finite union of sets of the form $G \times H$ where $G \in \mathcal{G}$ and $H \in \mathcal{H}$, and $\mathcal{U}$ the set of those $U \in \mathcal{U}_0$ such that $\pi_1[U] = X$, where π_1 is the projection from $X \times Y$ onto X. Consider

$$\mathcal{D}' = \{D : D \subseteq X \times Y, \text{ for every } U_0 \in \mathcal{U} \text{ there is a } U \in \mathcal{U} \text{ such that } U \subseteq U_0 \setminus D\}.$$

$\mathcal{D} \subseteq \mathcal{D}'$. **P** Suppose that $D \in \mathcal{D}$ and $U_0 \in \mathcal{U}$, and consider $\mathcal{U}_1 = \{U : U \in \mathcal{U}_0,\ U \subseteq U_0 \setminus D\}$. For every $x \in X$ the section $U_0[\{x\}]$ is open and not empty and the section $D[\{x\}]$ is nowhere dense, so there is a y such that $(x, y) \in U_0 \setminus D$; now there are $G \in \mathcal{G}$, containing x, and an open H containing y such that $G \times H \subseteq U_0 \setminus D$. Let $H' \in \mathcal{H}$ be such that $\emptyset \neq H' \subseteq H$. Then $U = G \times H' \in \mathcal{U}_1$ and $x \in \pi_1[U]$. As x is arbitrary, $\{\pi_1[U] : U \in \mathcal{U}_1\}$ is an open cover of X; as X is compact and $\mathcal{U}_1$ is upwards-directed, there is a $U \in \mathcal{U}_1$ such that $\pi_1[U] = X$; in which case $U \in \mathcal{U}$ and $U \subseteq U_0 \setminus D$. As U is arbitrary, $D \in \mathcal{D}'$; as D is arbitrary, $\mathcal{D} \subseteq \mathcal{D}'$. **Q**

$\mathcal{D}$ is cofinal with $\mathcal{D}'$. **P** Let $D \in \mathcal{D}'$. For each $H \in \mathcal{H} \setminus \{\emptyset\}$, $X \times H \in \mathcal{U}$, so there is a $U_H \in \mathcal{U}$ such that $U_H \subseteq (X \times H) \setminus D$; try $D_1 = (X \setminus Y) \setminus \bigcup_{H \in \mathcal{H} \setminus \{\emptyset\}} U_H$. D_1 is closed. Since $\mathcal{U} \subseteq \mathcal{U}_0 \subseteq \mathcal{W}$, $(X \times Y) \setminus D_1 \in \mathcal{W}$. If $x \in X$, then $D_1[\{x\}]$ is a closed set not including any member of the π-base $\mathcal{H}$, so is nowhere dense in Y; thus $D_1 \in \mathcal{D}_0$ and (being closed) belongs to $\mathcal{D}$. Of course $D \subseteq D_1$. As D is arbitrary, $\mathcal{D}$ is cofinal with $\mathcal{D}'$. **Q**

(v) Because $\mathcal{U}$ is countable, 526Hd tells us that $\mathcal{D}' \preccurlyeq_{\mathrm{T}} \mathcal{N}wd$, where $\mathcal{N}wd$ is the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$; while of course $\mathcal{D} \equiv_{\mathrm{T}} \mathcal{D}'$ (513E(d-ii)). Let $\phi : \mathcal{L} \to \mathcal{N}(\mu) \times \mathcal{D}^{\mathbb{N}}$ be such that if $\phi(V) = (E, \langle D_n \rangle_{n \in \mathbb{N}})$ then $V \subseteq (E \times Y) \cup \bigcup_{n \in \mathbb{N}} D_n$; such a function exists by (iii), and is evidently a Tukey function.

Note that the measure algebra of μ, being a totally finite measure algebra with countable Maharam type, can be regularly embedded in the measure algebra of Lebesgue measure on either $[0, 1]$ or on $\mathbb{R}$. As μ is a Radon measure (416G), $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}$ (524K) and

$$\mathcal{L} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu) \times \mathcal{D}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{N} \times \mathcal{N}wd^{\mathbb{N}} \cong \mathcal{N} \times \mathcal{N}wd$$

(513Eg, 526Ha). Accordingly

$$
\begin{aligned}
(\mathcal{L}, \subseteq, \mathcal{L}) &\equiv_{\mathrm{GT}} (\mathcal{L}, \subseteq', [\mathcal{L}]^{\le\omega})\\
&\text{(513Id)}\\
&\preccurlyeq_{\mathrm{GT}} (\mathcal{N} \times \mathcal{N}wd, \le', [\mathcal{N} \times \mathcal{N}wd]^{\le\omega})\\
&\text{(512Gb)}\\
&\equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq', [\mathcal{N}]^{\le\omega}) \times (\mathcal{N}wd, \subseteq', [\mathcal{N}wd]^{\le\omega})\\
&\text{(512Hd)}\\
&\equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq \mathcal{N}) \times (\mathcal{M}, \subseteq, \mathcal{M})\\
&\text{(513Id, 526Hb, 512Hb)}\\
&\preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N}) \times (\mathcal{N}, \subseteq, \mathcal{N})\\
&\text{(522P)}\\
&\equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})
\end{aligned}
$$

(513Eh), and $\mathcal{L} \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

(c) This proves the theorem when X is compact and metrizable, μ is totally finite and Y is a Baire space. Now suppose that Y is still a Baire space, while (X, μ) is any totally finite quasi-Radon measure space with countable Maharam type.

(i) There is a compact metrizable Radon measure space (Z, λ) such that λ and μ have isomorphic measure algebras. **P** Because the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ is totally finite, it is isomorphic to the simple product of a countable family $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ of homogeneous totally finite measure algebras (332B). Because μ has countable Maharam type, every $\mathfrak{A}_i$ is either $\{0\}$, $\{0,1\}$ or isomorphic to the measure algebra of Lebesgue measure on an interval; in any case it is isomorphic to the measure algebra of a compact Radon measure space (Z_i,λ_i). Take (Z',λ') to be the direct sum of the measure spaces $\langle(Z_i,\lambda_i)\rangle_{i\in I}$; then the measure algebra of (Z',λ') is isomorphic to $\mathfrak{A}$. If we give Z' its disjoint union topology, it is a locally compact σ-compact metrizable space, and its one-point compactification Z is second-countable, therefore metrizable; taking λ to be the trivial extension of λ', (Z,λ) is a compact metrizable Radon measure space with measure algebra $(\mathfrak{B},\bar\lambda)\cong(\mathfrak{A},\bar\mu)$. **Q**

(ii) Let $f:X\to Z$ be an inverse-measure-preserving function inducing an isomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ of the measure algebras (416Wb). By 418J, f is almost continuous, so there is a Borel measurable function which is equal almost everywhere to f (418V[9]); this function will still represent π, so we may suppose that f itself is Borel measurable. Now if $V\in\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, there is a $V'\in\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)$ such that $\{x:V[\{x\}]\ne V'[\{f(x)\}]\}\in\mathcal{N}(\lambda)$. **P** Let $\tilde{\mathcal{V}}$ be the family of subsets V of $X\times Y$ for which there is a $V'\in\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)$ such that $\{x:V[\{x\}]\ne V'[\{f(x)\}]\}\in\mathcal{N}(\lambda)$. Then $\tilde{\mathcal{V}}$ is a σ-algebra. If $E\in\mathcal{B}(X)$ and $H\in\mathcal{B}(Y)$, then there must be an $F\in\mathcal{B}(Z)$ such that $F^\bullet=\pi E^\bullet$ in $\mathfrak{B}$, so that $E\triangle f^{-1}[F]\in\mathcal{N}(\mu)$; now $F\times H$ witnesses that $E\times H$ belongs to $\tilde{\mathcal{V}}$. Accordingly $\tilde{\mathcal{V}}$ must include $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. **Q**

(iii) We know that $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\lambda)$ (524Sa), so there is a Tukey function $\theta:\mathcal{N}(\mu)\to\mathcal{N}(\lambda)$. Set $\mathcal{L}'=\mathcal{N}(\lambda)\ltimes_{\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)}\mathcal{M}(Y)$. Define a function $\phi:\mathcal{L}\to\mathcal{L}'$ as follows. First, for $V\in\mathcal{L}$, choose $\phi_0(V)\in\mathcal{L}\cap(\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y))$ including V ((a) above). Next, by (ii) here, we can choose $\phi_1(V)\in\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)$ such that $N_V=\{x:\phi_0(V)[\{x\}]\ne\phi_1(V)[\{f(x)\}]\}$ belongs to $\mathcal{N}(\mu)$. Set $F=\{z:z\in Z,\ \phi_1(V)[\{z\}]\text{ is not meager}\}$; then F is a Borel set, by 4A3Ta[10], and $f^{-1}[F]\subseteq N_V\cup\{x:\phi_0(V)[\{x\}]\notin\mathcal{M}(Y)\}\in\mathcal{N}(\mu)$; so $F\in\mathcal{N}(\lambda)$ and $\phi_1(V)\in\mathcal{L}'$. Finally, set $\phi(V)=(\theta(N_V)\times Y)\cup\phi_1(V)\in\mathcal{L}'$.

ϕ is a Tukey function from $\mathcal{L}$ to $\mathcal{L}'$. **P** Take $W\in\mathcal{L}'$ and consider $\mathcal{E}=\{V:V\in\mathcal{L},\ \phi(V)\subseteq W\}$. If $Y=\emptyset$ then of course $\mathcal{E}$ is bounded above in $\mathcal{L}$. Otherwise, $N^*=\{z:W[\{z\}]=Y\}$ must be λ-negligible, while $\theta(N_V)\subseteq N^*$ for every $V\in\mathcal{E}$; because θ is a Tukey function, $\tilde N=\bigcup\{N_V:V\in\mathcal{E}\}$ is μ-negligible. Take $W_1\in\mathcal{L}'\cap(\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y))$ including W, and set $\tilde W=\{(x,y):(f(x),y)\in W_1\}$; then $\tilde W\in\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ because f is Borel measurable. As

$$\{x:\tilde W[\{x\}]\notin\mathcal{M}(Y)\}=f^{-1}\{z:W_1[\{z\}]\notin\mathcal{M}(Y)\}$$

is negligible, $\tilde W\in\mathcal{L}$. So $V_0=(\tilde N\times Y)\cup\tilde W$ belongs to $\mathcal{L}$. Now take any $V\in\mathcal{E}$. If $x\in X\setminus\tilde N$, then $x\notin N_V$, so

$$\begin{aligned}V[\{x\}]&\subseteq\phi_0(V)[\{x\}]=\phi_1(V)[\{f(x)\}]\subseteq W[\{f(x)\}]\\&\subseteq W_1[\{f(x)\}]=\tilde W[\{x\}]=V_0[\{x\}].\end{aligned}$$

This shows that $V\subseteq V_0$; as V is arbitrary, V_0 is an upper bound for $\mathcal{E}$ in $\mathcal{L}$; as W is arbitrary, ϕ is a Tukey function. **Q**

(iv) By (b), we know that $\mathcal{L}'\preccurlyeq_{\mathrm{T}}\mathcal{N}$, so (iii) tells us that $\mathcal{L}\preccurlyeq_{\mathrm{T}}\mathcal{N}$, and the theorem is true in this case also.

(d) We are nearly home. If Y is a Baire space and (X,μ) is a σ-finite quasi-Radon measure space with countable Maharam type, which is not totally finite, then there is a measurable function $f:X\to\,]0,\infty[$ such that $\int fd\mu=1$ (215B(ix)). Let ν be the indefinite-integral measure defined by f. Then ν has the same negligible sets as μ (234Lc), and is a quasi-Radon measure (415Ob), so

$$\mathcal{L}=\mathcal{N}(\nu)\ltimes_{\mathcal{B}(X\times Y)}\mathcal{M}(Y)\preccurlyeq_{\mathrm{T}}\mathcal{N},$$

by (c).

(e) Finally, suppose that Y is not a Baire space. In this case, let H^* be the smallest comeager regular open subset of Y (4A3Sa once more), and set $\mathcal{L}^*=\mathcal{N}(\mu)\ltimes_{\mathcal{B}(X\times H^*)}\mathcal{M}(H^*)$. Then $\mathcal{L}\preccurlyeq_{\mathrm{T}}\mathcal{L}^*$. **P** For every $V\in\mathcal{L}$, let V' be such that $V\subseteq V'\in\mathcal{B}(X\times Y)\cap(\mathcal{N}(\mu)\ltimes\mathcal{M}(Y))$, and set $\phi(V)=V'\cap(X\times H^*)$. Then

[9] Later editions only.
[10] Formerly 4A3Sa.

$\phi(V)$ is a Borel subset of $X \times H^*$, and $\phi(V)[\{x\}] = V'[\{x\}] \cap H^*$ is meager in H^* whenever $V'[\{x\}]$ is meager in Y, so $\phi(V) \in \mathcal{L}^*$. To see that $\phi : \mathcal{L} :\to \mathcal{L}^*$ is a Tukey function, take any $W \in \mathcal{L}^*$. There is a Borel set $W' \in \mathcal{L}^*$ including W, and now $V' = W' \cup (X \times (Y \setminus H^*))$ is a Borel subset of $X \times Y$; since $V'[\{x\}]$ is meager in Y whenever $W'[\{x\}]$ is meager in H^*, $V' \in \mathcal{L}$. Of course V' is an upper bound of $\{V : V \in \mathcal{L}, \phi(V) \subseteq W\}$; as W is arbitrary, ϕ is a Tukey function and $\mathcal{L} \preccurlyeq_{\mathrm{T}} \mathcal{L}^*$. **Q**

By (d), $\mathcal{L} \preccurlyeq_{\mathrm{T}} \mathcal{N}$ in this case also, and the proof is complete.

527K Corollary $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M} \equiv_{\mathrm{T}} \mathcal{N}$.

proof By 527J, $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M} \preccurlyeq_{\mathrm{T}} \mathcal{N}$. On the other hand, $E \mapsto E \times \mathbb{R}$ is a Tukey function from $\mathcal{N}$ to $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$, so $\mathcal{N} \preccurlyeq_{\mathrm{T}} \mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$.

527L There are some interesting questions concerning the saturation of skew products. Here and in 527O I give results which will be useful later.

Theorem Let X be a set, Σ a σ-ideal of subsets of X, and $\mathcal{I} \triangleleft \mathcal{P}X$ a σ-ideal; suppose that $\Sigma/\Sigma \cap \mathcal{I}$ is ccc. Let (Y, T, ν) be a σ-finite measure space. Then $(\Sigma\widehat{\otimes}\mathrm{T})/((\Sigma\widehat{\otimes}\mathrm{T}) \cap (\mathcal{I} \ltimes \mathcal{N}(\nu)))$ is ccc.

proof (a) The case $\nu Y = 0$ is trivial, as then $\mathcal{I} \ltimes \mathcal{N}(\nu) = \mathcal{P}(X \times Y)$. Otherwise, there is a probability measure on Y with the same domain and null ideal as ν (215B(vii)), so we may suppose that $\nu Y = 1$.

(b) The family $\mathcal{W}$ of sets $W \subseteq X \times Y$ such that $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and $x \mapsto \nu W[\{x\}]$ is Σ-measurable is a Dynkin class (136A), and contains $E \times F$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$; by the Monotone Class Theorem (136B) it includes $\Sigma\widehat{\otimes}\mathrm{T}$.

(c) Now suppose that $\langle W_\xi \rangle_{\xi<\omega_1}$ is a disjoint family in $\Sigma\widehat{\otimes}\mathrm{T}$. For $n \in \mathbb{N}$ and $\xi < \kappa$ set

$$E_{n\xi} = \{x : \nu W_\xi[\{x\}] \ge 2^{-n}\};$$

then $\#(\{\xi : x \in E_{n\xi}\}) \le 2^{-n}$ for every $x \in X$. It follows that $A_n = \{\xi : \xi < \omega_1,\ E_{n\xi} \notin \mathcal{I}\}$ is countable. **P** **?** Otherwise, write $\mathfrak{A}$ for the ccc algebra $\Sigma/\Sigma \cap \mathcal{I}$, and $a_\xi = E^\bullet_{n\xi}$ for $\xi < \omega_1$. Then $\mathfrak{A}$ is Dedekind complete; set $b_\xi = \sup_{\xi\le\eta<\omega_1} a_\eta$ for $\xi < \omega_1$ and $b = \inf_{\xi<\omega_1} b_\xi$. Because $\mathfrak{A}$ is ccc, there is a $\zeta < \omega_1$ such that $b = b_\xi$ for every $\xi \ge \zeta$; because A_n is uncountable, $b \ne 0$. Choose $\langle c_i \rangle_{i\in\mathbb{N}}$ and $\langle \xi_i \rangle_{i\in\mathbb{N}}$ inductively such that $c_0 = b$ and, given that $0 \ne c_i \subseteq b$, ξ_i is to be such that $c_{i+1} = a_{\xi_i} \cap c_i \ne 0$ and $\xi_i > \xi_j$ for every $j < i$. Now $\inf_{i\le 2^n} a_{\xi_i} \supseteq c_{2^n+1}$ is non-zero, so there is an $x \in \bigcap_{i\le 2^n} E_{n\xi_i}$; but this is impossible. **XQ**

(d) This is true for every $n \in \mathbb{N}$, so there is a $\xi < \omega_1$ such that $\xi \notin A_n$ for every n, that is, $E_{n\xi} \in \mathcal{I}$ for every n. But in this case

$$\{x : W_\xi[\{x\}] \notin \mathcal{N}(\nu)\} = \bigcup_{n\in\mathbb{N}} E_{n\xi}$$

belongs to $\mathcal{I}$ and $W_\xi \in \mathcal{I} \ltimes \mathcal{N}(\nu)$. As $\langle W_\xi \rangle_{\xi<\omega_1}$ is arbitrary, $(\Sigma\widehat{\otimes}\mathrm{T}) \cap (\mathcal{I} \ltimes \mathcal{N}(\nu))$ is ω_1-saturated in $\Sigma\widehat{\otimes}\mathrm{T}$ and $(\Sigma\widehat{\otimes}\mathrm{T})/(\Sigma\widehat{\otimes}\mathrm{T}) \cap (\mathcal{I} \ltimes \mathcal{N}(\nu))$ is ccc (316C).

527M The final theorem of this section provides me with an opportunity to introduce a concept which will be needed in §547.

Definition A Boolean algebra $\mathfrak{A}$ is **harmless** (cf. Just 92) if it is ccc and whenever $\mathfrak{B}$ is a countable subalgebra of $\mathfrak{A}$, there is a regularly embedded countable subalgebra of $\mathfrak{A}$ including $\mathfrak{B}$.

527N Lemma (a) A Boolean algebra with a harmless order-dense subalgebra is itself harmless.

(b) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then it is harmless iff every order-closed subalgebra of $\mathfrak{A}$ with countable Maharam type has countable π-weight.

(c) For any set I, the regular open algebra $\mathrm{RO}(\{0,1\}^I)$ of $\{0,1\}^I$ is harmless, so the category algebra of $\{0,1\}^I$ is harmless.

(d) If $\mathfrak{A}$ has countable π-weight it is harmless.

(e) If $\mathfrak{A}$ is a harmless Boolean algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\mathfrak{B}$ is harmless. In particular, any principal ideal of a harmless Boolean algebra is harmless.

proof (a) Let $\mathfrak{A}$ be a Boolean algebra with a harmless order-dense subalgebra $\mathfrak{D}$. By 316Xj or 513E(e-iii), $\mathfrak{A}$ is ccc. Let $\mathfrak{B}$ be a countable subalgebra of $\mathfrak{A}$. For each $b \in \mathfrak{B}$ let $D_b \subseteq \mathfrak{D}$ be a countable set with supremum b (313K, 316E). Let $\mathfrak{D}_0$ be the subalgebra of $\mathfrak{D}$ generated by $\bigcup_{b\in\mathfrak{B}} D_b$. Then $\mathfrak{D}_0$ is countable, so there is a countable subalgebra $\mathfrak{D}_1$ of $\mathfrak{D}$, including $\mathfrak{D}_0$, which is regularly embedded in $\mathfrak{D}$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \mathfrak{D}_1$. Then $\mathfrak{C}$ is countable. Now every member of $\mathfrak{C}$ is the supremum of the members of $\mathfrak{D}_1$ it includes. **P** Set

$$C = \{c : c \in \mathfrak{C},\, c = \sup\{d : d \in \mathfrak{D}_1,\, d \subseteq c\} = \inf\{d : d \in \mathfrak{D}_1,\, c \subseteq d\}\}.$$

Then C is closed under union (use 313Bd) and complementation (313A), and includes $\mathfrak{B} \cup \mathfrak{D}_1$, so $C = \mathfrak{C}$. **Q**

It follows that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$, because if $C \subseteq \mathfrak{C}$ has supremum 1 in $\mathfrak{C}$ then $\bigcup_{c\in C}\{d : d \in \mathfrak{D}_1,\, d \subseteq c\}$ must have supremum 1 in $\mathfrak{C}$ and therefore in $\mathfrak{D}_1$ (because $\mathfrak{D}_1 \subseteq \mathfrak{C}$) and in $\mathfrak{D}$ (because $\mathfrak{D}_1$ is regularly embedded in $\mathfrak{D}$) and in $\mathfrak{A}$ (because $\mathfrak{D}$ is regularly embedded in $\mathfrak{A}$). But this means that $\sup C$ must be 1 in $\mathfrak{A}$. As C is arbitrary, $\mathfrak{C}$ is regularly embedded. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ is harmless.

(b)(i) Suppose that $\mathfrak{A}$ is harmless and that $\mathfrak{B} \subseteq \mathfrak{A}$ is an order-closed subalgebra of countable Maharam type. Let $B \subseteq \mathfrak{B}$ be a countable set which τ-generates $\mathfrak{B}$, and $\mathfrak{B}_0$ the algebra generated by B; let $\mathfrak{C}$ be a countable subalgebra of $\mathfrak{A}$, including $\mathfrak{B}_0$, which is regularly embedded in $\mathfrak{A}$. Let $\mathfrak{D}$ be the set

$$\{d : d \in \mathfrak{A},\, d = \sup\{c : c \in \mathfrak{C},\, c \subseteq d\} = \inf\{c : c \in \mathfrak{C},\, d \subseteq c\}\}.$$

Then $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{A}$. **P** As in (a) just above, it is a subalgebra. If $D \subseteq \mathfrak{D}$ is a non-empty set with supremum a in $\mathfrak{A}$, set $C = \{c : c \in \mathfrak{C},\, c \subseteq a\}$, $C' = \{c : c \in \mathfrak{C},\, a \subseteq c\}$. Then a is an upper bound for C and a lower bound for C'. **?** If either a is not the least upper bound of C, or a is not the greatest lower bound of C', then $A = \{c' \setminus c : c' \in C',\, c \in C\}$ is a subset of $\mathfrak{C}$ with a non-zero lower bound in $\mathfrak{A}$, so A has a non-zero lower bound c^* in $\mathfrak{C}$. Now if $d \in D$, $c \in \mathfrak{C}$ and $c \subseteq d$, then $c \in C$ so $c \cap c^* = 0$; as $d = \sup\{c : c \in \mathfrak{C},\, c \subseteq d\}$, $d \cap c^* = 0$. This is true for every $d \in D$, so $a \cap c^* = 0$ and $1 \setminus c^* \in C'$; but c^* was chosen to be included in every member of C'. **X** Thus $a \in \mathfrak{D}$; as D is arbitrary, $\mathfrak{D}$ is order-closed in $\mathfrak{A}$. **Q**

Now $B \subseteq \mathfrak{C} \subseteq \mathfrak{D}$. As $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$ (314Ga), $\mathfrak{B} \cap \mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{B}$ including B, so is the whole of $\mathfrak{B}$, and $\mathfrak{B} \subseteq \mathfrak{D}$. It follows that $\pi(\mathfrak{B}) \le \pi(\mathfrak{D})$ (514Eb). But $\mathfrak{C}$ is countable and order-dense in $\mathfrak{D}$, so $\pi(\mathfrak{D})$ and $\pi(\mathfrak{B})$ are countable. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ satisfies the declared condition.

(ii) Now suppose that $\mathfrak{A}$ satisfies the condition. Note first that $\mathfrak{A}$ is ccc. **P?** Suppose, if possible, otherwise; let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a disjoint family in $\mathfrak{A} \setminus \{0\}$. Replacing a_0 by $a_0 \cup (1 \setminus \sup_{\xi<\omega_1} a_\xi)$ if necessary, we may suppose that $\sup_{\xi<\omega_1} a_\xi = 1$. The map $I \mapsto \sup_{\xi\in I} a_\xi : \mathcal{P}\omega_1 \to \mathfrak{A}$ is an injective order-continuous Boolean homomorphism, so its image $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{P}\omega_1$. Now $\tau(\mathfrak{B}) = \tau(\mathcal{P}\omega_1) = \omega$ (514Ef, or otherwise), but $\pi(\mathfrak{B}) = \omega_1$; which is supposed to be impossible. **XQ**

If $\mathfrak{B}$ is a countable subalgebra of $\mathfrak{A}$, let $\mathfrak{B}_1$ be the order-closed subalgebra of $\mathfrak{A}$ which it generates. Then $\tau(\mathfrak{B}_1) \le \omega$ so $\pi(\mathfrak{B}_1) \le \omega$, and there is a countable subalgebra $\mathfrak{C}$ of $\mathfrak{B}_1$ which is order-dense in $\mathfrak{B}_1$; of course we may suppose that $\mathfrak{B} \subseteq \mathfrak{C}$. Now the identity maps from $\mathfrak{C}$ to $\mathfrak{B}_1$ and from $\mathfrak{B}_1$ to $\mathfrak{A}$ are both order-continuous, so their composition also is, and $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ is harmless.

(c) All regular open algebras are Dedekind complete. If $\mathfrak{B} \subseteq \mathrm{RO}(\{0,1\}^I)$ is an order-closed subalgebra with countable Maharam type, let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$ which τ-generates $\mathfrak{B}$. Every regular open subset of $\{0,1\}^I$ is determined by coordinates in some countable set (4A2E(b-i)), so there is a countable $J \subseteq I$ such that every G_n is determined by coordinates in J. Let $\pi_J : \{0,1\}^I \to \{0,1\}^J$ be the restriction map; then we have an injective order-continuous Boolean homomorphism $H \mapsto \pi_J^{-1}[H] : \mathrm{RO}(\{0,1\}^J) \to \mathrm{RO}(\{0,1\}^I)$ (4A2B(f-iii)). Let $\mathfrak{C}$ be the image of this homomorphism, so that $\mathfrak{C}$ is an order-closed subalgebra of $\mathrm{RO}(\{0,1\}^I)$. If $H_n = \pi_J[G_n]$ then H_n is regular and open for each n (4A2B(f-iii) again, because π_J is surjective and $\pi_J^{-1}[H_n] = G_n$ is regular and open), so $G_n = \pi_J^{-1}[H_n] \in \mathfrak{C}$; accordingly $\mathfrak{B} \subseteq \mathfrak{C}$. Now $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{C}$ so

$$\pi(\mathfrak{B}) \le \pi(\mathfrak{C}) = \pi(\{0,1\}^J) \le \omega.$$

As $\mathfrak{B}$ is arbitrary, $\mathrm{RO}(\{0,1\}^I)$ satisfies the condition of (b) and is harmless.

Of course it follows at once that the category algebra is harmless, because it is isomorphic to the regular open algebra (514If-514Ig).

(d) Let D be a countable order-dense set in $\mathfrak{A}$. If $\mathfrak{B}$ is a countable subalgebra of $\mathfrak{A}$, let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $D \cup \mathfrak{B}$; then $\mathfrak{C}$ is countable, includes $\mathfrak{B}$ and is order-dense, therefore regularly embedded in $\mathfrak{A}$. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ is harmless.

(e) Let $\mathfrak{D} \subseteq \mathfrak{B}$ be a countable subalgebra. Because $\pi[\mathfrak{A}] = \mathfrak{B}$, there is a countable subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ such that $\pi[\mathfrak{C}] = \mathfrak{D}$. Let $\mathfrak{C}_1 \supseteq \mathfrak{C}$ be a countable regularly embedded subalgebra of $\mathfrak{A}$. Then $\mathfrak{D}_1 = \pi[\mathfrak{C}_1]$ is regularly embedded in $\mathfrak{B}$. **P** Let $D \subseteq \mathfrak{D}_1$ be a non-empty set such that 1 is not the least upper bound of D in $\mathfrak{B}$. Set $C = \mathfrak{C}_1 \cap \pi^{-1}[D \cup \{0\}]$; then 1 is not the least upper bound of $\pi[C]$ in $\mathfrak{B}$, so (because π is order-continuous) 1 is not the least upper bound of C in $\mathfrak{A}$. Because $\mathfrak{C}_1$ is regularly embedded in $\mathfrak{A}$, there is a non-zero $c_0 \in \mathfrak{C}_1$ such that $c_0 \cap c = 0$ for every $c \in C$. In particular, $c_0 \notin C$ and $\pi c_0 \neq 0$. But we also have $\pi c \cap \pi c_0 = 0$ for every $c \in C$, that is, $d \cap \pi c_0 = 0$ for every $d \in D$, and 1 is not the least upper bound of D in $\mathfrak{D}_1$. As D is arbitrary, $\mathfrak{D}_1$ is regularly embedded. **Q** Of course $\mathfrak{D}_1$ is countable. As $\mathfrak{D}$ is arbitrary, $\mathfrak{B}$ is harmless.

If $c \in \mathfrak{A}$ then $a \mapsto a \cap c$ is an order-continuous homomorphism onto the principal ideal $\mathfrak{A}_c$ generated by c, so $\mathfrak{A}_c$ is harmless.

527O Theorem Let (X, Σ, μ) be a σ-finite measure space and Y a topological space such that the category algebra $\mathfrak{G}$ of Y is harmless. Write $\mathcal{L}$ for $(\Sigma \widehat{\otimes} \mathcal{B}(Y)) \cap (\mathcal{N}(\mu) \ltimes \mathcal{M}(Y))$ and $\mathfrak{A}$ for the measure algebra of μ. Then $\mathfrak{C} = (\Sigma \widehat{\otimes} \mathcal{B}(Y))/\mathcal{L}$ is ccc, and is isomorphic to the Dedekind completion of the free product $\mathfrak{A} \otimes \mathfrak{G}$. If neither $\mathfrak{A}$ nor $\mathfrak{G}$ is trivial, the isomorphism corresponds to embeddings $E^\bullet \mapsto (E \times Y)^\bullet : \mathfrak{A} \to \mathfrak{C}$ and $F^\bullet \mapsto (X \times F)^\bullet : \mathfrak{B} \to \mathfrak{C}$.

proof Write $\mathfrak{S}$ for the topology of Y.

(a) Let $\mathcal{W}$ be the family of all sets of the form $\bigcup_{n\in\mathbb{N}} E_n \times H_n$, where $E_n \in \Sigma$ and $H_n \subseteq Y$ is open for every n. Then for any $W \in \mathcal{W}$ there is a $W' \in \mathcal{W}$ such that $W' \triangle ((X \times Y) \setminus W) \in \mathcal{L}$. **P** Express W as $\bigcup_{n\in\mathbb{N}} E_n \times H_n$ where $E_n \in \Sigma$ and $H_n \in \mathfrak{S}$ for each n. Let $\mathfrak{D}$ be the order-closed subalgebra of $\mathfrak{G}$ generated by $\{H_n^\bullet : n \in \mathbb{N}\}$. Because $\mathfrak{G}$ is harmless and Dedekind complete, $\pi(\mathfrak{D}) \leq \omega$ (527Nb); let $\langle G_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{S}$ such that $\{G_n^\bullet : n \in \mathbb{N}\}$ is a π-base for $\mathfrak{D}$; we may suppose that any non-empty open subset of any G_n is non-meager. Let $\mathfrak{S}_1$ be the second-countable topology on Y generated by $\{H_n : n \in \mathbb{N}\} \cup \{G_n : n \in \mathbb{N}\}$, and $\mathcal{B}_1(Y) \subseteq \mathcal{B}(Y)$ the corresponding Borel σ-algebra. Then $V^\bullet \in \mathfrak{D}$ for every $V \in \mathfrak{S}_1$, because V is the union of a countable family of sets all with images in $\mathfrak{D}$. If $V \in \mathfrak{S}_1$ is dense for $\mathfrak{S}_1$, and $n \in \mathbb{N}$ is such that G_n is non-empty, $V \cap G_n \neq \emptyset$ so $V^\bullet \cap G_n^\bullet \neq 0$, by the choice of the G_n. But this means that $V^\bullet = 1$, that is, V is comeager for the original topology of Y.

Now W and $(X \times Y) \setminus W$ belong to $\Sigma \widehat{\otimes} \mathcal{B}_1(Y)$. By 527I, there are W' and $\langle D_n \rangle_{n\in\mathbb{N}}$ such that

$((X \times Y) \setminus W) \triangle W' \subseteq \bigcup_{n\in\mathbb{N}} D_n$,

W' is expressible as $\bigcup_{n\in\mathbb{N}} F_n \times V_n$ where $F_n \in \Sigma$ and $V_n \in \mathfrak{S}_1$ for every n,

every D_n belongs to $\Sigma \widehat{\otimes} \mathcal{B}_1(Y)$,

for every $x \in X$ and $n \in \mathbb{N}$, $D_n[\{x\}]$ is closed and nowhere dense for $\mathfrak{S}_1$.

Evidently $W' \in \mathcal{W}$; but we have just seen that sets which are closed and nowhere dense for $\mathfrak{S}_1$ are meager for $\mathfrak{S}$. So every D_n belongs to $\mathcal{L}$ and $((X \times Y) \setminus W) \triangle W' \in \mathcal{L}$. **Q**

(b) It follows (as in the proof of 527I) that $\mathcal{V} = \{W \triangle D : W \in \mathcal{W}, D \in \mathcal{L}\}$ is a σ-algebra of sets, and as $E \times H \in \mathcal{W}$ for every $E \in \Sigma$ and $H \in \mathfrak{S}$, $\mathcal{V} = \Sigma \widehat{\otimes} \mathcal{B}(Y)$.

(c) $\mathfrak{C}$ is ccc. **P?** Otherwise, there is a disjoint family $\langle e_\xi \rangle_{\xi<\omega_1}$ in $\mathfrak{C} \setminus \{0\}$. For each $\xi < \omega_1$, there is a $V_\xi \in (\Sigma \widehat{\otimes} \mathcal{B}(Y)) \setminus \mathcal{L}$ such that $V_\xi^\bullet = e_\xi$, and a $W_\xi \in \mathcal{W}$ such that $V_\xi \triangle W_\xi \in \mathcal{L}$. Express W_ξ as $\bigcup_{n\in\mathbb{N}} E_{\xi n} \times H_{\xi n}$; as $W_\xi \notin \mathcal{L}$, there must be an n_ξ such that $E_\xi = E_{\xi,n_\xi} \notin \mathcal{N}(\mu)$ and $H_\xi = H_{\xi,n_\xi}$ is non-meager. Since the measure algebra of μ satisfies Knaster's condition (525Tb), there is an uncountable $A \subseteq \omega_1$ such that $E_\xi \cap E_\eta \notin \mathcal{N}(\mu)$ for all ξ, $\eta \in A$; because $\mathfrak{G}$ is ccc, there are distinct ξ, $\eta \in A$ such that $H_\xi \cap H_\eta$ is non-meager. But also

$$(E_\xi \cap E_\eta) \times (H_\xi \cap H_\eta) \subseteq W_\xi \cap W_\eta \in \mathcal{L}$$

because $(W_\xi \cap W_\eta)^\bullet = e_\xi \cap e_\eta = 0$. So this is impossible. **XQ**

Thus $\mathfrak{C}$ is ccc. As it is Dedekind σ-complete (314C), it is Dedekind complete (316Fa).

(d) If either $\mu X = 0$ or Y is meager, then $\mathfrak{A} \otimes \mathfrak{G}$ and $\mathfrak{C}$ are trivially isomorphic, and we can stop. Otherwise, the map $E \mapsto (E \times Y)^\bullet : \Sigma \to \mathfrak{C}$ is a Boolean homomorphism with kernel $\Sigma \cap \mathcal{N}(\mu)$, so induces a Boolean homomorphism $\pi_1 : \mathfrak{A} \to \mathfrak{C}$. Similarly, we have a Boolean homomorphism $\pi_2 : \mathfrak{G} \to \mathfrak{C}$ defined by setting $\pi_2(F^\bullet) = (X \times F)^\bullet$ for $F \in \mathcal{B}(Y)$. These now give us a Boolean homomorphism $\phi : \mathfrak{A} \otimes \mathfrak{G} \to \mathfrak{C}$ defined by saying that

$$\psi(E^\bullet \otimes F^\bullet) = \pi_1(E^\bullet) \cap \pi_2(F^\bullet) = (E \times F)^\bullet$$

for $E \in \Sigma$ and $F \in \mathcal{B}(Y)$ (315Jb). If $E \in \Sigma \setminus \mathcal{N}(\mu)$ and $F \in \mathcal{B}(Y) \setminus \mathcal{M}(Y)$, then $E \times F \notin \mathcal{L}$; so ϕ is injective (use 315Kb). If $c \in \mathfrak{C}$ is non-zero, it is expressible as $W^\bullet$ for some $W \in \mathcal{W} \setminus \mathcal{L}$; there must now be $E \in \Sigma$, $F \in \mathcal{B}(Y)$ such that $E \times F \subseteq W$ and $E \times F \notin \mathcal{L}$, so that $\phi(E^\bullet \otimes F^\bullet)$ is non-zero and included in w. Thus $\phi[\mathfrak{A} \otimes \mathfrak{G}]$ is isomorphic to $\mathfrak{A} \otimes \mathfrak{G}$ and is an order-dense subalgebra of the Dedekind complete Boolean algebra $\mathfrak{C}$; it follows that $\mathfrak{C}$ can be identified with the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}$.

527X Basic exercises >(a) Show that there is a set belonging to $\mathcal{N} \ltimes \mathcal{N}$ which has full outer measure for Lebesgue measure in the plane. (*Hint*: enumerate the compact non-negligible subsets of the plane as $\langle K_\xi \rangle_{\xi < \mathfrak{c}}$ (4A3Fa); note that the projection L_ξ of K_ξ onto the first coordinate is always non-negligible, therefore uncountable, therefore of cardinal $\mathfrak{c}$ (423L); choose $s_\xi \in L_\xi \setminus \{s_\eta : \eta < \xi\}$ and $t_\xi \in K_\xi[\{s_\xi\}]$ for each ξ; consider $\{(s_\xi, t_\xi) : \xi < \mathfrak{c}\}$.)

>(b) Show that there is a unique construction of iterated skew products $\mathcal{I}_0 \ltimes \mathcal{I}_1 \ltimes \ldots \ltimes \mathcal{I}_n$ such that

(i) whenever $X_0, \ldots, X_n$ are sets and $\mathcal{I}_j$ is an ideal of subsets of X_j for every j, then $\mathcal{I}_0 \ltimes \ldots \ltimes \mathcal{I}_n$ is an ideal of subsets of $X_0 \times \ldots \times X_n$;

(ii) whenever $X_0, \ldots, X_n$ are sets, $\mathcal{I}_j$ is an ideal of subsets of X_j for every j, and $k < n$, then the natural identification of $X_0 \times \ldots \times X_n$ with $(X_0 \times \ldots \times X_k) \times (X_{k+1} \times \ldots \times X_n)$ identifies $\mathcal{I}_0 \ltimes \ldots \ltimes \mathcal{I}_n$ with $(\mathcal{I}_0 \ltimes \ldots \ltimes I_k) \ltimes (\mathcal{I}_{k+1} \ltimes \ldots \ltimes I_n)$ as defined in 527B.

(c) Complete the analysis in 527Bb by describing what happens if one of X, Y is empty or one of the ideals is not proper.

(d) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ an ideal of subsets of X; let Y be a topological space, $\mathcal{B}$ its Borel σ-algebra, $\widehat{\mathcal{B}}$ its Baire-property algebra, and $\mathcal{M}$ its meager ideal. Show that $\mathcal{I} \ltimes_{\Sigma \widehat{\otimes} \mathcal{B}} \mathcal{M} = \mathcal{I} \ltimes_{\Sigma \widehat{\otimes} \widehat{\mathcal{B}}} \mathcal{M}$.

>(e) Let Z be the Stone space of the measure algebra of Lebesgue measure on $[0, 1]$, and $f : Z \to [0, 1]$ the canonical inverse-measure-preserving continuous function (416V). Let $F \subseteq [0, 1]$ be a nowhere dense set which is not negligible, and set $W = \{(x, z) : x \in [0, 1], z \in Z, x + f(z) \in F\}$. Show that W is a nowhere dense closed set in $[0, 1] \times Z$ but does not belong to $\mathcal{M}([0, 1]) \ltimes \mathcal{M}(Z)$. (*Hint*: meager subsets of Z are negligible (321K).)

>(f) Suppose that I and J are sets, $X = \{0, 1\}^I$ and $Y = \{0, 1\}^J$. Show that $\mathcal{M}(X) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{M}(Y) = \mathcal{M}(X \times Y)$.

(g) Write $\mathfrak{X}$ for the class of topological spaces which have category algebras which are atomless and with countable π-weight. (i) Show that the Sorgenfrey line (415Xc) belongs to $\mathfrak{X}$. (ii) Show that the split interval (419L) belongs to $\mathfrak{X}$. (iii) Show that if the regular open algebra of a topological space X is atomless and has countable π-weight, then $X \in \mathfrak{X}$. (iv) Show that any open subspace of a space in $\mathfrak{X}$ belongs to $\mathfrak{X}$. (v) Show that any dense subspace of a space in $\mathfrak{X}$ belongs to $\mathfrak{X}$. (vi) Show that any comeager subspace of a space in $\mathfrak{X}$ belongs to $\mathfrak{X}$. (vii) Show that the product of countably many spaces in $\mathfrak{X}$ belongs to $\mathfrak{X}$.

(h) Show that a measurable algebra is harmless iff it is purely atomic.

527Y Further exercises (a) Show that $\mathcal{I} \ltimes \mathcal{J} \ne \mathcal{I} \rtimes \mathcal{J}$ for any of the four cases in which $\{\mathcal{I}, \mathcal{J}\} \subseteq \{\mathcal{M}, \mathcal{N}\}$.

(b) Extend the idea of 527Xb to define an ideal $\ltimes_{\xi < \zeta} \mathcal{I}_\xi$ of subsets of $\prod_{\xi < \zeta} X_\zeta$ when ζ is any ordinal and I_ξ is an ideal of subsets of X_ξ for every $\xi < \zeta$.

(c) Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I}$ a σ-ideal of Σ such that $\Sigma/\mathcal{I}$ is ccc. Let (Y, T, ν) be a probability space. Show that $(\Sigma\widehat{\otimes}\mathrm{T})/(\Sigma\widehat{\otimes}\mathrm{T}) \cap (\mathcal{I} \ltimes \mathcal{N}(\nu))$ is ccc.

(d) Let $(Y, \mathfrak{T})$ be a topological space. Show that there is a topology $\mathfrak{S}$ on Y, coarser than $\mathfrak{T}$, such that the weight of $(Y, \mathfrak{S})$ is equal to the π-weight of $(Y, \mathfrak{T})$ and the two topologies have the same nowhere dense sets, the same meager ideal and the same Baire-property algebras.

(e) Let X be a topological space with a σ-finite measure μ such that μ has countable Maharam type and every measurable set can be expressed as the symmetric difference of a Borel set and a negligible set. Let Y be a topological space with a countable π-base. Show that $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{M}(Y) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu) \times \mathcal{N}$.

(f) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of harmless Boolean algebras satisfying Knaster's condition, and $\mathfrak{A}$ their free product (315I). Show that $\mathfrak{A}$ is harmless.

527 Notes and comments Skew products of ideals have been used many times for special purposes, and we are approaching the point at which it would be worth developing a general theory of such products. I am not really attempting to do this here, though the language of 527B is supposed to point to the right questions. My primary aim in this section is to show that $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$ and $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$ are very different (527H, 527K). Of course the difference appears only when the continuum hypothesis is false (513Xf, 513Xr).

The version of the Kuratowski-Ulam theorem given in 527D is a natural one from the point of view of this chapter, but you should be aware that there are many more cases in which $\mathcal{M}^* = \mathcal{M}(X \times Y)$; see 527Xf and Fremlin Natkaniec & Recław 00. The statement of 527J includes the phrase 'quasi-Radon measure'. Actually we do not really need either τ-additivity or inner regularity with respect to closed sets. What we need is a measure μ such that $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}$ and the Borel sets generate the measure algebra (527Ye). The argument for 527J betrays its origin in the case $X = Y = [0, 1]$, which is of course also the natural home of 527C-527F. Some of the complications of the argument are due to its being written out for spaces of countable π-weight; an alternative approach would start with a reduction to the case in which Y is second-countable (527Yd).

It is interesting that all four of the quotient algebras

$$\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2) \cap (\mathcal{M} \ltimes \mathcal{M}), \quad \mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2) \cap (\mathcal{M} \ltimes \mathcal{N}),$$

$$\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2) \cap (\mathcal{N} \ltimes \mathcal{M}), \quad \mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2) \cap (\mathcal{N} \ltimes \mathcal{N})$$

are ccc (see 527E, 527Yc, 527O, 527Bc and also 527L). This should not be taken for granted; for a variety of examples of quotient algebras associated with σ-ideals see Fremlin 03.

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528 Amoeba algebras

In the course of investigating the principal consequences of Martin's axiom, Martin & Solovay 70 introduced the partially ordered set of open subsets of $\mathbb{R}$ with measure strictly less than γ, for $\gamma > 0$ (528O). Elementary extensions of this idea lead us to a very interesting class of partially ordered sets, which I study here in terms of their regular open algebras, the 'amoeba algebras' (528A). Of course the most important ones are those associated with Lebesgue measure, and these are closely related to 'localization posets' (528I), themselves intimately connected with the localization relations of 522K. In the second half of the section I look at the cardinal functions of these algebras, of which the most interesting seems to be Maharam type (528V).

As elsewhere in this chapter, I will write $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for the measure algebra of the usual measure on $\{0, 1\}^\kappa$. In any measure algebra $(\mathfrak{A}, \bar{\mu})$ I will write $\mathfrak{A}^f = \{a : a \in \mathfrak{A}, \bar{\mu}a < \infty\}$.

528A Amoeba algebras Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) If $0 < \gamma \le \bar{\mu}1$, the **amoeba algebra** $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is the regular open algebra $\mathrm{RO}^{\uparrow}(P)$ where $P = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma\}$, ordered by $\subseteq$.

(b) The **variable-measure amoeba algebra** $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ (Truss 88) is the regular open algebra $\mathrm{RO}^{\uparrow}(P')$ where

$$P' = \{(a, \alpha) : a \in \mathfrak{A},\ \alpha \in]\bar{\mu}a, \bar{\mu}1]\},$$

ordered by saying that

$$(a, \alpha) \le (b, \beta) \text{ if } a \subseteq b \text{ and } \beta \le \alpha.$$

528B It may help to have the following simple facts set out straight away.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $0 < \gamma \le \bar{\mu}1$. Set $P = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma\}$.

(a) Two elements a, $b \in P$ are compatible upwards in P iff $\bar{\mu}(a \cup b) < \gamma$.

(b) Suppose that $(\mathfrak{A}, \bar{\mu})$ is semi-finite and atomless.

(i) P is separative upwards, so $[a, \infty[\in \mathrm{RO}^{\uparrow}(P)$ for every $a \in P$.

(ii) If $A \subseteq P$ is non-empty, then the infimum $\inf_{a \in A} [a, \infty[$ is empty unless $\sup A$ is defined in $\mathfrak{A}$ and belongs to P, and in this case $\inf_{a \in A} [a, \infty[= [\sup A, \infty[$.

proof (a) $[a, \infty[\cap [b, \infty[= \{c : a \cup b \subseteq c \in P\}$ is non-empty iff $a \cup b \in P$.

(b)(i) Let a, $b \in P$ be such that $a \not\subseteq b$. If $\bar{\mu}(a \cup b) \ge \gamma$ then a and b are already incompatible upwards. Otherwise, $\bar{\mu}(1 \setminus (a \cup b)) \ge \gamma - \bar{\mu}(a \cup b)$. Because $(\mathfrak{A}, \bar{\mu})$ is atomless and semi-finite, there is a $d \subseteq 1 \setminus (a \cup b)$ such that $\bar{\mu}d = \gamma - \bar{\mu}(a \cup b)$. Set $c = b \cup d$. Then

$$\bar{\mu}c = \gamma - \bar{\mu}(a \setminus b) < \gamma = \bar{\mu}(a \cup c),$$

so $c \in [b, \infty[\subseteq P$, while a and c are incompatible upwards in P. As a and b are arbitrary, P is separative upwards.

By 514Me, it follows that $[a, \infty[$ is a regular up-open set for every $a \in P$.

(ii) This is a re-phrasing of 514Mf.

528C Proposition Suppose that (X, Σ, μ) is a measure space, $(\mathfrak{A}, \bar{\mu})$ its measure algebra and $0 < \gamma \le \mu X$. If $\mathcal{E} \subseteq \Sigma$ is any family such that μ is outer regular with respect to $\mathcal{E}$, then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is isomorphic to $\mathrm{RO}^{\uparrow}(\{E : E \in \mathcal{E},\ \mu E < \gamma\})$.

proof Set $P = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma\}$, $Q = \{E : E \in \mathcal{E},\ \mu E < \gamma\}$. Because μ is outer regular with respect to $\mathcal{E}$, the map $G \mapsto G^{\bullet} : Q \to \mathfrak{A}$ maps Q onto a cofinal subset P' of P. Moreover, two elements E_0 and E_1 of Q are compatible upwards in Q iff $\mu(E_0 \cup E_1) < \gamma$ iff $E_0^{\bullet}$ and $E_1^{\bullet}$ are compatible upwards in P. By 514R, $\mathrm{RO}^{\uparrow}(P)$ and $\mathrm{RO}^{\uparrow}(Q)$ are isomorphic.

528D Proposition (a) (Truss 88) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless homogeneous probability algebra. Then the amoeba algebras $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma')$ are isomorphic for all γ, $\gamma' \in]0, 1[$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a non-totally-finite atomless quasi-homogeneous measure algebra (definition: 374G). Then all the amoeba algebras $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$, for $\gamma > 0$, are isomorphic.

proof (a)(i) Set $P = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma\}$, and let κ be the Maharam type of $\mathfrak{A}$. Then the upwards cellularity of P is at most κ. **P?** Otherwise, there is an up-antichain $A \subseteq P$ with cardinality κ^+. Let $\epsilon > 0$ be such that $A' = \{a : a \in A,\ \bar{\mu}a \le \gamma - \epsilon\}$ has cardinal κ^+. Because the topological density of $\mathfrak{A}$ is κ (521Ea), there must be distinct a, $a' \in A'$ such that $\bar{\mu}(a \triangle a') < \epsilon$; but in this case $\bar{\mu}(a \cup a') < \gamma$, so that $a \cup a'$ is an upper bound for $\{a, a'\}$ in P. **XQ**

(ii) If $1 - \sqrt{1 - \gamma} \le \alpha < \gamma$ and D is a countable subset of $]\alpha, \gamma[$ such that $\sup D = \gamma$, then there is a maximal up-antichain $\langle a_{t\xi} \rangle_{(t,\xi) \in D \times \kappa}$ in P such that $\bar{\mu}a_{t\xi} = t$ for every $t \in D$, $\xi < \kappa$. **P** Start with a stochastically independent family $\langle c_{t\xi} \rangle_{(t,\xi) \in D \times \kappa}$ of elements of $\mathfrak{A}$ with $\bar{\mu}c_{t\xi} = t$ for all $t \in D$, $\xi < \kappa$. Because $\alpha \ge 1 - \sqrt{1 - \gamma}$, $A = \langle c_{t\xi} \rangle_{(t,\xi) \in D \times \kappa}$ is an up-antichain in P. Next, because $\sup D = \gamma$, $Q = \{a : a \in P,\ \bar{\mu}a \in D\}$ is cofinal with P. So there is a maximal up-antichain $A' \supseteq A$ such that $A' \subseteq Q$ (513Aa). Now (because $c^{\uparrow}(P) \le \kappa$) $\{a : a \in A',\ \bar{\mu}a = t\}$ has cardinal κ for every $t \in D$, so we can enumerate A' as $\langle a_{t\kappa} \rangle_{(t,\xi) \in D \times \kappa}$ in P where $\bar{\mu}a_{t\xi} = t$ for every $t \in D$ and $\xi < \kappa$. **Q**

(iii) There are α, $\alpha' \in]0,1[$ such that

$$1-\sqrt{1-\gamma} \le \alpha < \gamma, \quad 1-\sqrt{1-\gamma'} \le \alpha' < \gamma', \quad \frac{\gamma-\alpha}{1-\alpha} = \frac{\gamma'-\alpha'}{1-\alpha'}.$$

P We need consider only the case $\gamma \le \gamma'$. Set

$$\beta = \frac{1}{\sqrt{1-\gamma}} - 1, \quad \alpha = \gamma - \beta(1-\gamma), \quad \alpha' = \gamma' - \beta(1-\gamma').$$

Then $\frac{\gamma-\alpha}{1-\gamma} = \beta = \frac{\gamma'-\alpha'}{1-\gamma'}$. Of course $\alpha \le \gamma$ and $\alpha' \le \gamma'$. On the other side, $\alpha = 1-\sqrt{1-\gamma}$, while $\beta \le \frac{1}{\sqrt{1-\gamma'}} - 1$ so $\alpha' \ge 1-\sqrt{1-\gamma'}$. **Q**

(iv) If $a \in P$, $\{b : a \subseteq b \in P\}$ is isomorphic, as partially ordered set, to $\{b : b \in \mathfrak{A},\ \bar\mu b < \frac{\gamma-\bar\mu a}{1-\bar\mu a}\}$. **P** The principal ideal $\mathfrak{A}_{1\setminus a}$ generated by $1\setminus a$ is isomorphic, up to a scalar multiple of the measure, to $\mathfrak{A}$, and $\{b : a \subseteq b \in P\}$ is isomorphic, as partially ordered set, to $\{b : b \subseteq 1\setminus a,\ \bar\mu b < \gamma - \bar\mu a\}$. **Q**

(v) For each $n \in \mathbb{N}$, set $\alpha_n = \gamma - 2^{-n}(\gamma-\alpha)$, $\alpha'_n = \gamma' - 2^{-n}(\gamma'-\alpha')$; then

$$\frac{1-\alpha'_n}{\gamma'-\alpha'_n} = 1 + \frac{1-\gamma'}{\gamma'-\alpha'_n} = 1 + 2^n\frac{1-\gamma'}{\gamma'-\alpha'} = \frac{1-\alpha_n}{\gamma-\alpha_n}, \quad \frac{\gamma'-\alpha'_n}{1-\alpha'_n} = \frac{\gamma-\alpha_n}{1-\alpha_n}$$

for every $n \in \mathbb{N}$. Set $P' = \{a : a \in \mathfrak{A},\ \bar\mu a < \gamma'\}$. By (b), we have a maximal up-antichain $\langle a_{n\xi}\rangle_{(n,\xi)\in\mathbb{N}\times\xi}$ in P such that $\bar\mu a_{n\xi} = \alpha_n$ for all $n \in \mathbb{N}$ and $\xi < \kappa$; similarly, there is a maximal up-antichain $\langle a'_{n\xi}\rangle_{(n,\xi)\in\mathbb{N}\times\xi}$ in P' such that $\bar\mu a'_{n\xi} = \alpha'_n$ for all $n \in \mathbb{N}$ and $\xi < \kappa$. Now, for each $n \in \mathbb{N}$ and $\xi < \kappa$, $[a_{n\xi},\infty[$, taken in P, is isomorphic, as partially ordered set, to $[a'_{n\xi},\infty[$, taken in P', by (d). So

$$\mathrm{RO}^\uparrow(P) \cong \prod_{n\in\mathbb{N},\xi<\kappa} \mathrm{RO}^\uparrow([a_{n\xi},\infty[)$$

(514Nf)

$$\cong \prod_{n\in\mathbb{N},\xi<\kappa} \mathrm{RO}^\uparrow([a'_{n\xi},\infty[) \cong \mathrm{RO}^\uparrow(P').$$

(b) Suppose that β, $\gamma > 0$. As in Lemma 332I, we have a partition D of unity in $\mathfrak{A}$ such that $\bar\mu a = \beta$ for every $a \in D$. Similarly, we have a partition D' of unity such that $\bar\mu a = \gamma$ for every $a \in D'$. By 332E, $\#(D) = \#(D') = c(\mathfrak{A})$. Let $h : D \to D'$ be a bijection. If $d \in D$, the principal ideals $\mathfrak{A}_d$, $\mathfrak{A}_{h(d)}$ have the same Maharam type, because $(\mathfrak{A},\bar\mu)$ is quasi-homogeneous (374H), and are therefore isomorphic as measure algebras, up to a scalar factor of the measure; let $\pi_d : \mathfrak{A}_d \to \mathfrak{A}_{h(d)}$ be a Boolean isomorphism such that $\bar\mu(\pi_d a) = \frac{\gamma}{\beta}\bar\mu a$ for every $a \subseteq d$. Now we have a function $\pi : \mathfrak{A}^f \to \mathfrak{A}^f$ defined by saying that $\pi a = \sup_{d\in D}\pi_d(a \cap d)$ whenever $\bar\mu a < \infty$, and π is a Boolean ring automorphism such that $\bar\mu\pi a = \frac{\gamma}{\beta}\bar\mu a$ for every $a \in \mathfrak{A}^f$. But now π includes an isomorphism between the partially ordered sets $\{a : \bar\mu a < \beta\}$ and $\{a : \bar\mu a < \gamma\}$, so their regular open algebras $\mathrm{AM}(\mathfrak{A},\bar\mu,\beta)$ and $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ are isomorphic.

528E Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra. Then there is a family $\langle c_\alpha\rangle_{\alpha\in[0,\bar\mu 1]}$ in $\mathfrak{A}$ such that $c_\alpha \subseteq c_\beta$ and $\bar\mu c_\alpha = \alpha$ whenever $0 \le \alpha \le \beta \le \bar\mu 1$, and $\alpha \mapsto c_\alpha$ is continuous for the measure-algebra topology of $\mathfrak{A}$.

proof Because $(\mathfrak{A},\bar\mu)$ is semi-finite, there is a non-decreasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^f$ such that $\sup_{n\in\mathbb{N}}\bar\mu e_n = \bar\mu 1$, starting from $e_0 = 0$; set $e = \sup_{n\in\mathbb{N}} e_n$, so that $\bar\mu e = \bar\mu 1$. Then $(\mathfrak{A}_e, \bar\mu\restriction\mathfrak{A}_e)$ is σ-finite and atomless. Let $(\mathfrak{C},\bar\lambda)$ be the measure algebra of Lebesgue measure on $[0,\bar\mu 1[$. For each $n \in \mathbb{N}$ set $e'_n = e_{n+1}\setminus e_n$ and $d_n = [\bar\mu e_n, \bar\mu e_{n+1}[^\bullet \in \mathfrak{C}$.

Because $\mathfrak{A}$ is atomless, 332P tells us that there is for each $n \in \mathbb{N}$ a measure-preserving Boolean homomorphism π_n from the principal ideal $\mathfrak{C}_{d_n}$ to a principal ideal of $\mathfrak{A}_{e'_n}$, which must be $\mathfrak{A}_{e'_n}$ itself because

$\bar{\mu}e'_n = \bar{\lambda}d_n$; by 324Kb, π_n is order-continuous. Assembling these, we have an order-continuous measure-preserving Boolean homomorphism $\pi : \mathfrak{C} \to \mathfrak{A}_e$ defined by setting $\pi d = \sup_{n\in\mathbb{N}} \pi_n(d \cap d_n)$ for every $d \in \mathfrak{C}$. Now set $c_\alpha = \pi\,[0,\alpha[^\bullet$ for $\alpha \le \bar{\mu}1$. Because π is continuous for the measure-algebra topologies of $\mathfrak{C}$ and $\mathfrak{A}_e$ (324Fc), or otherwise, $\alpha \mapsto c_\alpha$ is continuous.

528F Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and $\gamma \in\,]0,\infty[$.

(a) Suppose that $e \in \mathfrak{A}$ and $\bar{\mu}e \ge \gamma$. If $\mathfrak{A}_e$ is atomless, then $\mathrm{AM}(\mathfrak{A}_e, \bar{\mu}{\restriction}\mathfrak{A}_e, \gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$.

(b) Suppose that $\mathfrak{A}$ is atomless, and that $\gamma < \bar{\mu}1$. Let $\langle e_k\rangle_{k\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}$ with supremum 1, and suppose that $\bar{\mu}e_k \ge \gamma$ for every $k \in \mathbb{N}$. Then we have a sequence $\langle \pi_k\rangle_{k\in\mathbb{N}}$ such that $\pi_k : \mathrm{AM}(\mathfrak{A}_{e_k}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k}, \gamma) \to \mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is a regular embedding for every $k \in \mathbb{N}$, and $\bigcup_{k\in\mathbb{N}} \pi_k[\mathrm{AM}(\mathfrak{A}_{e_k}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k}, \gamma)]$ τ-generates $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$.

(c) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is atomless and quasi-homogeneous, and that $\gamma < \bar{\mu}1$. Then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$.

proof (a) Set $P = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma\}$ and $Q = P \cap \mathfrak{A}_e$. By 528E, we have a continuous order-preserving function $\alpha \mapsto c_\alpha : [0, \bar{\mu}e] \to \mathfrak{A}_e$ such that $\bar{\mu}c_\alpha = \alpha$ for each α. If $a \in \mathfrak{A}$, then the function $\beta \mapsto \bar{\mu}(c_\beta \setminus a)$ is a continuous non-decreasing function from $[0, \bar{\mu}e]$ onto $[0, \bar{\mu}(c_{\bar{\mu}e} \setminus a)]$, and we can set $\delta(a,\alpha) = \min\{\beta : \bar{\mu}(c_\beta \setminus a) = \alpha\}$ whenever $0 \le \alpha \le \bar{\mu}(c_{\bar{\mu}e} \setminus a)$. In this case,

$$\bar{\mu}((a \cap e) \cup c_{\delta(a,\alpha)}) = \bar{\mu}(a \cap e) + \bar{\mu}(c_{\delta(a,\alpha)} \setminus a) = \alpha + \bar{\mu}(a \cap e).$$

Note that $\delta(a,\alpha) \le \delta(a',\alpha')$ whenever $a \subseteq a'$ and $\alpha \le \alpha' \le \bar{\mu}(c_{\bar{\mu}1} \setminus a')$.

If $a \in P$, then

$$\bar{\mu}(c_{\bar{\mu}e} \setminus a) = \bar{\mu}c_{\bar{\mu}e} - \bar{\mu}(a \cap c_{\bar{\mu}e}) \ge \bar{\mu}e - \bar{\mu}(a \cap e) \ge \bar{\mu}a - \bar{\mu}(a \cap e) = \bar{\mu}(a \setminus e).$$

So $\delta(a, \bar{\mu}(a \setminus e))$ is defined, and we have a function f given by the formula

$$f(a) = (a \cap e) \cup c_{\delta(a,\bar{\mu}(a\setminus e))}$$

for $a \in P$. In this case $\bar{\mu}f(a) = \bar{\mu}a$, so $f(a) \in Q$, for each a, and f, like δ, is order-preserving. Of course $f(a) = a$ for $a \in Q$.

If $a \in P$, $b \in Q$ and $f(a) \subseteq b$, there is an $a' \in P$ such that $a \subseteq a'$ and $b = f(a')$. **P** Set $a' = a \cup (b \setminus f(a))$. Then

$$\bar{\mu}a' = \bar{\mu}a + \bar{\mu}(b \setminus f(a)) = \bar{\mu}f(a) + \bar{\mu}(b \setminus f(a)) = \bar{\mu}b < \gamma,$$

so $a' \in P$. Also $b \subseteq f(a) \cup (a' \cap e) \subseteq f(a')$; as $\bar{\mu}b = \bar{\mu}a' = \bar{\mu}f(a')$, $b = f(a')$. **Q** So if $Q_0 \subseteq Q$ is cofinal with Q, $f^{-1}[Q_0]$ will be cofinal with P (as in the proof of 514P), and we have an order-continuous Boolean homomorphism $\pi : \mathrm{RO}^\uparrow(Q) \to \mathrm{RO}^\uparrow(P)$ defined by setting $\pi H = \mathrm{int}\,\overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}^\uparrow(Q)$. Finally, $f[P] = f[Q] = Q$. So π is injective and is a regular embedding of $\mathrm{AM}(\mathfrak{A}_e, \bar{\mu}{\restriction}\mathfrak{A}_e, \gamma) = \mathrm{RO}^\uparrow(Q)$ into $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma) = \mathrm{RO}^\uparrow(P)$.

(b)(i) For each $k \in \mathbb{N}$, set $Q_k = P \cap \mathfrak{A}_{e_k}$ and choose functions $f_k : P \to Q_k$ and $\pi_k : \mathrm{RO}^\uparrow(Q_k) \to \mathrm{RO}^\uparrow(P)$ as in (a) above. If we write $[c,\infty[\, = \{a : c \subseteq a \in P\}$ for every $c \in P$, then $\mathfrak{A}_{e_k} \cap [c,\infty[\, = \{b : c \subseteq b \in Q_k\}$ for $k \in \mathbb{N}$ and $c \in Q_k$; in this case, $\mathfrak{A}_{e_k} \cap [c,\infty[\, \in \mathrm{RO}^\uparrow(Q_k)$, by 528B(b-i).

(ii) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^\uparrow(P)$ generated by $\bigcup_{k\in\mathbb{N}} \pi_k[\mathrm{RO}^\uparrow(Q_k)]$. If $a \in P$, there is a non-empty $G \in \mathfrak{G}$ included in $[a,\infty[\, \in \mathrm{RO}^\uparrow(P)$. **P** Because $a \subseteq \sup_{k\in\mathbb{N}} e_k$ and $\langle e_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, there is an infinite $I \subseteq \mathbb{N}$ such that $\sum_{k\in I} \bar{\mu}(a \setminus e_k) < \gamma - \bar{\mu}a$. Set $b = \sup_{k\in I} f_k(a)$. Then

$$\bar{\mu}b \le \bar{\mu}a + \textstyle\sum_{k\in I} \bar{\mu}(a \setminus f_k(a)) \le \bar{\mu}a + \sum_{k\in I} \bar{\mu}(a \setminus e_k) < \gamma$$

because $f_k(a) \supseteq a \cap e_k$ for every k, by the construction in (a). Thus $b \in P$. Also

$$\bar{\mu}(a \setminus b) \le \inf_{k\in\mathbb{N}} \bar{\mu}(a \setminus f_k(a)) = 0,$$

so $a \subseteq b$.

Set

$$V_k = \mathfrak{A}_{e_k} \cap [f_k(a),\infty[\, \in \mathrm{RO}^\uparrow(Q_k)$$

for every k. Then $\pi_k V_k = \operatorname{int}\overline{f_k^{-1}[V_k]}$ belongs to $\mathfrak{G}$ for each k, and $G = \inf_{k\in\mathbb{N}}\pi_k V_k = \bigcap_{k\in\mathbb{N}}\pi_k V_k$ (514M(d-ii)) belongs to $\mathfrak{G}$. Because every f_k is order-preserving, $f_k(b') \supseteq f_k(a)$ and $f_k(b') \in V_k$ for every $b' \supseteq b$; thus $b \in \operatorname{int} f_k^{-1}[V_k]$ for every k, and $b \in G$. This shows that $G \ne \emptyset$.

? Suppose, if possible, that $G \not\subseteq [a,\infty[$. Then there is a $c \in G$ such that $a \setminus c \ne 0$. If $\bar\mu(c \cup a) > \gamma$, set $c' = c$. Otherwise, let $\delta > 0$ be such that

$$\delta + \bar\mu c < \gamma < \delta + \bar\mu(c \cup a) < \bar\mu 1.$$

Because $(\mathfrak{A},\bar\mu)$ is atomless and semi-finite, there is a $d \subseteq 1 \setminus (c \cup a)$ such that $\bar\mu d = \delta$. Set $c' = c \cup d$; then $c \subseteq c' \in P$ so $c' \in G$, while $\bar\mu(c' \cup a) > \gamma$, as in the previous case.

Because I is infinite, $\sup_{k\in I} e_k = 1$ and there is a $k \in I$ such that $\bar\mu((c' \cup a) \cap e_k) \ge \gamma$. In this case, $c' \in \pi_k V_k \subseteq \overline{f_k^{-1}[V_k]}$, so $[c',\infty[$ meets $f_k^{-1}[V_k]$ and there is a $c'' \supseteq c'$ such that $c'' \in P$ and $f_k(c'') \in V_k$, that is, $f_k(c'') \supseteq f_k(a)$. Now, however,

$$f_k(c'') \supseteq (c' \cap e_k) \cup (f_k(a) \cap e_k) \supseteq (c' \cup a) \cap e_k$$

has measure at least γ, and cannot belong to Q_k. **XQ**

(iii) Since $\{[a,\infty[\,: a \in P\}$ is a base for the topology of P, it is a π-base for $\mathrm{RO}^\uparrow(P)$, and $\mathfrak{G}$ includes a π-base for $\mathrm{RO}^\uparrow(P)$. But this means that every member of $\mathrm{RO}^\uparrow(P)$ is the supremum of the members of $\mathfrak{G}$ it includes, and belongs to $\mathfrak{G}$. Thus $\mathfrak{G} = \mathrm{RO}^\uparrow(P)$, as claimed.

(c)(i) This time, let $\langle c_\alpha\rangle_{\alpha\in[0,\bar\mu 1]}$ be a family in $\mathfrak{A}$ such that $c_\alpha \subseteq c_\beta$ and $\bar\mu c_\alpha = \alpha$ whenever $0 \le \alpha \le \beta \le \bar\mu 1$. Set $P = \{(a,\alpha) : a \in \mathfrak{A},\ \alpha \in \,]\bar\mu a, \bar\mu 1]\}$. Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence with supremum $\bar\mu 1$ and $\gamma_0 = 0$. For each $n \in \mathbb{N}$, set $P_n = \{(a,\alpha) : \gamma_n \le \bar\mu a < \alpha \le \gamma_{n+1}\}$ and $Q_n = \{a : a \in \mathfrak{A},\ \bar\mu a < \gamma_{n+1}\}$, so that P_n is an up-open set in P. Note that $\bigcup_{n\in\mathbb{N}} P_n$ is dense in P for the up-topology, since if $(a,\alpha) \in P$ then $(a, \min(\alpha,\gamma_{n+1})) \in P_n$ where $\gamma_n \le \bar\mu a < \gamma_{n+1}$. Also

$$\mathrm{RO}^\uparrow(Q_n) = \mathrm{AM}(\mathfrak{A},\bar\mu,\gamma_{n+1}) \cong \mathrm{AM}(\mathfrak{A},\bar\mu,\gamma).$$

P If $\bar\mu 1 = \infty$, this is 528Db. If $(\mathfrak{A},\bar\mu)$ is totally finite, then $\mathfrak{A}$ is homogeneous, so we can apply 528Da to an appropriate multiple of the measure $\bar\mu$. **Q**

For $a \in \mathfrak{A}^f$, the function $\alpha \mapsto \bar\mu(c_\alpha \setminus a) : [0,\bar\mu 1] \to [0,\infty]$ is continuous and non-decreasing, and

$$\bar\mu(c_{\bar\mu 1} \setminus a) \ge \bar\mu c_{\bar\mu 1} - \bar\mu a = \bar\mu 1 - \bar\mu a = \bar\mu(1 \setminus a).$$

So we can define $\delta(a,\alpha)$, for $a \in \mathfrak{A}^f$ and $0 \le \alpha \le \bar\mu(1 \setminus a)$, by saying that

$$\delta(a,\alpha) = \min\{\beta : \bar\mu(c_\beta \setminus a) = \alpha\} = \min\{\beta : \bar\mu(a \cup c_\beta) = \bar\mu a + \alpha\}.$$

As in (a), $\delta(a,\alpha) \le \delta(a',\alpha')$ whenever $a \subseteq a'$ and $\alpha \le \alpha'$. For $(a,\alpha) \in P_n$, set

$$f_n(a,\alpha) = a \cup c_{\delta(a,\gamma_{n+1}-\alpha)},$$

so that $\bar\mu f_n(a,\alpha) = \bar\mu a + \gamma_{n+1} - \alpha < \gamma_{n+1}$ and $f_n(a,\alpha) \in Q_n$. Of course $f_n(a,\gamma_{n+1}) = a$ if $(a,\gamma_{n+1}) \in P_n$, that is, if $a \in Q_n$ and $\bar\mu a \ge \gamma_n$.

(ii)(α) $f_n : P_n \to Q_n$ is order-preserving. **P** If $(a,\alpha) \le (a',\alpha')$ in P_n, then $\delta(a,\gamma_{n+1}-\alpha) \le \delta(a',\gamma_{n+1}-\alpha')$, so $f_n(a,\alpha) \subseteq f_n(a',\alpha')$. **Q**

(β) If $p \in P_n$, $b \in Q_n$ and $f_n(p) \subseteq b$, there is a $p' \in P_n$ such that $p \le p'$ and $b \subseteq f_n(p')$. **P** Express p as (a,α). Consider $a' = a \cup (b \setminus f_n(p))$. Then

$$\bar\mu a' = \bar\mu a + \bar\mu b - \bar\mu f_n(p) = \bar\mu a + \bar\mu b - \bar\mu a - \gamma_{n+1} + \alpha < \alpha,$$

so $(a',\alpha) \in P$. Of course $(a,\alpha) \le (a',\alpha)$, so $p' = (a',\alpha) \in P_n$. Also $f_n(p') \supseteq f_n(p)$ and

$$f_n(p') \supseteq a' \supseteq b \setminus f_n(p),$$

so $b \subseteq f_n(p')$. **Q**

(γ) $f_n[P_n]$ is cofinal with Q_n. **P** If $b \in Q_n$, take $b' \in \mathfrak{A}$ such that $b \subseteq b'$ and $\gamma_n \le \bar\mu b' < \gamma_{n+1}$. Then $(b',\gamma_{n+1}) \in P_n$ and

$$b \subseteq b' = f_n(b',\gamma_{n+1}) \in f_n[P_n].\ \mathbf{Q}$$

(iii) By 514P, $\mathrm{RO}^{\uparrow}(Q_n)$ can be regularly embedded in $\mathrm{RO}^{\uparrow}(P_n)$. Now $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ is isomorphic to $\mathrm{RO}^{\uparrow}(Q_n)$, so there is an injective order-continuous Boolean homomorphism $\pi_n:\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)\to\mathrm{RO}^{\uparrow}(P_n)$. Putting these together, we have an injective order-continuous Boolean homomorphism $\pi:\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)\to\prod_{n\in\mathbb{N}}\mathrm{RO}^{\uparrow}(P_n)$ defined by setting $\pi u=\langle\pi_n(u)\rangle_{n\in\mathbb{N}}$ for $u\in\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$. On the other hand, since $\langle P_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of up-open subsets of P with dense union,

$$\prod_{n\in\mathbb{N}}\mathrm{RO}^{\uparrow}(P_n)\cong\mathrm{RO}^{\uparrow}(P)=\mathrm{AM}^*(\mathfrak{A},\bar\mu)$$

by 315H. So we have a regular embedding of $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ into $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$, as claimed.

528G Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathfrak{C}$ a σ-subalgebra of $\mathfrak{A}$ such that $\sup(\mathfrak{C}\cap\mathfrak{A}^f)=1$ in $\mathfrak{A}$. Then $\mathrm{AM}^*(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$.

proof (a) For each $a\in\mathfrak{A}^f$ we have a 'conditional expectation' $u_a\in L^1(\mathfrak{C})$ defined by saying that $\int_c u_a=\bar\mu(a\cap c)$ for every $c\in\mathfrak{C}^f$. (Apply 365O[11] to the identity map from $\mathfrak{C}^f$ to $\mathfrak{A}^f$.) Note that as the supremum of $\mathfrak{C}^f$ in $\mathfrak{A}$ is 1,

$$\int u_a=\sup_{c\in\mathfrak{C}^f}\int_c u_a=\sup_{c\in\mathfrak{C}^f}\bar\mu(a\cap c)=\bar\mu a.$$

Also, of course, $0\le\bar\mu(a\cap c)\le\bar\mu c$ for every $c\in\mathfrak{C}^f$, so $0\le u_a\le\chi 1$ in $L^\infty(\mathfrak{C})$. Next, let u_a^* be the decreasing rearrangement of u_a, that is, the element of $L^\infty(\mathfrak{A}_L)$ (where $\mathfrak{A}_L$ is the measure algebra of Lebesgue measure on $[0,\infty[$) such that $[\![u^*>\alpha]\!]=[0,\bar\mu[\![u>\alpha]\!][^\bullet$ for every $\alpha\ge 0$ (373Da).

(b) Set

$$P=\{(a,\alpha):a\in\mathfrak{A},\,\alpha\in\,]\bar\mu a,\bar\mu 1]\},\qquad Q=\{(c,\alpha):c\in\mathfrak{C},\,\alpha\in\,]\bar\mu c,\bar\mu 1]\}.$$

Define a function f on P by saying that $f(a,\alpha)=(c,\beta)$ if

$$c=[\![u_a=1]\!]=\max\{d:d\in\mathfrak{C},\,d\subseteq a\},$$

$$\beta=\max\{\beta':\beta'\ge 0,\,\beta'+\int_{\beta'}^\infty u_a^*\le\alpha\}.$$

Note that $\beta>\bar\mu c$ because

$$\bar\mu c+\int_{\bar\mu c}^\infty u_a^*=\int u_a^*=\int u_a<\alpha,$$

using 373Fa for the equality in the middle, while $\beta\le\alpha\le\bar\mu 1$; so (c,β) belongs to Q.

(c)(i) If $p\le p'$ in P, then $f(p)\le f(p')$ in Q. **P** Express p, p', $f(p)$ and $f(p')$ as (a,α), (a',α'), (c,β), (c',β') respectively. Then $c\subseteq a\subseteq a'$ so $c\subseteq c'$. Next, $\chi a\le\chi a'$ so $u_a\le u_{a'}$ and $u_a^*\le u_{a'}^*$ (373Db); accordingly

$$\alpha'\le\alpha=\beta+\int_\beta^\infty u_a^*\le\beta+\int_\beta^\infty u_{a'}^*,$$

and $\beta'\le\beta$. **Q**

(ii) If $p\in P$, $q\in Q$ and $f(p)\le q$, then there is a $p'\ge p$ such that $f(p')\ge q$. **P** Express p, $f(p)$ and q as (a,α), (c,β) and (d,γ) respectively. Set $a'=a\cup d$. Then

$$\bar\mu a'=\bar\mu a+\bar\mu d-\bar\mu(a\cap d)=\int u_a+\bar\mu d-\int_d u_a\ge\int u_a^*+\bar\mu d-\int_0^{\bar\mu d}u_a^*$$

(apply 373E with $v=\chi d$)

$$=\bar\mu d+\int_{\bar\mu d}^\infty u_a^*<\beta+\int_\beta^\infty u_a^*$$

(because $[\![u_a^*=1]\!]=[0,\bar\mu c]^\bullet$ and $\bar\mu c\le\bar\mu d<\gamma\le\beta$)

$$=\alpha.$$

So $(a',\alpha)\in P$. Next, computing the integrals $\int_b u_a\vee\chi d$ for b belonging to $\mathfrak{C}^f$ and either included in d or disjoint from it, we see that $u_{a'}=u_a\vee\chi d$ so that $[\![u_{a'}=1]\!]=[\![u_a=1]\!]\cup d=d$. Accordingly

[11] Formerly 365P.

$$\bar{\mu}a' = \int u_{a'} = \int u^*_{a'} = \bar{\mu}d + \int_{\bar{\mu}d}^{\infty} u^*_{a'} < \gamma + \int_{\gamma}^{\infty} u^*_{a'}$$

(as noted above for u_a, we have $[\![u^*_{a'} = 1]\!] = [0, \bar{\mu}d]^\bullet$), and if we set $\alpha' = \min(\alpha, \gamma + \int_\gamma^\infty u^*_{a'})$ then $p' = (a', \alpha') \ge p$ and $f(p') \ge q$, as required. **Q**

(iii) Since P and Q have a common least element $(0, \bar{\mu}1)$ which is invariant under f, f satisfies the second condition of 514P and $\mathrm{AM}^*(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C}) = \mathrm{RO}^\uparrow(Q)$ is regularly embedded in $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu}) = \mathrm{RO}^\uparrow(P)$.

528H Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, not $\{0\}$, and let $\kappa \ge \max(\omega, \tau(\mathfrak{A}), c(\mathfrak{A}))$ be a cardinal. Then $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \frac{1}{2})$.

proof (a) To begin with (down to the end of (g) below), assume that $\mathfrak{A}$ is atomless. Let $(\mathfrak{A}^{\mathbb{N}}, \bar{\mu}_\infty)$ be the simple product of a sequence of copies of $(\mathfrak{A}, \bar{\mu})$ (322L), so that $\bar{\mu}_\infty \boldsymbol{a} = \sum_{n=0}^\infty \mu a_n$ if $\boldsymbol{a} = \langle a_n \rangle_{n\in\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$. Note that as $\mathfrak{A}$ is certainly infinite, $\tau(\mathfrak{A}^{\mathbb{N}}) = \tau(\mathfrak{A})$ and $c(\mathfrak{A}^{\mathbb{N}}) = c(\mathfrak{A})$ (514Ef). By 526D, there is a function $\theta : \mathfrak{A}^{\mathbb{N}} \to \mathfrak{B}_\kappa$ such that

$\theta(\sup A) = \sup \theta[A]$ for every non-empty $A \subseteq \mathfrak{A}^{\mathbb{N}}$ with a supremum in $\mathfrak{A}$,

$\bar{\nu}_\kappa \theta(\boldsymbol{a}) = 1 - \exp(-\bar{\mu}_\infty \boldsymbol{a})$ for every $\boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}}$,

whenever $\langle \boldsymbol{a}^{(i)} \rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}^{\mathbb{N}}$ and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{\theta(\boldsymbol{a}) : \boldsymbol{a} \subseteq \boldsymbol{a}^{(i)}\}$ for each i, then $\langle \mathfrak{C}_i \rangle_{i\in I}$ is stochastically independent.

(b) For $b \in \mathfrak{B}_\kappa$, set $g(b) = \sup\{\boldsymbol{a} : \boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}},\ \theta(\boldsymbol{a}) \subseteq b\}$.

(i) It is immediate from its definition that $g : \mathfrak{B}_\kappa \to \mathfrak{A}^{\mathbb{N}}$ is order-preserving.

(ii) Because θ is supremum-preserving, $\theta(g(b)) \subseteq b$ for every $b \in \mathfrak{B}_\kappa$.

(iii) If $b \in \mathfrak{B}_\kappa \setminus \{1\}$ then

$$1 - \bar{\nu}_\kappa b \le 1 - \bar{\nu}_\kappa \theta(g(b)) = \exp(-\bar{\mu}_\infty g(b)),$$

so $\bar{\mu}_\infty g(b) \le -\ln(1 - \bar{\nu}_\kappa b)$ is finite.

(iv) $\boldsymbol{a} \subseteq g(\theta(\boldsymbol{a}))$ for every $\boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}}$; and if $\boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}}$ has finite measure then $g(\theta(\boldsymbol{a})) = \boldsymbol{a}$, because if $\boldsymbol{a}' \not\subseteq \boldsymbol{a}$ then $\bar{\nu}_\kappa \theta(\boldsymbol{a} \cup \boldsymbol{a}') > \bar{\nu}_\kappa \theta(\boldsymbol{a})$.

(v) If $b \in \mathfrak{B}_\kappa$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\bar{\mu}_\infty(g(b') \setminus g(b)) \le \epsilon$ whenever $\bar{\nu}_\kappa(b' \setminus b) \le \delta$. **P?** Otherwise, $g(b) \ne 1_{\mathfrak{A}^{\mathbb{N}}}$ so $b \ne 1_{\mathfrak{B}_\kappa}$ and we can find a sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}_\kappa$ such that $\bar{\nu}_\kappa(b_n \setminus b) \le 2^{-n-2}(1 - \bar{\nu}_\kappa b)$ and $\bar{\mu}_\infty(g(b_n) \setminus g(b)) \ge \epsilon$ for every $n \in \mathbb{N}$. For each n, set $b^*_n = b \cup \sup_{m\ge n} b_m$; then

$$\bar{\mu}_\infty g(b^*_n) = \bar{\mu}_\infty g(b) + \bar{\mu}_\infty(g(b^*_n) \setminus g(b)) \ge \bar{\mu}_\infty g(b) + \bar{\mu}_\infty(g(b_n) \setminus g(b)) \ge \epsilon + \bar{\mu}_\infty g(b).$$

Note that $\bar{\nu}_\kappa b^*_0 < 1$ so $g(b^*_0)$ has finite measure.

The sequences $\langle b^*_n \rangle_{n\in\mathbb{N}}$, $\langle g(b^*_n) \rangle_{n\in\mathbb{N}}$ and $\langle \theta(g(b^*_n)) \rangle_{n\in\mathbb{N}}$ are all non-increasing. Set $\boldsymbol{a} = \inf_{n\in\mathbb{N}} g(b^*_n)$, so that

$$\theta(\boldsymbol{a}) \subseteq \inf_{n\in\mathbb{N}} \theta(g(b^*_n)) \subseteq \inf_{n\in\mathbb{N}} b^*_n = b$$

because $\bar{\nu}_\kappa(b^*_n \setminus b) \le 2^{-n-1}$ for every n. It follows that $\boldsymbol{a} \subseteq g(b)$. At the same time,

$$\bar{\mu}_\infty \boldsymbol{a} = \lim_{n\to\infty} \bar{\mu}_\infty g(b^*_n) > \bar{\mu}_\infty g(b),$$

which is impossible. **XQ**

(c) Define $\psi : \mathfrak{A}^{\mathbb{N}} \to \mathfrak{A}$ by setting $\psi(\boldsymbol{a}) = \sup_{n\in\mathbb{N}} a_n$ whenever $\boldsymbol{a} = \langle a_n \rangle_{n\in\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$.

(i) ψ is supremum-preserving and $\psi(0) = 0$.

(ii) If $\boldsymbol{a}$, $\boldsymbol{a}' \in \mathfrak{A}^{\mathbb{N}}$ then

$$\bar{\mu}(\psi(\boldsymbol{a}) \mathbin{\triangle} \psi(\boldsymbol{a}')) \le \bar{\mu}_\infty(\boldsymbol{a} \mathbin{\triangle} \boldsymbol{a}') \qquad \bar{\mu}(\psi(\boldsymbol{a}) \setminus \psi(\boldsymbol{a}')) \le \bar{\mu}_\infty(\boldsymbol{a} \setminus \boldsymbol{a}').$$

(iii) Now if $b \in \mathfrak{B}_\kappa$, $a \in \mathfrak{A}$, $a \supseteq \psi(g(b))$ and $\bar{\mu}a < \alpha \in \mathbb{R}$, there is a $b' \supseteq b$ such that $a \subseteq \psi(g(b'))$, $\bar{\mu}\psi(g(b')) < \alpha$ and $\bar{\nu}_\kappa(b' \setminus b) \le 1 - \exp(-\bar{\mu}(a \setminus \psi(g(b))))$.

P Take α' such that $\bar{\mu}a < \alpha' < \alpha$. By (b-v), there is a $\delta > 0$ such that $\bar{\mu}_\infty(g(b') \setminus g(b)) \le \alpha' - \bar{\mu}a$ whenever $\bar{\nu}_\kappa(b' \setminus b) \le \delta$. Set $\boldsymbol{a}^{(n)} = \langle a_{ni} \rangle_{i\in\mathbb{N}}$ for each $n \in \mathbb{N}$, where $a_{ni} = a \setminus \psi(g(b))$ if $i = n$, 0 otherwise. For each $n \in \mathbb{N}$, let $\mathfrak{C}_n$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by

$$\{\theta(\boldsymbol{a}) : \boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}},\ \boldsymbol{a} \cap \boldsymbol{a}^{(m)} = 0 \text{ for every } m \ge n\},$$

and let $T_n : L^1(\mathfrak{B}_\kappa, \bar{\nu}_\kappa) \to L^1(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ be the corresponding conditional-expectation operator (365Q[12]). Then $\langle \mathfrak{C}_n \rangle_{n\in\mathbb{N}}$ is non-decreasing; also $\bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)} \cap c) = \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}) \cdot \bar{\nu}_\kappa c$ for every $n \in \mathbb{N}$ and $c \in \mathfrak{C}_n$, by the final clause of (a). By Lévy's martingale theorem (275I, 367Jb), $\langle T_n(\chi b)\rangle_{n\in\mathbb{N}}$ is $\| \ \|_1$-convergent. We can therefore find an $n \in \mathbb{N}$ such that $\|T_n(\chi b) - T_{n+1}(\chi b)\|_1 \le \delta \exp(-\bar{\mu}a)$. Set $b' = b \cup \theta(\boldsymbol{a}^{(n)})$. Then $g(b') \supseteq g(b) \cup \boldsymbol{a}^{(n)}$, so $\psi(g(b')) \supseteq a_{nn} \cup \psi(g(b)) = a$. Also

$$\bar{\nu}_\kappa(b' \setminus b) \le \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}) = 1 - \exp(-\bar{\mu}_\infty \boldsymbol{a}^{(n)}) = 1 - \exp(-\bar{\mu}(a \setminus \psi(g(b))).$$

? If $\bar{\mu}\psi(g(b')) \ge \alpha$, set $\boldsymbol{e} = g(b') \setminus (g(b) \cup \sup_{m\in\mathbb{N}} \boldsymbol{a}^{(m)})$. Since

$$\psi(g(b) \cup \sup_{m\in\mathbb{N}} \boldsymbol{a}^{(m)}) = \psi(g(b)) \cup a = a,$$

$\psi(\boldsymbol{e}) \supseteq \psi(g(b')) \setminus a$ and

$$\bar{\mu}_\infty \boldsymbol{e} \ge \alpha - \bar{\mu}a > \alpha' - \bar{\mu}a;$$

as $\boldsymbol{e} \subseteq g(\theta(\boldsymbol{e}))$ and $\boldsymbol{e} \cap g(b) = 0$, $\bar{\nu}_\kappa(\theta(\boldsymbol{e}) \setminus b) > \delta$. On the other hand,

$$(1 - \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}))\bar{\nu}_\kappa(b \cap \theta(\boldsymbol{e})) = (1 - \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)})) \int_{\theta(\boldsymbol{e})} T_n(\chi b)$$

(because $\boldsymbol{e} \cap \boldsymbol{a}^{(m)} = 0$ for every m, so $\theta(\boldsymbol{e}) \in \mathfrak{C}_n$)

$$= \int \chi(1 \setminus \theta(\boldsymbol{a}^{(n)})) \cdot \int T_n(\chi b) \times \chi\theta(\boldsymbol{e})$$

$$= \int \chi(1 \setminus \theta(\boldsymbol{a}^{(n)})) \times T_n(\chi b) \times \chi\theta(\boldsymbol{e})$$

(because $T_n(\chi b) \times \chi\theta(\boldsymbol{e}) \in L^0(\mathfrak{C}_n)$ and $\chi(1 \setminus \theta(\boldsymbol{a}^{(n)}))$ are stochastically independent)

$$= \int_{\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)})} T_n(\chi b),$$

$$(1 - \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}))\bar{\nu}_\kappa(\theta(\boldsymbol{e})) = \bar{\nu}_\kappa(\theta(\boldsymbol{e}) \setminus \theta(\boldsymbol{a}^{(n)}) = \bar{\nu}_\kappa(b \cap \theta(\boldsymbol{e}) \setminus \theta(\boldsymbol{a}^{(n)})$$

(because $\theta(\boldsymbol{e}) \subseteq \theta(g(b')) \subseteq b' = b \cup \theta(\boldsymbol{a}^{(n)})$)

$$= \int_{\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)})} T_{n+1}(\chi b)$$

because $\theta(\boldsymbol{e})$ and $\theta(\boldsymbol{a}^{(n)})$ both belong to $\mathfrak{C}_{n+1}$. So

$$\begin{aligned}\delta \exp(-\bar{\mu}a) &= \delta \exp(-\bar{\mu}_\infty \boldsymbol{a}^{(n)}) < \bar{\nu}_\kappa(\theta(\boldsymbol{e}) \setminus b) \exp(-\bar{\mu}_\infty \boldsymbol{a}^{(n)}) \\ &= (1 - \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}))\bar{\nu}_\kappa(\theta(\boldsymbol{e}) \setminus b)) \\ &= \int_{\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)})} T_{n+1}(\chi b) - T_n(\chi b) \\ &\le \|T_n(\chi b) - T_{n+1}(\chi b)\|_1 \le \delta \exp(-\bar{\mu}a),\end{aligned}$$

which is impossible. **X**

So $\bar{\mu}\psi(g(b')) < \alpha$, as required. **Q**

(d) Fix $c \in \mathfrak{B}_\kappa$ with measure $\frac{1}{2}$; then the principal ideal of $\mathfrak{B}_\kappa$ generated by c is isomorphic to $\mathfrak{B}_\kappa$ with the measure halved. We therefore have a Boolean isomorphism $\pi : \mathfrak{B}_\kappa \to (\mathfrak{B}_\kappa)_c$ such that $\bar{\nu}_\kappa\pi b = \frac{1}{2}\bar{\nu}_\kappa b$ for every $b \in \mathfrak{B}_\kappa$. Set $h(b) = \psi(g(\pi^{-1}(b \cap c)))$ for $b \in \mathfrak{B}_\kappa$. Then $h : \mathfrak{B}_\kappa \to \mathfrak{A}$ is order-preserving and $h(b) = h(b \cap c)$ for every $b \in \mathfrak{B}_\kappa$. Translating the results of (b) and (c), we see that

if $b \in \mathfrak{B}_\kappa$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\bar{\mu}(h(b') \setminus h(b)) \le \epsilon$ whenever $\bar{\nu}_\kappa(b' \setminus b) \le \delta$,

if $b \in \mathfrak{B}_\kappa$, $a \in \mathfrak{A}$, $a \supseteq h(b)$ and $\bar{\mu}a < \alpha \in \mathbb{R}$, there is a $b' \supseteq b$ such that $a \subseteq h(b')$, $\bar{\mu}h(b') < \alpha$ and $\bar{\nu}_\kappa(b' \setminus b) \le \frac{1}{2}(1 - \exp(-\bar{\mu}(a \setminus h(b)))$.

[12]Formerly 365R.

Note also that

$$\bar\mu h(b)=\bar\mu\psi(g(\pi^{-1}(b\cap c)))\le\bar\mu_\infty g(\pi^{-1}(b\cap c))$$
$$\le-\ln(1-\bar\nu_\kappa\pi^{-1}(b\cap c))\le-\ln(1-2\bar\nu_\kappa(b\cap c))$$

if we take $\ln(0)$ to be $-\infty$.

(e)(i) Set $\gamma_0=\frac12(1-\exp(-\bar\mu1))$, interpreting $\exp(-\infty)$ as 0, so that $0<\gamma_0\le\frac12$. Let P be the partially ordered set $\{(a,\alpha):a\in\mathfrak A,\ \alpha\in\,]\bar\mu a,\bar\mu1]\}$ and Q the partially ordered set $\{b:b\in\mathfrak B_\kappa,\ \bar\nu_\kappa b<\gamma_0\}$, so that $\mathrm{AM}^*(\mathfrak A,\bar\mu)=\mathrm{RO}^\uparrow(P)$ and $\mathrm{AM}(\mathfrak B_\kappa,\bar\nu_\kappa,\gamma_0)=\mathrm{RO}^\uparrow(Q)$. For $b\in Q$, set $\alpha_b=\sup\{\bar\mu h(b'):b\subseteq b'\in Q\}$. Then $\alpha_b>\bar\mu(h(b))$. **P** We have

$$\bar\mu h(b)\le-\ln(1-2\bar\nu_\kappa(b\cap c))<-\ln(1-2\gamma_0)=\bar\mu1,$$

so $h(b)\ne1$. Because $\mathfrak A$ is atomless, there is an $a\in\mathfrak A$, disjoint from $h(b)$, such that $0<\bar\mu a<-\ln(2\bar\nu_\kappa b)$. Set $\boldsymbol{a}=\langle a_n\rangle_{n\in\mathbb N}$ where $a_0=a$, $a_n=0$ for $n\ge1$. Then

$$\bar\nu_\kappa\theta(\boldsymbol{a})=1-\exp(-\bar\mu a)<1-2\bar\nu_\kappa b,$$

so $b'=b\cup\pi\theta(\boldsymbol{a})\in Q$, while $h(b')\supseteq h(b)\cup a\supset h(b)$. **Q**

(ii) If $b\in Q$ and $\bar\mu h(b)<\alpha$, there is a $b_1\in Q$ such that $b\subseteq b_1$, $h(b_1)=h(b)$ and $\alpha_{b'}\le\alpha$. **P** Let $\delta>0$ be such that $\bar\mu(h(b')\setminus h(b))\le\alpha-\bar\mu h(b)$ whenever $\bar\nu_\kappa(b'\setminus b)\le\delta$. Because $\gamma_0\le\frac12$, there is a $b_1\in Q$ such that $b\subseteq b_1$, $b\cap c=b_1\cap c$ and $\bar\mu b_1\ge\gamma_0-\delta$. Then $h(b_1)=h(b)$. If $b'\in Q$ and $b'\supseteq b_1$, then $\bar\nu_\kappa(b'\cap c\setminus b)\le\delta$, so

$$\bar\mu(h(b')\setminus h(b))=\bar\mu(h(b'\cap c)\setminus h(b))\le\alpha-\bar\mu h(b)$$

and $\bar\mu h(b')\le\alpha$; thus $\alpha_{b_2}\le\alpha$. **Q**

(f) By (e-i), we can define $f:Q\to P$ by setting $f(b)=(h(b),\alpha_b)$ for $b\in Q$.

(i) f is order-preserving because h is.

(ii) If $P_1\subseteq P$ is up-open and cofinal with P, $f^{-1}[P_1]$ is cofinal with Q. **P** Take any $b\in Q$. Set

$$\alpha=\min(\alpha_b,\bar\mu h(b)-\ln(1-2\gamma_0+2\bar\nu_\kappa b))>\bar\mu h(b),$$

so that $f(b)\le(h(b),\alpha)$ in P. Then there is an $(a,\beta)\in P_1$ such that $(h(b),\alpha)\le(a,\beta)$, that is, $h(b)\subseteq a$ and $\beta\le\alpha$. In this case, there is a $b_1\in\mathfrak B_\kappa$ such that $b_1\supseteq b$, $h(b_1)\supseteq a$, $\bar\mu h(b_1)<\beta$ and

$$\bar\nu_\kappa(b_1\setminus b)\le\frac12(1-\exp(-\bar\mu(a\setminus h(b))))<\gamma_0-\bar\nu b$$

because $\bar\mu(a\setminus h(b))<-\ln(1-2\gamma_0+2\bar\nu_\kappa b)$. So $b_1\in Q$. By (e-ii), there is a $b_2\in Q$ such that $b_2\supseteq b_1$, $h(b_2)=h(b_1)$ and $\alpha_{b_2}\le\beta$. Now $b\subseteq b_2$, while $f(b_2)=(h(b_1),\alpha_{b_2})\ge(a,\beta)$. As P_1 is up-open, $f(b_2)\in P_1$; as b is arbitrary, $f^{-1}[P_1]$ is cofinal with Q. **Q**

(iii) $f[Q]$ is cofinal with P. **P** Take $(a,\alpha)\in P$. Set $\boldsymbol{a}=\langle a_n\rangle_{n\in\mathbb N}\in\mathfrak A^{\mathbb N}$ where $a_0=a$ and $a_n=0$ for $n\ge1$, and set $b=\pi\theta(\boldsymbol{a})$. Note that

$$\bar\nu_\kappa b=\frac12(1-\exp(-\bar\mu a))<\gamma_0,$$

so $b\in Q$. Because $\bar\mu_\infty\boldsymbol{a}<\infty$, $g(\pi^{-1}b)=\boldsymbol{a}$ and $h(b)=a$. By (e-ii) again, we can now find a $b_1\supseteq b$ in Q such that $h(b_1)=h(b)$ and $\alpha_{b_1}\le\alpha$. So $f(b_1)\ge(a,\alpha)$. As (a,α) is arbitrary, $f[Q]$ is cofinal. **Q**

(g) By 514O, $\mathrm{AM}^*(\mathfrak A,\bar\mu)=\mathrm{RO}^\uparrow(P)$ can be regularly embedded in

$$\mathrm{RO}^\uparrow(Q)=\mathrm{AM}(\mathfrak B_\kappa,\bar\nu_\kappa,\gamma_0)\cong\mathrm{AM}(\mathfrak B_\kappa,\nu_\kappa,\tfrac12)$$

by 528Da.

(h) All this has been done on the assumption that $\mathfrak A$ is atomless, as required in (e). For the general case, consider the localizable measure algebra free product $(\mathfrak C,\bar\lambda)$ of $(\mathfrak A,\bar\mu)$ and $(\mathfrak B_\omega,\bar\nu_\omega)$ (325E). By 521Qa, we have

$$\max(\omega,c(\mathfrak C),\tau(\mathfrak C))\le\max(\omega,c(\mathfrak A),c(\mathfrak B_\omega),\tau(\mathfrak A),\tau(\mathfrak B_\omega))\le\kappa.$$

Also $\mathfrak{C}$ is atomless because $\mathfrak{B}_\omega$ is isomorphic to a closed subalgebra of $\mathfrak{C}$ (325Dd) and is atomless (316R[13]). By (a)-(g), $\mathrm{AM}^*(\mathfrak{C},\bar\lambda)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac12)$. Now consider the canonical embedding $\varepsilon_1:\mathfrak{A}\to\mathfrak{C}$. This is order-continuous and measure-preserving (325Da), so identifies the Dedekind σ-complete Boolean algebra $\mathfrak{A}$ with a σ-subalgebra of $\mathfrak{C}$; also $\mathfrak{A}^f$ has supremum 1 both in $\mathfrak{A}$ and $\mathfrak{C}$. By 528G, $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{C},\bar\lambda)$ and therefore in $\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac12)$.

528I Definition For any set I, the **(I,∞)-localization poset** is the set

$$\mathcal{S}_I^\infty=\{p:p\subseteq\mathbb{N}\times I,\ \#(p[\{n\}])\le 2^n\text{ for every }n,\ \sup_{n\in\mathbb{N}}\#(p[\{n\}])\text{ is finite}\},$$

ordered by $\subseteq$. For $p\in\mathcal{S}_I^\infty$ set $\|p\|=\max_{n\in\mathbb{N}}\#(p[\{n\}])$. I will write $\mathcal{S}^\infty$ for $\mathcal{S}_{\mathbb{N}}^\infty$, already introduced in the proof of 522T.

528J Proposition Let κ be an infinite cardinal, $\mathcal{S}_\kappa^\infty$ the (κ,∞)-localization poset, and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra, not $\{0\}$, with $\kappa\ge\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))$. Then the variable-measure amoeba algebra $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

proof (a) To begin with (down to the end of (d) below), suppose that $\mathfrak{A}$ is atomless. Let P be the partially ordered set $\{(a,\alpha):a\in\mathfrak{A},\ \alpha\in\,]\bar\mu a,\mu 1]\}$, so that $\mathrm{AM}^*(\mathfrak{A},\bar\mu)=\mathrm{RO}^\uparrow(P)$. Give $\mathfrak{A}^f$ its measure metric, so that its topological density is at most κ (521Eb). Set $\gamma_0=\frac12\bar\mu 1$ and for $n\ge 1$ set $\gamma_n=2^{-2n-1}\bar\mu 1$ if $\bar\mu 1<\infty$, 4^{-n} otherwise. For each n, let D_n be a dense subset of $\{a:a\in\mathfrak{A}^f,\ \bar\mu a\le\gamma_n\}$, containing 0, with cardinal at most κ, and let $\langle d_{n\xi}\rangle_{\xi<\kappa}$ be a family running over D_n with cofinal repetitions.

(b) If $p\in\mathcal{S}_\kappa^\infty$ set

$$a_p=\sup_{(n,\xi)\in p}d_{n\xi},\quad \alpha_p=\bar\mu a_p+\sum_{n=0}^\infty(2^n-\#(p[\{n\}]))\gamma_n.$$

Then

$$\bar\mu a_p<\alpha_p\le\sum_{n=0}^\infty 2^n\gamma_n=\bar\mu 1,$$

so we can define $f:\mathcal{S}_\kappa^\infty\to P$ by setting $f(p)=(a_p,\alpha_p)$. f is order-preserving, because if $p\subseteq p'$ in $\mathcal{S}_\kappa^\infty$ then

$$\begin{aligned}\alpha_{p'}&=\bar\mu a_{p'}+\sum_{n=0}^\infty(2^n-\#(p'[\{n\}]))\gamma_n\\&\le\bar\mu a_p+\sum_{(n,\xi)\in p'\setminus p}\bar\mu d_{n\xi}+\sum_{n=0}^\infty(2^n-\#(p'[\{n\}]))\gamma_n\\&\le\bar\mu a_p+\sum_{n=0}^\infty\#(p'[\{n\}]\setminus p[\{n\}])\gamma_n+\sum_{n=0}^\infty(2^n-\#(p'[\{n\}]))\gamma_n=\alpha_p.\end{aligned}$$

(c) Suppose that $p\in\mathcal{S}_\kappa^\infty$ and $f(p)\le(a,\alpha)\in P$. Take $\alpha'\in\,]\bar\mu a,\alpha[$. For $n\in\mathbb{N}$, set $k_n=2^n-\#(p[\{n\}])$. Then $a_p\subseteq a$ and

$$\bar\mu a<\alpha\le\alpha_p=\bar\mu a_p+\sum_{n=0}^\infty k_n\gamma_n.$$

So there is an $r\in\mathbb{N}$ such that

$$\bar\mu a<\bar\mu a_p+\sum_{n=0}^\infty\gamma_n\min(r,k_n);$$

take r so large that, in addition, $\sum_{n=r+1}^\infty 2^n\gamma_n\le\alpha-\alpha'$.

For each n, set $k'_n=\min(r,k_n)$ and $C_n=\{\sup D:D\in[D_n]^{\le k'_n}\}$. Then (because $\mathfrak{A}$ is atomless) C_n is dense in $\{c:c\in\mathfrak{A}^f,\ \bar\mu c\le k'_n\gamma_n\}$. We can therefore choose $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively in such a way that

$$c_n\in C_n,\quad\bar\mu(a\cup\sup_{m<n}c_m)<\alpha',$$

$$\bar\mu(a\setminus(a_p\cup\sup_{m<n}c_m))<\sum_{m=n}^\infty k'_m\gamma_m$$

[13] Formerly 316Xi.

for every $n \in \mathbb{N}$. **P** For the inductive step to $n \geq 0$, set $b = a \setminus (a_p \cup \sup_{m<n} c_m)$. Take $b' \subseteq b$ such that $\bar{\mu}b' = \min(k'_n\gamma_n, \bar{\mu}b)$, so that

$$\bar{\mu}(a \cup \sup_{m<n} c_m \cup b') = \bar{\mu}(a \cup \sup_{m<n} c_m) < \alpha',$$

$$\bar{\mu}(b \setminus b') < \sum_{m=n+1}^{\infty} k'_m\gamma_m.$$

Let $c_n \in C_n$ be such that

$$\alpha' > \bar{\mu}(a \cup \sup_{m<n} c_m \cup b') + \bar{\mu}(c_n \setminus b') \geq \bar{\mu}(a \cup \sup_{m\leq n} c_m),$$

$$\sum_{m=n+1}^{\infty} k'_m\gamma_m > \bar{\mu}(b \setminus b') + \bar{\mu}(b' \setminus c_n) = \bar{\mu}(a \setminus (a_p \cup \sup_{m\leq n} c_m))$$

and the induction proceeds. **Q**

For each n, we can find a set $D'_n \subseteq D_n$, with cardinal k'_n, such that $c_n = \sup D'_n$. Because $\langle d_{n\xi}\rangle_{\xi<\kappa}$ runs over D_n with cofinal repetitions, we can find a set $I_n \subseteq \kappa \setminus p[\{n\}]$ such that $\#(I_n) = k'_n$ and $c_n = \sup_{\xi\in I_n} d_{n\xi}$. Set $q = p \cup \{(n,\xi) : n \in \mathbb{N},\, \xi \in I_n\}$. Then

$$\#(q[\{n\}]) \leq \#(p[\{n\}]) + k'_n \leq \min(2^n, \|p\| + r)$$

for every n, so $q \in \mathcal{S}_\kappa^\infty$ and $p \subseteq q$. Now

$$a_q = a_p \cup \sup_{n\in\mathbb{N},\xi\in I_n} d_{n\xi} = a_p \cup \sup_{n\in\mathbb{N}} c_n \supseteq a$$

because

$$\bar{\mu}(a \setminus a_q) \leq \inf_{n\in\mathbb{N}} \bar{\mu}(a \setminus (a_p \cup \sup_{m<n} c_m)) \leq \inf_{n\in\mathbb{N}} \sum_{m=n}^{\infty} 2^m\gamma_m = 0.$$

Also

$$\bar{\mu}a_q = \sup_{n\in\mathbb{N}} \bar{\mu}(a_p \cup \sup_{m<n} c_m) \leq \alpha' < \alpha,$$

while $\#(q[\{n\}]) = \#(p[\{n\}]) + k_n = 2^n$ whenever $n \leq r$, so

$$\alpha_q = \bar{\mu}a_q + \sum_{n=r+1}^{\infty}(2^n - \#(q[\{n\}]))\gamma_n \leq \alpha' + \sum_{n=r+1}^{\infty} 2^n\gamma_n \leq \alpha.$$

(d) What (c) shows is that if $p \in \mathcal{S}_\kappa^\infty$ and $f(p) \leq (a,\alpha)$ in P, then there is a $q \supseteq p$ in $\mathcal{S}_\kappa^\infty$ such that $(a,\alpha) \leq f(q)$. Next, $\mathcal{S}_\kappa^\infty$ has a least element $\emptyset$, and $f(\emptyset) = (0, \bar{\mu}1)$ is the least element of P. So 514P tells us that $\mathrm{RO}^\uparrow(P) = \mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

(e) As for the general case, we can use the same trick as in part (h) of the proof of 528H. Let $(\mathfrak{C}, \bar{\lambda})$ be the localizable measure algebra free product of $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}_\omega, \bar{\nu}_\omega)$; as before, $\mathfrak{C}$ is atomless, $\max(\omega, c(\mathfrak{C}), \tau(\mathfrak{C})) \leq \kappa$ and $(\mathfrak{A}, \bar{\mu})$ is embedded in $(\mathfrak{C}, \bar{\lambda})$ as a σ-subalgebra with sufficient elements of finite measure. So $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ is regularly embedded in $\mathrm{AM}^*(\mathfrak{C}, \bar{\lambda})$ and in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

528K Theorem (Truss 88) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless σ-finite measure algebra in which every non-zero principal ideal has Maharam type κ, and $0 < \gamma < \bar{\mu}1$. Then each of the algebras

$$\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma), \quad \mathrm{AM}^*(\mathfrak{A}, \bar{\mu}), \quad \mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \tfrac{1}{2})$$

can be regularly embedded in the other two, and all three can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

proof By 528H, $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \frac{1}{2})$. Take any $e \in \mathfrak{A}$ such that $\gamma < \bar{\mu}e < \infty$. Then the principal ideal $(\mathfrak{A}_e, \bar{\mu}\restriction\mathfrak{A}_e)$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$, so

$$\mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \tfrac{1}{2}) \cong \mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \frac{\gamma}{\bar{\mu}e})$$

(528Da)

$$\cong \mathrm{AM}(\mathfrak{A}_e, \bar{\mu}\restriction\mathfrak{A}_e, \gamma)$$

can be regularly embedded in $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ (528Fa). By 528Fc, $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$. Finally, by 528J, $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$. Because regular embeddability is transitive (313N), these facts are enough to prove the theorem.

528L It is possible without great effort to calculate many of the cardinal functions of these algebras.

Lemma $\mathfrak{m}(\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac{1}{2}))\le\operatorname{add}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

proof Set $P=\{a:a\in\mathfrak{B}_\omega,\,\bar\nu_\omega a<\frac{1}{2}\}$. Then $\operatorname{wdistr}(\mathfrak{B}_\omega)\ge\mathfrak{m}^\uparrow(P)$. **P** Take a family $\langle B_\xi\rangle_{\xi<\kappa}$ of maximal antichains in $\mathfrak{B}_\omega$, where $\kappa<\mathfrak{m}^\uparrow(P)$. Let $C\subseteq\mathfrak{B}_\omega$ be a maximal disjoint set such that $\{b:b\in B_\xi,\,b\cap c\ne 0\}$ is finite for every $\xi<\kappa$ and $c\in C$. **?** Suppose, if possible, that $c_0=1\setminus\sup C$ is not 0. Take $a_0\in P$ such that $\bar\nu_\omega(a_0\cup c_0)>\frac{1}{2}$. (If $\bar\nu_\omega c_0>\frac{1}{2}$, take $a_0=0$; otherwise, take $a_0\subseteq 1\setminus c_0$ such that $\frac{1}{2}-\bar\nu_\omega c_0<\bar\nu_\omega a_0<\frac{1}{2}$.) For each $\xi<\kappa$, set

$$Q_\xi=\{a:a\in P,\,\{b:b\in B_\xi,\,b\not\subseteq a\}\text{ is finite}\};$$

then Q_ξ is cofinal with P. There is therefore an upwards-directed $R\subseteq P$ such that $a_0\in R$ and R meets every Q_ξ. Set $e=\sup R$; then $\bar\nu_\omega e\le\frac{1}{2}$ so $c_1=c_0\setminus e=(a_0\cup c_0)\setminus e$ is non-zero.

If $\xi<\kappa$, there is an $a\in R\cap Q_\xi$, so that

$$\{b:b\in B_\xi,\,b\cap c_1\ne 0\}\subseteq\{b:b\in B_\xi,\,b\not\subseteq a\}$$

is finite. But this means that we ought to have added c_1 to C. **X**

Thus C is a maximal antichain. As $\langle B_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{wdistr}(\mathfrak{A})\ge\mathfrak{m}^\uparrow(P)$. **Q**

Now 524Mb tells us that $\operatorname{wdistr}(\mathfrak{B}_\omega)=\operatorname{add}\mathcal{N}$, so $\mathfrak{m}^\uparrow(P)\le\operatorname{add}\mathcal{N}$. Finally, by 517Db,

$$\mathfrak{m}(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\mathfrak{m}^\uparrow(P)\le\operatorname{add}\mathcal{N},$$

as claimed.

528M Lemma $\mathfrak{m}^\uparrow(\mathcal{S}^\infty)\ge\operatorname{add}\mathcal{N}$.

proof (a) Recall the definition of the supported relations $(\mathbb{N}^\mathbb{N},\subseteq^*,\mathcal{S}^{(\alpha)})$ from 522L, where $\mathcal{S}^{(\alpha)}=\{S:S\subseteq\mathbb{N}\times\mathbb{N},\,\#(S[\{n\}])\le\alpha(n)$ for every $n\in\mathbb{N}\}$ for $\alpha\in\mathbb{N}^\mathbb{N}$. Putting 522L, 522M and 512Db together, we have $\operatorname{add}(\mathbb{N}^\mathbb{N},\subseteq^*,\mathcal{S}^{(\alpha)})=\operatorname{add}\mathcal{N}$ whenever $\lim_{n\to\infty}\alpha(n)=\infty$.

(b) The core of the argument is the following fact. Suppose that $Q\subseteq\mathcal{S}^\infty$ is cofinal and up-open, $n\in\mathbb{N}$ and $\sigma\in[\mathbb{N}\times\mathbb{N}]^{<\omega}$. Let $G\subseteq\mathbb{N}\times\mathbb{N}$ be a set with finite vertical sections. Then there is a $k\in\mathbb{N}$ such that whenever $\sigma\subseteq p\in\mathcal{S}^\infty$, $p\subseteq\sigma\cup G$ and $\|p\|\le n$, there is a $q\in Q$ such that $p\subseteq q$ and $\|q\|\le k$.

P? Suppose, if possible, otherwise. Then for each $j\in\mathbb{N}$ we can find $p_j\in\mathcal{S}^\infty$ such that $\sigma\subseteq p_j\subseteq\sigma\cup G$, $\|p_j\|\le n$ and $\|q\|>j$ whenever $p\subseteq q\in Q$. Let p be a cluster point of $\langle p_j\rangle_{j\in\mathbb{N}}$ in $\mathcal{P}(\sigma\cup G)$. Then $\#(p[\{i\}])\le\sup_{j\in\mathbb{N}}\#(p_j[\{i\}])\le\min(2^i,n)$ for every i, so $p\in\mathcal{S}^\infty$. Because Q is cofinal with $\mathcal{S}^\infty$, there is a $q\in Q$ such that $p\subseteq q$. Set $k=n+\|q\|$. Then $(\sigma\cup G)\cap(k\times\mathbb{N})$ is finite, so there is an $i\ge k$ such that $p_i\cap(k\times\mathbb{N})=p\cap(k\times\mathbb{N})\subseteq q$. Set $q'=p_i\cup q$. Then

$$\begin{aligned}\#(q'[\{j\}])&=\#(q[\{j\}])\le\min(\|q\|,2^j)\text{ if }j<k,\\&\le\|p_i\|+\|q\|\le k\le 2^j\text{ otherwise.}\end{aligned}$$

So $q'\in\mathcal{S}^\infty$ and $\|q'\|\le k\le i$; because Q is up-open in $\mathcal{S}^\infty$, $q'\in Q$, while $p_i\subseteq q'$. But we chose p_i so that this could not happen. **XQ**

(c) We need to know that $\mathcal{S}^\infty$ is upwards-ccc. **P** For any $n\in\mathbb{N}$, finite $\sigma\subseteq\mathbb{N}\times\mathbb{N}$ the set $\{p:p\in\mathcal{S}^\infty,\,\|p\|\le 2^{n-1},\,p\cap(n\times\mathbb{N})=\sigma\}$ is upwards-linked. **Q**

(d) Now let $\langle Q_\xi\rangle_{\xi<\kappa}$ be any family of cofinal subsets of $\mathcal{S}^\infty$, where $\kappa<\operatorname{add}\mathcal{N}$, and $p_0\in\mathcal{S}^\infty$. For each $\xi<\kappa$ let $A_\xi\subseteq Q_\xi$ be a maximal up-antichain; by (c), A_ξ is countable. Set $Q'_\xi=\bigcup\{[q,\infty[\,:q\in A_\xi\}$, so that Q'_ξ is an up-open cofinal subset of $\mathcal{S}^\infty$. Set $A=\{p_0\}\cup\bigcup_{\xi<\kappa}A_\xi$. For $q\in A$, let $F_q\subseteq\mathbb{N}^\mathbb{N}$ be a finite set such that (identifying each member of F_q with its graph) $q\subseteq\bigcup F_q$; set $F=\bigcup_{q\in A}F_q$, so that

$$\#(F)\le\max(\omega,\kappa)<\operatorname{add}\mathcal{N}\le\mathfrak{b}$$

(522B). Let $g_0\in\mathbb{N}^\mathbb{N}$ be a strictly increasing function such that $\{i:f(i)>g_0(i)\}$ is finite for every $f\in F$, and also $p_0[\{i\}]\subseteq g_0(i)$ for every i. Set $G=\{(i,j):i\in\mathbb{N},\,j<g_0(i)\}$, so that $G\subseteq\mathbb{N}\times\mathbb{N}$ has finite vertical sections. Observe that if $q\in A$ then $q\setminus G$ is finite.

For each $\xi<\kappa$, $n\in\mathbb{N}$ and finite $\sigma\subseteq\mathbb{N}\times\mathbb{N}$, let $k(\xi,\sigma,n)\in\mathbb{N}$ be such that whenever $p\in\mathcal{S}^\infty$ and $\sigma\subseteq p\subseteq\sigma\cup G$ then there is a $q\in Q'_\xi$ such that $p\subseteq q$ and $\|q\|\le k(\xi,\sigma,n)$; such a k exists by (b) above. Set $k_\xi(n) = \sup\{k(\xi,\sigma,n) : \sigma \subseteq n \times g_0(n)\}$. Again because $\kappa < \mathfrak{b}$, there is a $g_1 \in \mathbb{N}^{\mathbb{N}}$ such that $\{n : k_\xi(n) > g_1(n)\}$ is finite for every $\xi < \kappa$. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be a non-decreasing function such that $\lim_{n\to\infty} \alpha(n) = \infty$ and

$$\alpha(2g_1(n)) \le n, \quad \alpha(n) + \#(p_0[\{n\}]) \le 2^n, \quad 2\alpha(n) \le n$$

for every n.

Because $\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}^{(\alpha)}) = \operatorname{add}\mathcal{N}$, there is an $S_0 \in \mathcal{S}^{(\alpha)}$ such that $f \subseteq^* S_0$ for every $f \in F$, so that $q \setminus S_0$ is finite for every $q \in A$. Replacing S_0 by $S_0 \cap G$ if necessary, we may suppose that $S_0 \subseteq G$.

(e) Let $\mathcal{S}$ be the family of subsets S of $\mathbb{N}\times\mathbb{N}$ such that $\#(S[\{n\}]) \le 2^n$ for every n, as in 522K. Note that $p_0 \cup S_0 \in \mathcal{S}$, because $\alpha(n) + \#(p_0(n)) \le 2^n$ for every n. Let C be the family of finite subsets σ of $\mathbb{N}\times\mathbb{N}$ such that $\sigma \cup S_0 \in \mathcal{S}$. For each $\xi < \kappa$, set

$$D_\xi = \{\sigma : \sigma \in C, \exists\, q \in A_\xi,\, q \subseteq \sigma \cup S_0\}.$$

Then D_ξ is cofinal with C. **P** Let $\sigma \in C$. Let n_0 be so large that $g_1(2^{n_0}) \ge k_\xi(2^{n_0})$ and $\sigma \subseteq n_0 \times g_0(n_0)$. Set $m = 2g_1(2^{n_0})$, $p = \sigma \cup (S_0 \cap (m \times \mathbb{N})) \in \mathcal{S}^\infty$. Then $\sigma \subseteq p \subseteq \sigma \cup G$ and $\|p\| \le \max(2^{n_0}, \alpha(m)) = 2^{n_0}$, so there is a $q \in Q'_\xi$ such that $p \subseteq q$ and

$$\|q\| \le k(\xi,\sigma,2^{n_0}) \le k_\xi(2^{n_0}) \le g_1(2^{n_0}) = \frac{m}{2}.$$

Let $q' \in A_\xi$ be such that $q' \subseteq q$. Let $m' \ge \max(m, n_0)$ be such that $q' \subseteq (m'\times\mathbb{N}) \cup S_0$, and set $\tau = q \cap (m'\times\mathbb{N})$, so that $\sigma \subseteq \tau$.

For $n < m$, we have

$$S_0[\{n\}] \subseteq p[\{n\}] \subseteq q[\{n\}] = \tau[\{n\}],$$

so $(\tau \cup S_0)[\{n\}] = q[\{n\}]$ has at most 2^n members. For $m \le n < m'$, we have

$$\#((\tau \cup S_0)[\{n\}]) \le \#(q[\{n\}]) + \#(S_0[\{n\}]) \le \|q\| + \alpha(n) \le \frac{m}{2} + \frac{n}{2} \le 2^n,$$

while for $n \ge m'$ we have

$$\#((\tau \cup S_0)[\{n\}]) = \#(S_0[\{n\}]) \le \alpha(n) \le 2^n.$$

So $\tau \cup S_0 \in \mathcal{S}$ and $\tau \in C$. Since

$$q' \subseteq (q' \cap (m' \times \mathbb{N})) \cup S_0 \subseteq (q \cap (m' \times \mathbb{N})) \cup S_0 = \tau \cup S_0,$$

$\tau \in D_\xi$. As σ is arbitrary, D_ξ is cofinal with C. **Q**

(f) Because $p_0 \cup S_0 \in \mathcal{S}$, $\sigma_0 = p_0 \setminus S_0$ belongs to C. Because $\kappa < \operatorname{add}\mathcal{N} \le \mathfrak{m}_{\text{countable}} \le \mathfrak{m}^\uparrow(C)$, there is an upwards-directed set $E \subseteq C$ meeting every D_ξ and containing σ_0. Set $S_1 = S_0 \cup \bigcup E$. Then, because E is upwards-directed,

$$\#(S_1[\{n\}]) = \sup_{\sigma\in E} \#((\sigma \cup S_0)[\{n\}]) \le 2^n$$

for every n, and $S_1 \in \mathcal{S}$. Set $R = \{p : p \in \mathcal{S}^\infty,\, p \subseteq S_1\}$; then $R \subseteq \mathcal{S}^\infty$ is upwards-directed (in fact, closed under $\cup$), and $p_0 \in R$ because $\sigma_0 \in E$. Now R meets Q_ξ for every $\xi < \kappa$. **P** There is a $\sigma \in D_\xi \cap E$. But this means that there is a $q \in A_\xi$ such that $q \subseteq \sigma \cup S_0 \subseteq S_1$ and $q \in R \cap Q_\xi$. **Q**

As p_0 and $\langle Q_\xi\rangle_{\xi<\kappa}$ are arbitrary, $\mathfrak{m}^\uparrow(\mathcal{S}^\infty) \ge \operatorname{add}\mathcal{N}$.

528N Theorem (Brendle 00, 2.3.10; Judah & Repický 95) Let $(\mathfrak{A}, \bar\mu)$ be an atomless σ-finite measure algebra with countable Maharam type, and $0 < \gamma < \bar\mu 1$. Then the algebras $\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)$ and $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ and the $(\mathbb{N}, \infty)$-localization poset $\mathcal{S}^\infty$ (active upwards) all have Martin numbers equal to $\operatorname{add}\mathcal{N}$.

proof By 517Ia and 528K, with 517Db again,

$$\begin{aligned}\mathfrak{m}(\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)) &= \mathfrak{m}(\mathrm{AM}^*(\mathfrak{A}, \bar\mu)) = \mathfrak{m}(\mathrm{AM}(\mathfrak{B}_\omega, \bar\nu_\omega, \tfrac{1}{2}))\\ &\ge \mathfrak{m}(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathfrak{m}^\uparrow(\mathcal{S}^\infty).\end{aligned}$$

As

$$\mathfrak{m}(\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\tfrac{1}{2}))\le\operatorname{add}\mathcal{N}\le\mathfrak{m}^{\uparrow}(\mathcal{S}^\infty)$$

(528L, 528M), all these are equal to $\operatorname{add}\mathcal{N}$.

528O Corollary Let $\gamma>0$. Let $\mathcal{G}$ be the partially ordered set

$$\{G:G\subseteq\mathbb{R}\text{ is open, }\mu_LG<\gamma\},$$

where μ_L is Lebesgue measure. Then $\mathfrak{m}^{\uparrow}(\mathcal{G})=\operatorname{add}\mathcal{N}$.

proof Put 528C and 528N together.

528P Proposition Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra, and $0<\gamma<\bar\mu 1$.

(a) For any integer $m\ge 2$,

$$c(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\operatorname{link}_m(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\max(c(\mathfrak{A}),\tau(\mathfrak{A})).$$

(b) $d(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\pi(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\max(\operatorname{cf}[c(\mathfrak{A})]^{\le\omega},\pi(\mathfrak{A}))$.

proof Set $P=\{a:a\in\mathfrak{A},\ \bar\mu a<\gamma\}$, so that $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)=\mathrm{RO}^{\uparrow}(P)$.

(a) Set $\kappa_0=\max(c(\mathfrak{A}),\tau(\mathfrak{A}))$, $\kappa_1=\operatorname{link}_m(\mathrm{RO}^{\uparrow}(P))=\operatorname{link}^{\uparrow}_m(P)$ and $\kappa_2=c(\mathrm{RO}^{\uparrow}(P))=c^{\uparrow}(P)$ (514N).

(i) The topological density of $\mathfrak{A}^f$ for its measure metric is κ_0 (521Eb), so P has a metrically dense subset D with cardinal at most κ_0. For $d\in D$, set

$$U_d=\{a:a\in P,\ \bar\mu(a\setminus d)<\frac{1}{m}(\gamma-\bar\mu d)\}.$$

Then U_d is upwards-m-linked in P. Also, if $a\in P$, there is a $d\in D$ such that $\bar\mu(a\mathbin{\triangle}d)<\frac{1}{m+1}(\gamma-\bar\mu a)$, and now $a\in U_d$. So P is κ_0-m-linked upwards and $\kappa_1\le\kappa_0$.

(ii) By 511Hb or 511Ia, $\kappa_2\le\kappa_1$.

(iii) We need to check that κ_2 is infinite. **P** Take $a\in\mathfrak{A}$ such that $\bar\mu a=\gamma$. For any $n\ge 1$, we can find disjoint $a_0,\dots,a_n\subseteq a$ all of measure $\frac{1}{n+1}\gamma$; now $\langle a\setminus a_i\rangle_{i\le n}$ is an up-antichain in P. So $\kappa_2=c^{\uparrow}(P)\ge n+1$; and this is true for every n. **Q**

Now if $(\mathfrak{A},\bar\mu)$ is totally finite, then $c(\mathfrak{A})=\omega\le\kappa_2$. Otherwise, there is a partition D of unity in $\mathfrak{A}$ such that $\bar\mu d=\frac{1}{2}\gamma$ for every $d\in D$; now D is an up-antichain in P and $\kappa_2\ge\#(D)=c(\mathfrak{A})$. So we see that in all cases $\kappa_2\ge c(\mathfrak{A})$.

(iv) If $e\in\mathfrak{A}^f$ and the principal ideal $\mathfrak{A}_e$ is homogeneous, then $\tau(\mathfrak{A}_e)\le\kappa_2$. **P?** Otherwise, set $\alpha=\bar\mu e$, $\kappa=\tau(\mathfrak{A}_e)$. Because $\bar\mu 1>\gamma$, there is a $d\subseteq 1\setminus e$ such that $\gamma<\bar\mu(e\cup d)<\gamma+\bar\mu e$, that is, $0<\gamma-\bar\mu d<\alpha$. Set $\beta=\sqrt{1-\frac{\gamma-\bar\mu d}{\alpha}}$. Because $\mathfrak{A}_e$ is isomorphic, up to a scalar multiple of the measure, to the measure algebra of the usual measure on $[0,1]^\kappa$, there is a family $\langle c_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}_e$ such that

$$\bar\mu c_\xi=\beta\alpha,\quad \bar\mu(c_\xi\cap c_\eta)=\beta^2\alpha$$

whenever ξ, $\eta<\kappa$ are distinct. Set $b_\xi=d\cup(e\setminus c_\xi)$ for $\xi<\kappa$. Then

$$\bar\mu(b_\xi\cup b_\eta)=\bar\mu d+\alpha-\beta^2\alpha=\gamma,$$

$$\bar\mu b_\xi=\bar\mu d+\alpha-\beta\alpha<\gamma$$

for all distinct ξ, $\eta<\kappa$. So $\langle b_\xi\rangle_{\xi<\kappa}$ is an up-antichain in P and witnesses that $\kappa_2\ge\kappa$. **XQ**

(v) Let E be a partition of unity in $\mathfrak{A}$ such that $0<\bar\mu e<\infty$ and $\mathfrak{A}_e$ is homogeneous for every $e\in E$. For $e\in E$, let $A_e\subseteq\mathfrak{A}_e$ be a set with cardinal at most κ_2 which τ-generates $\mathfrak{A}_e$. Then $A=\bigcup_{e\in E}A_e$ τ-generates $\mathfrak{A}$, so that

$$\kappa_0=\max(c(\mathfrak{A}),\tau(\mathfrak{A}))\le\max(c(\mathfrak{A}),\#(A))\le\max(c(\mathfrak{A}),\kappa_2)=\kappa_2,$$

and the three cardinals must be equal.

(b) Set $\kappa_3=\max(\pi(\mathfrak{A}),\operatorname{cf}[c(\mathfrak{A})]^{\le\omega})$, $\kappa_4=\pi(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))$ and $\kappa_5=d(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))$.

(i) By 323Mc, $\mathfrak{A}^f$ is complete in its measure metric. By 323Ma, $\cup : \mathfrak{A}^f \times \mathfrak{A}^f \to \mathfrak{A}^f$ is uniformly continuous for the measure metric, and P is an open subset of $\mathfrak{A}^f$, while the topological density of $\mathfrak{A}^f$ is κ_0. By 524C, $(P, \subseteq', [P]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa_0), \le, \ell^1(\kappa_0))$, where $\subseteq'$ is defined as in 512F. It follows that

$$\kappa_4 = \pi(\mathrm{RO}^{\uparrow}(P)) = \operatorname{cf} P$$

(514Nb)

$$= \operatorname{cov}(P, \subseteq, P) \le \max(\omega, \operatorname{cov}(P, \subseteq', [P]^{\le\omega}))$$

(512Gf)

$$\le \max(\omega, \operatorname{cov}(P, \subseteq', [P]^{<\omega}))$$

(512Gb)

$$\le \max(\omega, \operatorname{cf} \ell^1(\kappa_0))$$

(512Da)

$$= \operatorname{cf} \ell^1(\kappa_0) = \operatorname{cf} \mathcal{N}_{\kappa_0}$$

(where $\mathcal{N}_{\kappa_0}$ is the null ideal of the usual measure on $\{0,1\}^{\kappa_0}$, as in 524I)

$$= \max(\operatorname{cf} \mathcal{N}, \operatorname{cf}[\kappa_0]^{\le\omega})$$

(523N)

$$= \max(\operatorname{cf} \mathcal{N}, \operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}, \operatorname{cf}[c(\mathfrak{A})]^{\le\omega})$$

$$= \max(\operatorname{cf} \mathcal{N}, \operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}, c(\mathfrak{A}), \operatorname{cf}[c(\mathfrak{A})]^{\le\omega}) = \max(\pi(\mathfrak{A}), \operatorname{cf}[c(\mathfrak{A})]^{\le\omega})$$

(524Mc)

$$= \kappa_3.$$

(ii) By 514Nd, $d^{\uparrow}(P) = \kappa_5$. Let $\langle B_\xi \rangle_{\xi<\kappa_5}$ be a family of upwards-centered sets covering P. For each ξ, $b_\xi = \sup B_\xi$ is defined in $\mathfrak{A}$ (counting $\sup \emptyset$ as 0 if necessary), and

$$\bar\mu b_\xi = \sup_{I \in [B_\xi]^{<\omega}} \bar\mu(\sup I) \le \gamma.$$

Set $D = \{b_\xi \setminus b_\eta : \xi, \eta < \kappa_5\}$. Then D is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, take $a' \subseteq a$ such that $0 < \bar\mu a' < \gamma$. Then $a' \in P$, so there is some $\xi < \kappa_5$ such that $a' \in B_\xi$ and $a' \subseteq b_\xi$. Next, let $c \subseteq 1 \setminus b_\xi$ be such that

$$\gamma - \bar\mu b_\xi < \bar\mu c < \gamma - \bar\mu b_\xi + \bar\mu a'.$$

Then $c \cup (b_\xi \setminus a') \in P$, so there is an $\eta < \kappa_5$ such that $c \cup (b_\xi \setminus a') \subseteq b_\eta$. Now $d = b_\xi \setminus b_\eta \subseteq a'$; as $\bar\mu(b_\xi \cup c) > \gamma \ge \bar\mu b_\eta$, $b_\xi \not\subseteq b_\eta$ and $d \ne 0$. Of course $d \in D$ and $d \subseteq a$; as a is arbitrary, D is order-dense. **Q**

Accordingly $\pi(\mathfrak{A}) \le \#(D) \le \kappa_5$. At the same time, $\operatorname{cf}[c(\mathfrak{A})]^{\le\omega} \le \kappa_5$. **P** There is a disjoint set $E \subseteq \mathfrak{A} \setminus \{0\}$ with cardinal $c(\mathfrak{A})$ (332F). For each $\xi < \kappa_5$, let I_ξ be the countable set $\{e : e \in E,\ e \cap b_\xi \ne 0\}$. If $J \subseteq E$ is countable, let $\langle \epsilon_e \rangle_{e\in J}$ be a strictly positive family of real numbers with sum less than γ. For each $e \in J$ let $a_e \subseteq e$ be such that $0 < \bar\mu a_e \le \epsilon_e$, and set $a = \sup_{e\in J} a_e$. Then $a \in P$ so there is a $\xi < \kappa_5$ such that $a \subseteq b_\xi$ and $J \subseteq I_\xi$. As J is arbitrary, $\{I_\xi : \xi < \kappa_5\}$ is cofinal with $[E]^{\le\omega}$, and

$$\operatorname{cf}[c(\mathfrak{A})]^{\le\omega} = \operatorname{cf}[E]^{\le\omega} \le \kappa_5. \textbf{ Q}$$

Putting these together, we see that $\kappa_3 \le \kappa_5$.

(iii) By 514Da, $\kappa_5 \le \kappa_4$, so the three cardinals are equal.

528Q Proposition Let $\mathcal{S}^\infty$ be the $(\mathbb{N}, \infty)$-localization poset.
(a) $\pi(\mathrm{RO}^{\uparrow}(\mathcal{S}^\infty)) = \operatorname{cf} \mathcal{S}^\infty = \mathfrak{c}$.
(b) For every $m \ge 2$,

$$c(\mathrm{RO}^{\uparrow}(\mathcal{S}^\infty)) = c^{\uparrow}(\mathcal{S}^\infty) = \operatorname{link}_m(\mathrm{RO}^{\uparrow}(\mathcal{S}^\infty)) = \operatorname{link}^{\uparrow}_m(\mathcal{S}^\infty) = \omega.$$

(c) $d(\mathrm{RO}^{\uparrow}(\mathcal{S}^\infty)) = d^{\uparrow}(\mathcal{S}^\infty) = \operatorname{cf} \mathcal{N}$.

proof (a) If p, $q \in \mathcal{S}^\infty$ and $p \not\subseteq q$, take $(n,i) \in p\setminus q$; then there is a $q' \in \mathcal{S}^\infty$ such that $q' \supseteq q$, $\#(q'[\{n\}]) = 2^n$ and $(n,i) \notin q'$, in which case p and q' are incompatible upwards in $\mathcal{S}^\infty$. So $\mathcal{S}^\infty$ is separative upwards and 514Nb tells us that $\pi(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{cf}\,\mathcal{S}^\infty$.

Next, there is an almost-disjoint family $\langle h_\xi\rangle_{\xi<\mathfrak{c}}$ in $\mathbb{N}^\mathbb{N}$ (5A1Nc). Identifying each h_ξ with its graph, we can regard them as members of $\mathcal{S}^\infty$; and any member of $\mathcal{S}^\infty$ includes only finitely many of them. So $\mathrm{cf}\,\mathcal{S}^\infty \ge \mathfrak{c}$. On the other hand, of course, $\mathrm{cf}\,\mathcal{S}^\infty \le \#(\mathcal{S}^\infty) = \mathfrak{c}$. So $\pi(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{cf}\,\mathcal{S}^\infty = \mathfrak{c}$.

(b) If $m \ge 2$, let Q be the countable set of pairs (I,r) where $r \in \mathbb{N}$ and $I \in [\mathbb{N}\times\mathbb{N}]^{<\omega}$, and for $(I,r) \in Q$ set

$$A_{Ir} = \{p : p \in \mathcal{S}^\infty,\ p \cap (r\times\mathbb{N}) = I,\ \|p\| \le \frac{2^r}{m}\}.$$

Then $\bigcup_{i<m} p_i \in \mathcal{S}^\infty$ for any family $\langle p_i\rangle_{i<m}$ in A_{Ir}, that is, A_{Ir} is upwards-m-linked in $\mathcal{S}^\infty$. Also $\bigcup_{(I,r)\in Q} A_{Ir} = \mathcal{S}^\infty$, so $\mathrm{link}^\uparrow_m(\mathcal{S}^\infty) \le \omega$. Of course $c^\uparrow(\mathcal{S}^\infty)$ is infinite, and since $c^\uparrow(\mathcal{S}^\infty) \le \mathrm{link}^\uparrow_m(\mathcal{S}^\infty)$ (511Hb again), both must be ω. Now 514N tells us that

$$c(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{link}_m(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \omega.$$

(c) Consider the $\mathbb{N}$-localization relation $(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S})$ of 522K. We know from 522M and 512Da that

$$\mathrm{cov}(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}) = \mathrm{cov}(\mathcal{N}, \subseteq, \mathcal{N}) = \mathrm{cf}\mathcal{N}.$$

(i) Let $\mathcal{A} \subseteq \mathcal{S}$ be a set with cardinal $\mathrm{cf}\mathcal{N}$ such that for every $f \in \mathbb{N}^\mathbb{N}$ there is an $S \in \mathcal{A}$ such that $f \subseteq^* S$. Let $\mathcal{A}^*$ be

$$\{S : S \in \mathcal{S},\ S \setminus \bigcup\mathcal{A}' \text{ is finite for some finite } \mathcal{A}' \subseteq \mathcal{A}\};$$

then every member of $\mathcal{S}^\infty$ is included in some member of $\mathcal{A}^*$. But if $S \in \mathcal{A}^*$ then $\{p : p \in \mathcal{S}^\infty,\ p \subseteq S\}$ is upwards-directed. So

$$d^\uparrow(\mathcal{S}^\infty) \le \#(\mathcal{A}^*) \le \mathrm{cf}\mathcal{N}.$$

(ii) Now let $\mathcal{Q}$ be a family of upwards-centered subsets of $\mathcal{S}^\infty$ covering $\mathcal{S}^\infty$. For each $Q \in \mathcal{Q}$, $S_Q = \bigcup Q$ belongs to $\mathcal{S}$. Also every $f \in \mathbb{N}^\mathbb{N}$ belongs to $\mathcal{S}^\infty$ so is covered by some S_Q. So S_Q witnesses that $\mathrm{cf}\mathcal{N} = \mathrm{cov}(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}) \le \#(\mathcal{Q})$; as $\mathcal{Q}$ is arbitrary, $\mathrm{cf}\mathcal{N} \le d^\uparrow(\mathcal{S}^\infty)$.

(iii) 514Nd tells us that

$$d(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = d^\uparrow(\mathcal{S}^\infty),$$

so we have equality throughout.

528R Theorem Let κ be any cardinal, and $\mathcal{S}^\infty_\kappa$ the (κ,∞)-localization poset. Then $\mathrm{RO}^\uparrow(\mathcal{S}^\infty_\kappa)$ has countable Maharam type.

proof (a) If κ is finite then $\mathrm{cf}\,\mathcal{S}^\infty_\kappa$ is finite and the result is trivial. So let us suppose from now on that κ is infinite.

(b) $\mathcal{S}^\infty_\kappa$ is separative upwards. **P** If p, $q \in \mathcal{S}^\infty_\kappa$ and $p \not\subseteq q$, take $(n,\xi) \in p \setminus q$. Let $J \subseteq \kappa \setminus p[\{n\}]$ be a set of size $2^n - \#(q[\{n\}]$, and set $q' = q \cup (\{n\}\times J)$; then $q \subseteq q' \in \mathcal{S}^\infty$ and p, q' are incompatible upwards in $\mathcal{S}^\infty_\kappa$. **Q**

Accordingly $[p,\infty[\in \mathrm{RO}^\uparrow(\mathcal{S}^\infty_\kappa)$ for every $p \in \mathcal{S}^\infty_\kappa$ (514Me).

(c) For $n \in \mathbb{N}$, $m < 2^n$ and $\xi < \kappa$, set

$$G_{mn\xi} = \sup\{[p,\infty[: p \in \mathcal{S}^\infty_\kappa,\ \#(p[\{n\}]) = 2^n,\ (n,\xi) \in p \text{ and } \#(p[\{n\}]\cap\xi) = m\},$$

the supremum being taken in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty_\kappa)$.

(d) If $n \in \mathbb{N}$, $m < 2^n$ and $\xi < \eta < \kappa$ then $G_{mn\xi} \cap G_{mn\eta} = \emptyset$. **P** If p, $q \in \mathcal{S}^\infty_\kappa$, $\#(p[\{n\}]) = \#(q[\{n\}]) = 2^n$, $(n,\xi) \in p$, $(n,\eta) \in q$ and $\#(p[\{n\}]\cap\xi) = \#(q[\{n\}])\cap\eta) = m$ then $p[\{n\}] \ne q[\{n\}]$, $\#(p[\{n\}] \cup q[\{n\}]) > 2^n$ and $[p,\infty[\cap [q,\infty[$ is empty. **Q**

(e) If $\xi < \kappa$ then $\bigcup\{G_{mn\xi} : n \in \mathbb{N},\ m < 2^n\}$ is dense in $\mathcal{S}_\kappa^\infty$. **P** If $p \in \mathcal{S}_\kappa^\infty$, take $n \in \mathbb{N}$ such that $\#(p[\{n\}]) < 2^n$; then there is a $q \in S_\kappa^\infty$ such that $p \subseteq q$, $\xi \in q[\{n\}]$ and $\#(q[\{n\}]) = 2^n$. Set $m = \#(q[\{n\}] \cap \xi)$; then $[q, \infty[\ \subseteq [p, \infty[\ \cap G_{mn\xi}$. **Q**

Thus $\sup\{G_{mn\xi} : n \in \mathbb{N},\ m < 2^n\} = 1$ in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty)$ whenever $\xi < \kappa$.

(f) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$ generated by $\{G_{mn\xi} : n \in \mathbb{N},\ m < 2^n,\ \xi < \kappa\}$. For $n \in \mathbb{N}$ and $\xi < \kappa$ set $H_{n\xi} = [\{(n, \xi)\}, \infty[$; then $H_{n\xi} = \sup_{m<2^n} G_{mn\xi}$. **P** Certainly $G_{mn\xi} \subseteq H_{n\xi}$ whenever $m < 2^n$. If $\{(n, \xi)\} \subseteq p \in \mathcal{S}_\kappa^\infty$, let $q \in \mathcal{S}_\kappa^\infty$ be such that $p \subseteq q$ and $\#(q[\{n\}]) = 2^n$; set $m = \#(q[\{n\}] \cap \xi)$; then $[q, \infty[\ \subseteq H_{n\xi} \cap G_{mn\xi}$. Thus $\bigcup_{m<2^n} G_{mn\xi}$ is dense in $H_{n\xi}$ and $H_{n\xi} = \sup_{m<2^n} G_{mn\xi} \in \mathfrak{G}$. **Q** Consequently $H_{n\xi} \in \mathfrak{G}$.

(g) If $p \in \mathcal{S}_\kappa^\infty$ then $[p, \infty[\ = \inf_{(n,\xi)\in p} H_{n\xi}$ belongs to $\mathfrak{G}$, by 514Me. So $\mathfrak{G}$ includes an order-dense subset of $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$ and must be the whole of $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$; that is, $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$ is τ-generated by $\{G_{mn\xi} : n \in \mathbb{N},\ m < 2^n,\ \xi < \kappa\}$. With (iv) and (v), we see that the conditions of 514F are satisified with $J = \kappa$ and $I = \{(m, n) : n \in \mathbb{N},\ m < 2^n\}$, so that

$$\tau(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)) \le \max(\omega, \#(I)) = \omega.$$

528S The calculation of Maharam types of amoeba algebras seems to be a good deal harder. However it leads through an investigation of the structure of measure algebras, which is one of the things this book is about, so I take the space to give one of the main theorems. It depends on a special property of the standard generating families in algebras $\mathfrak{B}_\kappa$.

Definition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. I will say that a **well-spread basis** for $\mathfrak{A}$ is a non-decreasing sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathfrak{A}$ such that

(i) setting $D = \bigcup_{n\in\mathbb{N}} D_n$, $\#(D) \le \max(\omega, c(\mathfrak{A}), \tau(\mathfrak{A}))$;

(ii) if $a \in \mathfrak{A}$, $\gamma \in \mathbb{R}$ and $\bar\mu a < \gamma$, there is a set $D \subseteq \bigcup_{n\in\mathbb{N}} D_n$ such that $a \subseteq \sup D$ and $\bar\mu(\sup D) < \gamma$;

(iii) if $n \in \mathbb{N}$ and $\langle d_i\rangle_{i\in\mathbb{N}}$ is a sequence in D_n such that $\bar\mu(\sup_{i\in\mathbb{N}} d_i) < \infty$, there is an infinite set $J \subseteq \mathbb{N}$ such that $d = \sup_{i\in J} d_i$ belongs to D_n;

(iv) whenever $n \in \mathbb{N}$, $a \in \mathfrak{A}$ and $\bar\mu a \le \gamma' < \gamma < \bar\mu 1$, there is a $b \in \mathfrak{A}$ such that $a \subseteq b$ and $\gamma' \le \bar\mu b < \gamma$ and $\bar\mu(b \cup d) \ge \gamma$ whenever $d \in D_n$ and $d \not\subseteq a$.

528T Lemma (a) Let κ be an infinite cardinal, and $\langle e_\xi\rangle_{\xi<\kappa}$ the standard generating family in $\mathfrak{B}_\kappa$. For $n \in \mathbb{N}$ let C_n be the set of elements of $\mathfrak{B}_\kappa$ expressible as $\inf_{\xi\in I} e_\xi \cap \inf_{\xi\in J}(1 \setminus e_\xi)$ where I, $J \subseteq \kappa$ are disjoint and $\#(I \cup J) \le n$. Then $\langle C_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$. Moreover,

(*) for each $n \ge 1$, there is a set $C'_n \subseteq C_n$, with cardinal κ, such that $\bar\nu_\kappa c = 2^{-n}$ for every $c \in C'_n$, and whenever $a \in \mathfrak{B}_\kappa \setminus \{1\}$ and $I \subseteq C'_n$ is infinite, there is a $c \in I$ such that $c' \not\subseteq a \cup c$ whenever $c \subset c' \in C_n$.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $e \in \mathfrak{A}$. If $\langle C_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_e, \bar\mu\restriction\mathfrak{A}_e)$ and $\langle D_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_{1\setminus e}, \bar\mu\restriction\mathfrak{A}_{1\setminus e})$, then $\langle C_n \cup D_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}, \bar\mu)$.

proof (a)(i) $\langle C_n\rangle_{n\in\mathbb{N}}$ satisfies (i) of Definition 528S just because $\tau(\mathfrak{B}_\kappa) = \#(C_n) = \kappa$ for $n \ge 1$, while $C_0 = \{1\}$.

(ii) For $J \subseteq \kappa$, let $\mathfrak{C}_J$ be the order-closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\xi : \xi \in J\}$; recall that for every $a \in \mathfrak{B}_\kappa$ there is a countable set $\operatorname{supp} a \subseteq \kappa$ such that $a \in \mathfrak{C}_J$ iff $J \supseteq \operatorname{supp} a$ (254Rd/325Mb). Of course $\#(\operatorname{supp} c) \le n$ whenever $n \in \mathbb{N}$ and $c \in C_n$.

Suppose that $a \in \mathfrak{B}_\kappa$ and $\gamma > \bar\nu_\kappa a$. Then for each $k \in \mathbb{N}$ we can find an $a_k \in \mathfrak{B}_\kappa$, with finite support, such that $\bar\nu_\kappa(a \triangle a_k) \le 2^{-k-2}(\gamma - \bar\nu_\kappa a)$ (254Fe/325Jc). Set $b = \sup_{k\in\mathbb{N}} a_k$; then

$$\bar\nu_\kappa b \le \bar\nu_\kappa a + \sum_{k=0}^\infty \bar\nu_\kappa(a_k \setminus a) < \gamma,$$

$$\bar\nu_\kappa(a \setminus b) \le \inf_{k\in\mathbb{N}} \bar\nu_\kappa(a \setminus a_k) = 0,$$

so $a \subseteq b$. If $k \in \mathbb{N}$ and $\#(\operatorname{supp} a_k) = n_k$, then $a_k = \sup\{c : c \in C_{n_k},\ c \subseteq a_k\}$, so $b = \sup\{c : c \in \bigcup_{n\in\mathbb{N}} C_n,\ c \subseteq b\}$. Thus 528S(ii) is satisfied.

(iii) If $n \in \mathbb{N}$ and $\langle c_i \rangle_{i \in \mathbb{N}}$ is a sequence in C_n, there is an infinite $I \subseteq \mathbb{N}$ such that $\langle \operatorname{supp} c_i \rangle_{i \in I}$ is a Δ-system with root K say (5A1Jc). For $i \in I$, express c_i as $c'_i \cap c''_i$ where $c'_i \in \mathfrak{C}_K$ and $c''_i \in \mathfrak{C}_{(\operatorname{supp} c_i) \setminus K}$; as $\mathfrak{C}_K$ is finite, there is a c such that $J = \{i : c'_i = c\}$ is infinite. Now $c \in C_n$, and if $m \in \mathbb{N}$ then

$$\sup_{i \in J \setminus m} c_i = c \cap \sup_{i \in J \setminus m} c''_i = c$$

because $\langle c''_i \rangle_{i \in J \setminus m}$ is a stochastically independent family of elements of $\mathfrak{B}_\kappa$ all of measure at least 2^{-n}, so has supremum 1. In particular, 528S(iii) is satisfied.

(iv) Suppose that $n \in \mathbb{N}$ and $a \in \mathfrak{B}_\kappa$. Then there is a $\delta > 0$ such that $\bar{\nu}_\kappa(c \setminus a) \geq \delta$ whenever $c \in C_n$ and $c \not\subseteq a$. **P?** Otherwise, there is a sequence $\langle c_i \rangle_{i \in \mathbb{N}}$ in C_n such that $0 < \bar{\nu}_\kappa(c_i \setminus a) \leq 2^{-i}$ for every $i \in \mathbb{N}$. By (iii) just above, there is an infinite set $J \subseteq \mathbb{N}$ such that $c_j \subseteq \sup_{i \in J \setminus m} c_i$ for every $j \in J$. Set $j_0 = \min J$, and let m be such that $2^{-m+1} < \bar{\nu}_\kappa(c_{j_0} \setminus a)$; then

$$2^{-m+1} < \bar{\nu}_\kappa(\sup_{j \in J \setminus m} c_j \setminus a) \leq \sum_{j=m}^{\infty} \bar{\nu}_\kappa(c_j \setminus a) \leq 2^{-m+1},$$

which is absurd. **XQ**

(v) Suppose that $n \in \mathbb{N}$, $a \in \mathfrak{B}_\kappa$ and $\bar{\nu}_\kappa a \leq \gamma' < \gamma < 1$. Pick $\delta > 0$, $r > n$, $k^* \in \mathbb{N}$, $\epsilon > 0$ and $a' \in \mathfrak{B}_\kappa$ such that

$$\bar{\nu}_\kappa(c \setminus a) \geq \delta \text{ whenever } c \in C_n \text{ and } c \not\subseteq a,$$

$$2^{-r} \leq \gamma - \gamma', \quad (2^{-n} - 2^{-r})^n \delta \geq 2^{-r+2},$$

$$(1 - 2^{-r})^{k^*} \leq 1 - \gamma,$$

$$\epsilon \leq \frac{1}{2}\delta, \quad \epsilon \leq 2^{-r}(1 - 2^{-r})^{k^*},$$

$$\operatorname{supp} a' \text{ is finite}, \quad \bar{\nu}_\kappa(a \,\triangle\, a') \leq \epsilon.$$

Let $\langle K_i \rangle_{i \in \mathbb{N}}$ be a disjoint sequence in $[\kappa \setminus \operatorname{supp} a']^r$, and set $c_i = \inf_{\xi \in K_i} e_\xi$ for each $i \in \mathbb{N}$. Then $\sup_{i \in \mathbb{N}} c_i = 1$, so there is a first k such that $\bar{\nu}_\kappa(a \cup \sup_{i \leq k} c_i) \geq \gamma$; set $b_1 = \sup_{i<k} c_i$ and $b = a \cup b_1$. Surely $a \subseteq b$ and $\bar{\nu}_\kappa b < \gamma$; also

$$(1 - 2^{-r})^{k^*} \leq 1 - \gamma \leq 1 - \bar{\nu}_\kappa b_1 = (1 - 2^{-r})^k$$

so $k \leq k^*$. Moreover,

$$\gamma - \bar{\nu}_\kappa b \leq \bar{\nu}_\kappa(b \cup c_k) - \bar{\nu}_\kappa b \leq \bar{\nu}_\kappa(c_k \setminus b_1) = 2^{-r}(1 - 2^{-r})^k \leq 2^{-r} \leq \gamma - \gamma',$$

so that, in particular, $\bar{\nu}_\kappa b \geq \gamma'$.

If $c \in C_n$ and $c \not\subseteq a$ then $\bar{\nu}_\kappa(c \setminus a) \geq \delta$ so $\bar{\nu}_\kappa(c \setminus a') \geq \delta - \epsilon$. Express c as $\inf_{i \leq k} c'_i$ where $\operatorname{supp} c'_i \subseteq K_i$ for $i < k$ and $\operatorname{supp} c'_k \subseteq \kappa \setminus \bigcup_{i<k} K_i$. Set $J = \{i : i < k,\, c'_i \neq 1\}$; then $\#(J) \leq n$. Now

$$\begin{aligned} \bar{\nu}_\kappa(c \setminus (a' \cup b_1)) &= \bar{\nu}_\kappa((c'_k \setminus a') \cap \inf_{i \in J}(c'_i \setminus c_i) \cap \inf_{i \in k \setminus J}(1 \setminus c_i)) \\ &= \bar{\nu}_\kappa(c'_k \setminus a') \cdot \prod_{i \in J} \bar{\nu}_\kappa(c'_i \setminus c_i) \cdot \prod_{i \in k \setminus J} \bar{\nu}_\kappa(1 \setminus c_i) \end{aligned}$$

(because $\operatorname{supp}(c'_k \setminus a') \subseteq \operatorname{supp} c'_k \cup \operatorname{supp} a' \subseteq \kappa \setminus \bigcup_{i<k} K_i$, so we are taking an infimum of stochastically independent elements of $\mathfrak{B}_\kappa$)

$$\geq (\delta - \epsilon) \cdot \prod_{i \in J}(2^{-n} - 2^{-r}) \cdot \prod_{i \in k \setminus J}(1 - 2^{-r})$$

(of course every c'_i belongs to C_n)

$$\begin{aligned} &\geq \frac{1}{2}(2^{-n} - 2^{-r})^n (1 - 2^{-r})^k \delta \geq 2^{-r+1}(1 - 2^{-r})^k \\ &\geq 2^{-r}(1 - 2^{-r})^k + 2^{-r}(1 - 2^{-r})^{k^*} \geq \gamma - \bar{\nu}_\kappa b + \epsilon, \end{aligned}$$

and

$$\bar\nu_\kappa(c\setminus b)\ge\gamma-\bar\nu_\kappa b,$$

so $\bar\nu_\kappa(b\cup c)\ge\gamma$.

As n, a, γ' and γ are arbitrary, $\langle C_n\rangle_{n\in\mathbb{N}}$ satisfies 528S(iv) and is a well-spread basis.

(vi) As for (*), given $n\ge 1$, take a disjoint family $\langle K_\xi\rangle_{\xi<\kappa}$ in $[\kappa]^n$, and set $c_\xi=\inf_{\eta\in K_\xi}e_\eta$ for $\xi<\kappa$, $C'_n=\{c_\xi:\xi<\kappa\}$. If $I\subseteq\kappa$ is infinite and $\bar\nu_\kappa a<1$, take $\delta>0$ such that $\delta<2^{-n}(1-\bar\nu_\kappa a-\delta)$, and $a'\in\mathfrak{B}_\kappa$ such that $\operatorname{supp}a'$ is finite and $\bar\nu_\kappa(a\mathbin{\triangle}a')\le\delta$. Then there is a $\xi\in I$ such that $K_\xi\cap\operatorname{supp}a_\xi=\emptyset$. **?** If $c\in C_n$ is such that $c_\xi\subset c\subseteq a\cup c_\xi$, there must be a $d\in\mathfrak{B}_\kappa$, with support K_ξ, included in $c\setminus c_\xi$. But now $d\subseteq a$ and $\operatorname{supp}d\cap\operatorname{supp}a'=\emptyset$, so

$$2^{-n}(1-\bar\nu_\kappa a-\delta)\le 2^{-n}(1-\bar\nu_\kappa a')=\bar\nu_\kappa(d\setminus a')\le\delta+\bar\nu_\kappa(d\setminus a)=\delta.\ \mathbf{X}$$

(b)(i) We have

$$\begin{aligned}\#(\bigcup_{n\in\mathbb{N}}C_n\cup D_n)&\le\max(\omega,\#(\bigcup_{n\in\mathbb{N}}C_n),\#(\bigcup_{n\in\mathbb{N}}D_n))\\&\le\max(\omega,c(\mathfrak{A}_e),c(\mathfrak{A}_{1\setminus e}),\tau(\mathfrak{A}_e),\tau(\mathfrak{A}_{1\setminus e}))=\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))\end{aligned}$$

by 514E.

(ii) Suppose that $a\in\mathfrak{A}$ and $\bar\mu a<\gamma$. Then there are γ_1, γ_2 such that $\bar\mu(a\cap e)<\gamma_1$, $\bar\mu(a\setminus e)<\gamma_2$ and $\gamma_1+\gamma_2\le\gamma$. Let $C\subseteq\bigcup_{n\in\mathbb{N}}C_n$, $D\subseteq\bigcup_{n\in\mathbb{N}}D_n$ be such that $a\cap e\subseteq\sup C$, $a\setminus e\subseteq\sup D$, $\bar\mu(\sup C)<\gamma_1$ and $\bar\mu(\sup D)<\gamma_2$. Then $C\cup D\subseteq\bigcup_{n\in\mathbb{N}}C_n\cup D_n$, $a\subseteq\sup(C\cup D)$ and $\bar\mu(\sup(C\cup D))<\gamma$.

(iii) Suppose that $n\in\mathbb{N}$ and $\langle c_i\rangle_{i\in\mathbb{N}}$ is a sequence in $C_n\cup D_n$ such that $\bar\mu(\sup_{i\in\mathbb{N}}c_i)<\infty$. Then there is an infinite $J\subseteq\mathbb{N}$ such that either $c_i\in C_n$ for every $i\in J$, or $c_i\in D_n$ for every $i\in I$. In either case, there is an infinite $I\subseteq J$ such that $\sup_{i\in I}c_i$ belongs to $C_n\cup D_n$.

(iv) Thus $\langle C_n\cup D_n\rangle_{n\in\mathbb{N}}$ satisfies (i)-(iii) of Definition 528S. As for 528S(iv), suppose that $n\in\mathbb{N}$, $a\in\mathfrak{A}$ and $\bar\mu a\le\gamma'<\gamma<\bar\mu 1$. We need to find a $b\in\mathfrak{A}$ such that $a\subseteq b$ and

$$\gamma'\le\bar\mu b<\gamma\le\bar\mu(b\cup c)$$

whenever $c\in C_n\cup D_n$ and $c\not\subseteq a$.

case 1 If $e\subseteq a$, then $\bar\mu e$ is finite and

$$\bar\mu(a\setminus e)\le\gamma'-\bar\mu e<\gamma-\bar\mu e<\bar\mu(1\setminus e).$$

So there is a $b_2\in\mathfrak{A}_{1\setminus e}$ such that $a\setminus e\subseteq b_2$ and

$$\gamma'-\bar\mu e\le\bar\mu b_2<\gamma-\bar\mu e\le\bar\mu(b_2\cup d)$$

whenever $d\in D_n$ and $d\not\subseteq a\setminus e$; that is,

$$\gamma'\le\bar\mu(e\cup b_2)<\gamma\le\bar\mu(e\cup b_2\cup d)$$

whenever $d\in D_n$ and $d\not\subseteq a$. Since $c\subseteq a$ for every $c\in C_n$, we have $\bar\mu(e\cup b_2\cup c)\ge\gamma$ whenever $c\in C_n\cup D_n$ and $c\not\subseteq a$, and can take $b=e\cup b_2$.

case 2 Similarly, if $a\supseteq 1\setminus e$, we can take $b=(1\setminus e)\cup b_1$ for a suitable $b_1\subseteq e$.

case 3 If neither e nor $1\setminus e$ is included in a, we have

$$\max(\bar\mu(a\cap e),\gamma-\bar\mu(1\setminus e))<\min(\bar\mu e,\gamma-\bar\mu(a\setminus e)),$$

so we can find γ'_1, γ_1 such that

$$\max(\bar\mu(a\cap e),\gamma-\bar\mu(1\setminus e))<\gamma'_1<\gamma_1<\min(\bar\mu e,\gamma-\bar\mu(a\setminus e))$$

and $\gamma_1-\gamma'_1<\gamma-\gamma'$. Let $b_1\in\mathfrak{A}_e$ be such that $a\cap e\subseteq b_1$ and

$$\gamma'_1\le\bar\mu b_1<\gamma_1\le\bar\mu(b_1\cup c)$$

whenever $c\in C_n$ and $c\not\subseteq a\cap e$. Set $\gamma'_2=\gamma-\gamma_1$ and $\gamma_2=\gamma-\bar\mu b_1$, so that

$$\bar\mu(a\setminus e)<\gamma'_2<\gamma_2\le\gamma-\gamma'_1<\bar\mu(1\setminus e).$$

Let $b_2 \in \mathfrak{A}_{1\setminus e}$ be such that $a \setminus e \subseteq b_2$ and

$$\gamma_2' \le \bar\mu b_2 < \gamma_2 \le \bar\mu(b_2 \cup d)$$

whenever $d \in D_n$ and $d \not\subseteq a \setminus e$.

Try $b = b_1 \cup b_2$. Then $a \subseteq b$ and $\bar\mu b = \bar\mu b_1 + \bar\mu b_2$ belongs to

$$[\bar\mu b_1 + \gamma_2', \bar\mu b_1 + \gamma_2[\; \subseteq [\gamma_1' + \gamma - \gamma_1, \gamma[\; \subseteq [\gamma', \gamma[.$$

If $c \in C_n$ and $c \not\subseteq a$, then $c \not\subseteq a \cap e$, so

$$\bar\mu(b \cup c) = \bar\mu(b_1 \cup c) + \bar\mu b_2 \ge \gamma_1 + \gamma_2' = \gamma;$$

while if $d \in D_n$ and $d \not\subseteq a$, then

$$\bar\mu(b \cup d) = \bar\mu b_1 + \bar\mu(b_2 \cup d) \ge \mu b_1 + \gamma_2 = \gamma.$$

So in this case also we have found a suitable b.

528U Lemma Let $(\mathfrak{A}, \bar\mu)$ be an atomless semi-finite measure algebra and $0 < \gamma < \bar\mu 1$. Let E, ϵ, $\preccurlyeq$ and $\mathcal{F}$ be such that

E is a partition of unity in $\mathfrak{A}$ such that $\mathfrak{A}_e$ is homogeneous and $0 < \epsilon \le \bar\mu e < \infty$ for every $e \in E$;

$\preccurlyeq$ is a well-ordering of E such that $\tau(\mathfrak{A}_e) \le \tau(\mathfrak{A}_{e'})$ whenever $e \preccurlyeq e'$ in E;

$\mathcal{F}$ is a partition of E such that each member of $\mathcal{F}$ is either a singleton or a countable set with no $\preccurlyeq$-greatest member.

Let P_0 be

$$\{a : a \in \mathfrak{A},\ \bar\mu a < \gamma,\ \gamma \le \bar\mu(a \cup e) \text{ whenever } \{e\} \in \mathcal{F}\},$$

ordered by $\subseteq$. Then $\mathrm{RO}^{\uparrow}(P_0)$ has countable Maharam type.

proof (a)(i) For every $e \in E$, $(\mathfrak{A}_e, \bar\mu \restriction \mathfrak{A}_e)$ is a non-zero atomless homogeneous totally finite measure algebra, so is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ for some infinite cardinal κ (331L). So we can copy the well-spread basis for $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ described in 528Ta into a well-spread basis $\langle D_{en} \rangle_{n\in\mathbb{N}}$ for $(\mathfrak{A}_e, \bar\mu \restriction \mathfrak{A}_e)$ such that

$\#(\bigcup_{n\in\mathbb{N}} D_{en}) = \tau(\mathfrak{A}_e)$,

$\bar\mu d \ge 2^{-n} \bar\mu e$ whenever $n \in \mathbb{N}$ and $d \in D_{en}$,

$D_{e0} = \{e\}$,

for each $n \ge 1$ there is a set $D_{en}' \subseteq D_{en}$, with cardinal $\tau(\mathfrak{A}_e)$, such that $\bar\mu d = 2^{-n}\bar\mu e$ for every $d \in D_{en}'$, and whenever $a \in \mathfrak{A}_e \setminus \{e\}$ and $I \subseteq D_{en}'$ is infinite, there is a $d \in I$ such that $d' \not\subseteq a \cup d$ whenever $d' \in D_{en}$ and $d' \supset d$,

$(\bigcup_{n\in\mathbb{N}} D_{en}) \setminus (\bigcup_{n\ge 1} D_{en}')$ has cardinal $\tau(\mathfrak{A}_e)$.

(The last item is not mentioned in 528T, but is clearly achievable by thinning the sets D_{en}' appropriately, besides being automatic if we use the construction in (a-vi) of the proof of 528T.) Note that $\langle D_{en}' \rangle_{n\ge 1}$ is a disjoint sequence of subsets of $\mathfrak{A}_e$ for each e, so $\langle D_{en}' \rangle_{e\in E, n\ge 1}$ is disjoint.

(ii) For $e \in F \in \mathcal{F}$, set

$$D_e = \bigcup_{n\in\mathbb{N}} D_{en} \setminus \bigcup_{n\ge 1} D_{en}', \quad D_e^* = \bigcup_{e' \in F, e' \preccurlyeq e} D_e.$$

Because F is countable and $\tau(\mathfrak{A}_{e'}) \le \tau(\mathfrak{A}_e)$ whenever $e' \preccurlyeq e$, $\#(D_e^*) = \tau(\mathfrak{A}_e) = \#(D_{en}')$ for every $n \ge 1$. We therefore have a partition $\langle I_{ed} \rangle_{d\in D_e^*}$ of $\bigcup_{n\ge 1} D_{en}'$ into countably infinite sets such that $I_{ed} \cap D_{en}'$ is infinite whenever $d \in D_e^*$ and $n \ge 1$.

Let θ be a limit ordinal such that the set Ω of limit ordinals less than θ has cardinal $\#(\bigcup_{e\in E} D_e)$. (Of course we can take θ to be either an uncountable cardinal or the ordinal product $\omega \cdot \omega$ or 0.) Again because every member of $\mathcal{F}$ is countable, we have an enumeration $\langle d_\xi \rangle_{\xi<\theta}$ of $\bigcup_{e\in E, n\in\mathbb{N}} D_{en}$ such that whenever $\xi \in \Omega$ then there are $F \in \mathcal{F}$ and $e \in F$ such that

$$d_\xi \in D_e, \quad \{d_{\xi+i} : i \ge 1\} = \bigcup_{e' \in F, e' \succcurlyeq e} I_{e' d_\xi}.$$

This will mean that whenever $\xi \in \Omega$ and $F \in \mathcal{F}$, $e \in F$ are such that $d_\xi \in \mathfrak{A}_e$, then $\{i : d_{\xi+i} \in D_{e'n}'\}$ is infinite whenever $e' \in F$, $e \preccurlyeq e'$ and $n \in \mathbb{N}$.

(b)(i) Setting $P=\{a:a\in\mathfrak{A},\,\bar\mu a<\gamma\}$, $P_0\in\mathrm{RO}^\uparrow(P)$. **P** Evidently P_0 is up-open. If $a\in P\setminus P_0$, that is, there is some e such that $\{e\}\in\mathcal{F}$ and $\bar\mu(a\cup e)<\gamma$, set $b=a\cup e$; then $a\subseteq b\in P$, while $\bar\mu(b'\cup e)=\bar\mu b'<\gamma$ whenever $b'\in[b,\infty[$, so $[b,\infty[$ does not meet P_0. Accordingly $[a,\infty[\,\not\subseteq\overline{P}_0$ and $a\notin\operatorname{int}\overline{P}_0$. As a is arbitrary, $P_0=\operatorname{int}\overline{P}_0\in\mathrm{RO}^\uparrow(P)$. **Q**

It follows that $\mathrm{RO}^\uparrow(P_0)$ is the principal ideal of $\mathrm{RO}^\uparrow(P)$ generated by P_0 (314R(b-ii)). Moreover, for $a\in P_0$, $[a,\infty[$ is the same whether taken in P or P_0, and belongs to $\mathrm{RO}^\uparrow(P)$ by 528B(b-i).

(ii) For $a\in P_0$ and $n\in\mathbb{N}$, set $A_n(a)=\{d:d\in\bigcup_{e\in E}D_{en},\,d\subseteq a\}$. Then any sequence in $A_n(a)$ has a subsequence with an upper bound in $A_n(a)$. **P** Set $L=\{e:e\in E,\,\bar\mu(a\cap e)\ge 2^{-n}\epsilon\}$; then L is finite. If $e\in E\setminus L$ and $d\in D_{en}$, then $d\subseteq e$ and

$$\bar\mu d\ge 2^{-n}\bar\mu e\ge 2^{-n}\epsilon>\bar\mu(a\cap e)\ge\bar\mu(a\cap d),$$

so $d\not\subseteq a$. Thus $A_n(a)\subseteq\bigcup_{e\in L}D_{en}$. It follows that if $\langle c_i\rangle_{i\in\mathbb{N}}$ is any sequence in $A_n(a)$, there is an $e\in L$ such that $J=\{i:c_i\in D_{en}\}$ is infinite. Now there is an infinite $I\subseteq J$ such that $c=\sup_{i\in I}c_i$ belongs to D_{en}. In this case, $c\subseteq a$ so $c\in A_n(a)$ is an upper bound of $\{c_i:i\in I\}$. **Q**

It follows that $A_n(a)$ has only finitely many maximal elements, and any non-decreasing sequence in $A_n(a)$ has an upper bound in $A_n(a)$. Consequently, every member of $A_n(a)$ is included in a maximal element of $A_n(a)$. **P?** Otherwise, we should be able to find a strictly increasing family $\langle c_\xi\rangle_{\xi<\omega_1}$ in $A_n(a)$; but now there must be a $\xi<\omega_1$ such that $\bar\mu c_\xi=\bar\mu c_{\xi+1}<\gamma$ and $c_\xi=c_{\xi+1}$. **XQ**

Set $E_n(a)=\{\xi:d_\xi$ is a maximal element of $A_n(a)\}$, so that $E_n(a)$ is a finite subset of θ.

(iii) For $n\in\mathbb{N}$, set

$$Q_n=\{b:b\in P_0,\,A_n(b)=A_n(b')\text{ whenever }b\subseteq b'\in P_0\}.$$

Then whenever $a\in P_0$ and $n\in\mathbb{N}$ there is a $b\in Q_n$ such that $a\subseteq b$ and $A_n(a)=A_n(b)$. **P** Let L be a finite subset of E including $\{e:\bar\mu(a\cap e)\ge 2^{-n-1}\epsilon\}$ and such that $\bar\mu(\sup L)>\gamma$. Then $\langle\bigcup_{e\in L}D_{em}\rangle_{m\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_{\sup L},\bar\mu\restriction\mathfrak{A}_{\sup L})$. (Induce on $\#(L)$, using 528Tb for the inductive step.) Since

$$\bar\mu(a\cap\sup L)<\gamma-\bar\mu(a\setminus\sup L)<\bar\mu(\sup L),$$

there is a a $b_0\in\mathfrak{A}_{\sup L}$, including $a\cap\sup L$, such that

$$\gamma-\bar\mu(a\setminus\sup L)-2^{-n-1}\epsilon\le\bar\mu b_0<\gamma-\bar\mu(a\setminus\sup L)\le\bar\mu(b_0\cup d)$$

whenever $d\in\bigcup_{e\in L}D_{en}$ and $d\not\subseteq a$. Set $b=b_0\cup a$. Then $\bar\mu b=\bar\mu b_0+\bar\mu(a\setminus\sup L)<\gamma$, so $b\in P_0$. If $b\subseteq b'\in P_0$ and $d\in\bigcup_{e\in E}D_{en}\setminus A_n(a)$, then either $e\in L$ and

$$\bar\mu(b'\cup d)\ge\bar\mu(b\cup d)+\bar\mu(a\setminus\sup L)\ge\gamma>\bar\mu b',$$

or $e\notin L$,

$$\bar\mu(d\setminus a)\ge\bar\mu d-\bar\mu(a\cap e)\ge 2^{-n}\bar\mu e-2^{-n-1}\epsilon\ge 2^{-n-1}\epsilon$$

and

$$\bar\mu(b'\cup d)\ge\bar\mu b_0+\bar\mu(a\setminus\sup L)+2^{-n-1}\epsilon\ge\gamma>\bar\mu b';$$

in either case $d\not\subseteq b'$. Thus $A_n(b')=A_n(a)=A_n(b)$ whenever $b\subseteq b'\in P_0$, and $b\in Q_n$. **Q**

(c)(i) For $m,\,n,\,i\in\mathbb{N}$ and $\xi\in\Omega$, set

$$Q_{nmi\xi}=\{b:b\in Q_n,\,\xi+i\in E_n(b),\,\#(E_n(b)\cap\xi)=m\},$$

$$G_{nmi\xi}=\sup\{[b,\infty[\,:b\in Q_{nmi\xi}\}\in\mathrm{RO}^\uparrow(P_0).$$

(ii) For any $m,\,n,\,i\in\mathbb{N}$, $\langle G_{nmi\xi}\rangle_{\xi\in\Omega}$ is disjoint. **P** Suppose that $\xi<\eta$ in Ω. If $a\in Q_{nmi\xi}$ and $b\in Q_{nmi\eta}$, we see that $\xi+i<\eta$, $\xi+i\in E_n(a)$ and

$$\#(E_n(b)\cap\eta)=m=\#(E_n(a)\cap\xi)<\#(E_n(a)\cap\eta).$$

So $E_n(a)\ne E_n(b)$ and $A_n(a)\ne A_n(b)$. But both a and b are supposed to belong to Q_n, so $[a,\infty[$ must be disjoint from $[b,\infty[$. As b is arbitrary, $[a,\infty[\,\cap G_{nmi\eta}=\emptyset$; as a is arbitrary, $G_{nmi\xi}\cap G_{nmi\eta}=\emptyset$. **Q**

(iii) For any $\xi\in\Omega$ and $a\in P_0$, there are $m,\,n,\,i\in\mathbb{N}$ and $b\in Q_{nmi\xi}$ such that $a\subseteq b$. **P** Let $e\in E$ be such that $d_\xi\subseteq e$; let F be the member of $\mathcal{F}$ containing e. If $F=\{e\}$, then $\bar\mu(a\cup e)\ge\gamma>\bar\mu a$; set $e_0=e$, so that $e_0 \in F$, $e_0 \succcurlyeq e$ and $a \cap e_0 \ne e_0$. Otherwise, there are infinitely many members of F greater than e for the ordering $\preccurlyeq$, because F has no greatest member, so $\bar\mu(\sup_{e' \in F, e' \succcurlyeq e} e') = \infty$, and there must be an $e_0 \in F$ such that $e_0 \succcurlyeq e$ and $a \cap e_0 \ne e_0$.

Let $n \in \mathbb{N}$ be such that $2^{-n}\bar\mu e_0 < \min(\gamma - \bar\mu a, \bar\mu(e_0 \setminus a))$. Then $\{d_{\xi+i} : i \in \mathbb{N}\}$ meets $D'_{e_0 n}$ in an infinite set, and there is an $i \in \mathbb{N}$ such that $d_{\xi+i} \in D'_{e_0 n}$, $\bar\mu d_{\xi+i} = 2^{-n}\bar\mu e_0$, and $d \not\subseteq (a \cap e_0) \cup d_{\xi+i}$ whenever $d \in D_{e_0 n}$ and $d \supset d_{\xi+i}$. Set $a' = a \cup d_{\xi+i}$; then $d_{\xi+i}$ is a maximal member of $A_n(a')$. Let $b \in Q_n$ be such that $a' \subseteq b$ and $A_n(b) = A_n(a')$. Then $\xi + i \in E_n(b)$. Set $m = \#(E_n(b) \cap \xi)$. Then $b \in Q_{nmi\xi}$ and $a \subseteq b$. **Q**

Accordingly $b \in [a, \infty[\cap G_{nmi\xi}$. As a is arbitrary, $\bigcup_{m,n,i \in \mathbb{N}} G_{nmi\xi}$ is dense in P_0 and $\sup_{m,n,i \in \mathbb{N}} G_{nmi\xi} = P_0$ in $\mathrm{RO}^\uparrow(P_0)$.

(d)(i) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^\uparrow(P_0)$ generated by $\{G_{nmi\xi} : m, n, i \in \mathbb{N}, \xi \in \Omega\}$. By (c-ii) and (c-iii), the conditions of 514F are satisfied, and $\mathfrak{G}$ has countable Maharam type.

(ii) If $d \in P_0 \cap \bigcup_{e \in E, n \in \mathbb{N}} D_{en}$ then $[d, \infty[\in \mathfrak{G}$. **P** Set

$$H = \sup\{G_{nmi\xi} : m,\ n,\ i \in \mathbb{N},\ \xi \in \Omega \text{ and } G_{nmi\xi} \subseteq [d, \infty[\} \in \mathrm{RO}^\uparrow(P_0).$$

Then $H \in \mathfrak{G}$ and $H \subseteq [d, \infty[$. Suppose that $a \in P_0$ and $a \supseteq d$. Let $n \in \mathbb{N}$ be such that $d \in \bigcup_{e \in E} D_{en}$. Then there is a $b \in Q_n$ such that $a \subseteq b$. In this case, $d \in A_n(b)$ so there is a maximal $d' \in A_n(b)$ including d; let $\xi \in \Omega$, $i \in \mathbb{N}$ be such that $d' = d_{\xi+i}$, and set $m = \#(E_n(b) \cap \xi)$. Then $b \in Q_{nmi\xi}$. On the other hand, for any $b' \in Q_{nmi\xi}$, $d \subseteq d_{\xi+i} \subseteq b'$, so $[b', \infty[\subseteq [d, \infty[$; as b' is arbitrary, $G_{nmi\xi} \subseteq [d, \infty[$ and $G_{nmi\xi} \subseteq H$. Accordingly $b \in H \cap [a, \infty[$. As a is arbitrary, H is dense in $[d, \infty[$ and must be the whole of $[d, \infty[$; thus we have $[d, \infty[= H \in \mathfrak{G}$. **Q**

(iii) If $a \in P_0$ there is a $b \in P_0$ such that $a \subseteq b$ and $[b, \infty[\in \mathfrak{G}$. **P** Let E_0 be a countable subset of E such that $a \subseteq \sup E_0$ and $\bar\mu(\sup E_0) > \gamma$. Set $L = \{e : e \in E_0,\ a \supseteq e\}$. Then $E_0 \setminus L$ is non-empty, and

$$\textstyle\sum_{e \in E_0 \setminus L} \bar\mu(a \cap e) = \bar\mu a - \bar\mu(\sup L) < \gamma - \bar\mu(\sup L).$$

We therefore have a family $\langle \gamma_e \rangle_{e \in E_0 \setminus L}$ such that $\bar\mu(a \cap e) < \gamma_e \le \bar\mu e$ for every $e \in E_0 \setminus L$ and $\sum_{e \in E_0 \setminus L} \gamma_e < \gamma - \bar\mu(\sup L)$. For each $e \in E_0$ there is a $B_e \subseteq \bigcup_{n \in \mathbb{N}} D_{en}$ such that $a \cap e \subseteq \sup B_e$ and $\bar\mu(\sup B_e) \le \gamma_e$, by 528S(ii). Set

$$B = L \cup \textstyle\bigcup_{e \in E_0 \setminus L} B_e \subseteq \bigcup_{e \in E, n \in \mathbb{N}} D_{en}$$

and $b = \sup B$. Then $a \subseteq b$ and

$$\bar\mu b = \bar\mu(\sup L) + \textstyle\sum_{e \in E_0 \setminus L} \bar\mu(\sup B_e) \le \bar\mu(\sup L) + \sum_{e \in E_0 \setminus L} \gamma_e < \gamma,$$

so $b \in P_0$. On the other hand,

$$[b, \infty[= \textstyle\bigcap_{d \in B} [d, \infty[= \inf_{d \in B} [d, \infty[\in \mathfrak{G},$$

as required. **Q**

(iv) As a is arbitrary, $\mathfrak{G}$ includes a π-base for the Boolean algebra $\mathrm{RO}^\uparrow(P_0)$ and must be the whole of $\mathrm{RO}^\uparrow(P_0)$. Accordingly

$$\tau(\mathrm{RO}^\uparrow(P_0)) = \tau(\mathfrak{G}) \le \omega.$$

This completes the proof.

528V Theorem Let $(\mathfrak{A}, \bar\mu)$ be an atomless semi-finite measure algebra and $0 < \gamma < \bar\mu 1$. Then $\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)$ has countable Maharam type.

proof Throughout the proof, P will stand for $\{a : a \in \mathfrak{A},\ \bar\mu a < \gamma\}$.

(a) Suppose that there are a partition E of unity in $\mathfrak{A}$ and an $\epsilon > 0$ such that $\mathfrak{A}_e$ is homogeneous and $\epsilon \le \bar\mu e < \infty$ for every $e \in E$.

(i) Let $\preccurlyeq$ be a well-ordering of E such that $\tau(\mathfrak{A}_e) \le \tau(\mathfrak{A}_{e'})$ whenever $e \preccurlyeq e'$ in E. Let $\mathcal{F}_0$ be a maximal disjoint family of subsets of E of order type ω in the ordering induced by $\preccurlyeq$. Then $M = E \setminus \bigcup \mathcal{F}_0$ must be finite; set $\mathcal{F} = \mathcal{F}_0 \cup \{\{e\} : e \in M\}$.

(ii) For $L \subseteq M$, set

$$P_L = \{a : a \in P,\, a \supseteq \sup L,\, \bar{\mu}(a \cup e) \ge \gamma \text{ for } e \in M \setminus L\}.$$

Then $\langle P_L \rangle_{L \subseteq M}$ is a disjoint family of open subsets of P. Also $\bigcup_{L \subseteq M} P_L$ is dense in P. **P** If $a \in P$, let $L \subseteq M$ be a maximal set such that $\bar{\mu}(a \cup \sup L) < \gamma$, and set $b = a \cup \sup L$; then $a \subseteq b \in P_L$. **Q** So $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to the simple product $\prod_{L \subseteq M} \mathrm{RO}^{\uparrow}(P_L)$ (315H again).

(iii) If $L \subseteq M$, then $\mathrm{RO}^{\uparrow}(P_L)$ has countable Maharam type. **P** If $P_L = \emptyset$ this is trivial. Otherwise there is an $a \in P_L$ and $\bar{\mu}(\sup L) \le \bar{\mu}a < \gamma$. Consider $\mathfrak{A}' = \mathfrak{A}_{1 \setminus \sup L}$, $\gamma' = \gamma - \bar{\mu}(\sup L)$, $E' = E \setminus L$, $\mathcal{F}' = \mathcal{F} \setminus \{\{e\} : e \in L\}$ and $\preccurlyeq' = \preccurlyeq \cap (E' \times E')$. Then $(\mathfrak{A}', \bar{\mu}{\restriction}\mathfrak{A}')$, γ', E', ϵ, $\preccurlyeq'$ and $\mathcal{F}'$ satisfy the conditions of 528U. Setting

$$Q_0 = \{c : c \in \mathfrak{A}',\, \bar{\mu}c < \gamma' \le \bar{\mu}(c \cup e) \text{ for every } e \in M \setminus L\},$$

$\mathrm{RO}^{\uparrow}(Q_0)$ has countable Maharam type, by 528U. But the map $c \mapsto c \cup \sup L$ is an order-isomorphism between Q_0 and P_L, so $\mathrm{RO}^{\uparrow}(P_L)$ has countable Maharam type. **Q**

(iv) Thus $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma) = \mathrm{RO}^{\uparrow}(P)$ is isomorphic to the product of finitely many Boolean algebras with countable Maharam type, and has countable Maharam type (514Ef).

(b) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable.

(i) In this case, let E be a partition of unity in $\mathfrak{A}$ such that $\mathfrak{A}_e$ is homogeneous and $0 < \bar{\mu}e < \infty$ for every $e \in E$. Let $\epsilon > 0$ be such that $\sum_{e \in E, \bar{\mu}e \ge \epsilon} \bar{\mu}e > \gamma$. For each $k \in \mathbb{N}$, set

$$E_k = \{e : e \in E,\, \bar{\mu}e \ge 2^{-k}\epsilon\}, \quad e_k^* = \sup E_k.$$

By (a), $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)$ has countable Maharam type for every k.

(ii) Now 528Fb tells us that we have a sequence $\langle \pi_k \rangle_{k \in \mathbb{N}}$ such that π_k is a regular embedding of $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)$ into $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ for each k, and $\bigcup_{k \in \mathbb{N}} \pi_k[\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)]$ τ-generates $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$. So $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ has countable Maharam type. **P** For each k, we have a countable τ-generating set $D_k \subseteq \mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)$. Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ generated by $D = \bigcup_{k \in \mathbb{N}} \pi_k[D_k]$. For each $k \in \mathbb{N}$, $\pi_k^{-1}[\mathfrak{G}]$ is an order-closed subalgebra of $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)$ including D_k, so is the whole of $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)$, that is, $\pi_k[\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)] \subseteq \mathfrak{G}$. Since $\bigcup_{k \in \mathbb{N}} \pi_k[\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu}{\restriction}\mathfrak{A}_{e_k^*}, \gamma)]$ τ-generates $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$, $\mathfrak{G} = \mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\tau(\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)) \le \#(D) \le \omega$. **Q**

(c) Thus we have the result when $(\mathfrak{A}, \bar{\mu})$ is localizable. For the general case of atomless semi-finite $(\mathfrak{A}, \bar{\mu})$, let $(\widehat{\mathfrak{A}}, \tilde{\mu})$ be the localization of $(\mathfrak{A}, \bar{\mu})$ (322Q). Since the embedding $\mathfrak{A} \subsetneqq \widehat{\mathfrak{A}}$ identifies $\mathfrak{A}^f$ with $\widehat{\mathfrak{A}}^f$ (322P), $\{a : a \in \widehat{\mathfrak{A}},\, \tilde{\mu}a < \gamma\}$ can be identified with P, and the regular open algebras $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\mathrm{AM}(\widehat{\mathfrak{A}}, \tilde{\mu}, \gamma)$ are isomorphic. Again because $\mathfrak{A}^f$ and $\widehat{\mathfrak{A}}^f$ are isomorphic, $\widehat{\mathfrak{A}}$ is atomless. By (b), the common Maharam type of $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\mathrm{AM}(\widehat{\mathfrak{A}}, \tilde{\mu}, \gamma)$ is countable.

528X Basic exercises (a) Suppose that (X, Σ, μ) is a measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $\mathcal{E} \subseteq \Sigma$ be a family such that μ is outer regular with respect to $\mathcal{E}$, and P the set $\{(E, \alpha) : E \in \mathcal{E},\, \mu E < \alpha \le \mu X\}$, ordered by saying that $(E, \alpha) \le (F, \beta)$ if $E \subseteq F$ and $\beta \le \alpha$. Show that $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless quasi-homogeneous semi-finite measure algebra. Show that $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is homogeneous whenever $0 < \gamma < \bar{\mu}1$. (*Hint*: first check that $\mathfrak{A} \cong \mathfrak{A}_{1 \setminus a}$ whenever $a \in \mathfrak{A}$ and $0 < \bar{\mu}a < \bar{\mu}1$.)

(c)(i) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra. Show that $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \bar{\mu}1)$ is isomorphic to $\mathfrak{A}$. (ii) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless measure algebra and $e \in \mathfrak{A}$ a non-zero element of finite measure. Show that the principal ideal $\mathfrak{A}_e$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \bar{\mu}e)$.

(d) Show that if $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $0 < \gamma \le 1$ and $\kappa \ge \max(\omega, \tau(\mathfrak{A}))$ then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \gamma)$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(\mathfrak{A}, \bar{\mu})$ its amoeba algebra. Show that if $0 < \gamma < \mu X$ then the additivity of μ is not a precaliber of $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$.

(f) Let $(\mathfrak{A},\bar{\mu})$ be an atomless σ-finite measure algebra and $0<\gamma<\bar{\mu}1$. Show that $\mathfrak{m}(\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)) = \mathrm{wdistr}(\mathfrak{A})$.

(g) Let $(\mathfrak{A},\bar{\mu})$ be an atomless semi-finite measure algebra. (i) Show that

$$c(\mathrm{AM}^*(\mathfrak{A},\bar{\mu})) = \mathrm{link}_m(\mathrm{AM}^*(\mathfrak{A},\bar{\mu})) = \max(c(\mathfrak{A}),\tau(\mathfrak{A}))$$

for any integer $m\ge 2$. (ii) Show that

$$d(\mathrm{AM}^*(\mathfrak{A},\bar{\mu})) = \pi(\mathrm{AM}^*(\mathfrak{A},\bar{\mu})) = \max(\mathrm{cf}[c(\mathfrak{A})]^{\le\omega},\pi(\mathfrak{A})).$$

(h) Show that for any cardinal κ there is a probability algebra $(\mathfrak{A},\bar{\mu})$ such that $\mathrm{AM}(\mathfrak{A},\bar{\mu},\frac{1}{2})$ has Maharam type κ.

528Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be an atomless quasi-homogeneous semi-finite measure algebra. Show that $\mathrm{AM}^*(\mathfrak{A},\bar{\mu})$ is homogeneous.

(b) Let $(\mathfrak{A},\bar{\mu})$ be an atomless totally finite measure algebra, and suppose that $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A},\bar{\mu})$ for every $\gamma\in\,]0,\bar{\mu}1[$. Show that $\mathfrak{A}$ is homogeneous.

(c) Show that $\mathfrak{B}_{\omega_1}$ cannot be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\omega,\bar{\nu}_\omega,\frac{1}{2})$.

(d) Let $(\mathfrak{A},\bar{\mu})$ be an atomless probability algebra and $\gamma\in\,]0,1[$. Show that $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)$ is not weakly (σ,∞)-distributive.

(e) Let κ be an infinite cardinal. Show that (i) $\pi(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)) = \mathrm{cf}\,\mathcal{S}_\kappa^\infty$ is the cardinal power κ^ω; (ii) for every $m\ge 2$,

$$c(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)) = c^\uparrow(\mathcal{S}_\kappa^\infty) = \mathrm{link}_m(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)) = \mathrm{link}_m^\uparrow(\mathcal{S}_\kappa^\infty) = \kappa;$$

(iii) $d(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)) = d^\uparrow(\mathcal{S}_\kappa^\infty) = \max(\mathrm{cf}\,\mathcal{N},\mathrm{cf}[\kappa]^{\le\omega})$.

(f) Let $(\mathfrak{A},\bar{\mu})$ be a purely atomic semi-finite measure algebra of cellularity at most $\mathfrak{c}$, and $0<\gamma<\bar{\mu}1$. Show that $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)$ has countable Maharam type.

(g) Let $(\mathfrak{A},\bar{\mu})$ be an atomless semi-finite measure algebra and $0<\gamma<\bar{\mu}1$. Set $\kappa=\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))$ and $P=\{a:a\in\mathfrak{A},\ \bar{\mu}a<\gamma\}$; let $\mathbb{P}$ be the forcing notion $(P,\subseteq,0,\uparrow)$ (see 5A3A). Show that $\Vdash_{\mathbb{P}}\check{\kappa}<\omega_1$.

(h) Show that if $(\mathfrak{A},\bar{\mu})$ is a measure algebra with at most $\mathfrak{c}$ atoms, then $\tau(\mathrm{AM}^*(\mathfrak{A},\bar{\mu}))\le\omega$.

528Z Problems (a) Let $(\mathfrak{A}_L,\bar{\mu}_L)$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. Is the amoeba algebra $\mathrm{AM}(\mathfrak{A}_L,\bar{\mu}_L,1)$ isomorphic to the amoeba algebra $\mathrm{AM}(\mathfrak{B}_\omega,\bar{\nu}_\omega,\frac{1}{2})$?

(b) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$, and $0<\gamma<1$. Is it necessarily true that $\mathrm{AM}(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B},\gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)$? (See 528Xd and 528G.)

528 Notes and comments The ideas of 528A-528K are based on Truss 88. The original amoeba algebras of Martin & Solovay 70, used in their proof that $\mathrm{add}\,\mathcal{N}\ge\mathfrak{m}$ (528L), are closest to 528C. For some more about the amoeba algebras derived from Lebesgue measure, see Bartoszyński & Judah 95, §3.4. In this section I have been willing to assume that the measure algebras involved are atomless; amoeba algebras are surely still interesting for other measure algebras, but the new questions seem to be combinatoric rather than measure-theoretic. It seems still to be unknown whether the algebras $\mathrm{AM}(\mathfrak{A}_L,\bar{\mu}_L,1)$ and $\mathrm{AM}(\mathfrak{B}_\omega,\bar{\nu}_\omega,\frac{1}{2})$ are actually isomorphic, rather than just mutually embeddable (528K, 528Za).

If we think of the partially ordered sets of 528A and 528I as forcing notions, we can study them in terms of the forcing universes they lead to. This is associated with the prominence of 'regular embeddings' in this section. I will not attempt to use such methods here, but I mention them because results such as 528Yg have been part of the impulse for studying amoeba algebras, and led naturally to 528Ya, 528R and 528U-528V.

Version of 26.5.11

529 Further partially ordered sets of measure theory

I end the chapter with notes on some more structures which can be approached by the methods used earlier. The Banach lattices of Chapter 36 are of course partially ordered sets, and many of them can easily be assigned places in the Tukey classification (529C, 529D, 529Xa). More surprising is the fact that the Novák numbers of $\{0,1\}^I$, for large I, are supported by the additivity of Lebesgue measure (529F); this is associated with an interesting property of the localization poset from the last section (529E). There is a similarly unexpected connexion between the covering number of Lebesgue measure and 'reaping numbers' $\mathfrak{r}(\omega_1,\lambda)$ for large λ (529H).

529A Notation As in previous sections, I will write $\mathcal{N}(\mu)$ for the null ideal of μ in a measure space (X,Σ,μ), and $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$.

529B Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra.

(a) For $p\in[1,\infty[$, give $L^p=L^p(\mathfrak{A},\bar\mu)$ (definition: 366A) its norm topology. Then its topological density is

$$\begin{aligned}d(L^p)&=1 \text{ if } \mathfrak{A}=\{0\},\\&=\omega \text{ if } 0<\#(\mathfrak{A})<\omega,\\&=\max(c(\mathfrak{A}),\tau(\mathfrak{A})) \text{ if } \mathfrak{A} \text{ is infinite.}\end{aligned}$$

(b) Give $L^0=L^0(\mathfrak{A})$ its topology of convergence in measure (367L). Then

$$\begin{aligned}d(L^0)&=1 \text{ if } \mathfrak{A}=\{0\},\\&=\omega \text{ if } 0<\#(\mathfrak{A})<\omega,\\&=\tau(\mathfrak{A}) \text{ if } \mathfrak{A} \text{ is infinite.}\end{aligned}$$

proof (a)(i) The case in which $\mathfrak{A}$ is finite is elementary, since in this case $L^p\cong\mathbb{R}^n$, where n is the number of atoms of $\mathfrak{A}$. So henceforth let us suppose that $\mathfrak{A}$ is infinite.

(ii) If $\mathfrak{A}^f$ is the set of elements of $\mathfrak{A}$ of finite measure, we have a natural injection $a\mapsto\chi a:\mathfrak{A}^f\to L^p$, and $\|\chi a-\chi b\|_p=\mu(a\triangle b)^{1/p}$, so χ is a homeomorphism for the measure metric on $\mathfrak{A}^f$ and the norm metric on L^p. It follows that the density $d(A)$ of $A=\{\chi a:a\in\mathfrak{A}^f\}$ for the norm topology is equal to the density of $\mathfrak{A}^f$ for the strong measure-algebra topology, which is $\max(c(\mathfrak{A}),\tau(\mathfrak{A}))$, by 521Eb. So

$$\max(c(\mathfrak{A}),\tau(\mathfrak{A}))=d(A)\le d(L^p)$$

by 5A4B(h-ii). In the other direction, if A_0 is a dense subset of A with cardinal $d(A)$ and D is the set of rational linear combinations of members of A_0, $\overline{D}\supseteq S(\mathfrak{A}^f)$ is dense in L^p (366C), so

$$d(L^p)\le\#(D)\le\max(\omega,\#(A_0))=\max(c(\mathfrak{A}),\tau(\mathfrak{A})).$$

(b) Again, the case of finite $\mathfrak{A}$ is trivial, so we need consider only infinite $\mathfrak{A}$. In this case, $\tau(\mathfrak{A})$ is equal to the topological density $d_{\mathfrak{T}}(\mathfrak{A})$ of $\mathfrak{A}$ with its measure-algebra topology $\mathfrak{T}$ (521Ea).

(i) Let $A\subseteq\mathfrak{A}$ be a topologically dense set of cardinal $\tau(\mathfrak{A})$. Set

$$D=\{\sum_{i=0}^n q_i\chi a_i:q_0,\dots,q_n\in\mathbb{Q},\ a_0,\dots,a_n\in\mathfrak{A}\},$$

so that $D\subseteq L^0$ has cardinal $\tau(\mathfrak{A})$. Because $a\mapsto\chi a:\mathfrak{A}\to L^0$ is continuous (367Ra), the closure $\overline{D}$ of D includes $\{\chi a:a\in\mathfrak{A}\}$. Because $\overline{D}$ is a linear subspace of L^0, it includes $S(\mathfrak{A})$. Because $S(\mathfrak{A})$ is dense in L^0 (367Nc), $\overline{D}=L^0$ and $d(L^0)\le\#(D)=\tau(\mathfrak{A})$.

(ii) Let $B\subseteq L^0$ be a dense set with cardinal $d(L^0)$. Set

$$A=\{[\![u>\tfrac12]\!]:u\in B\},$$

so that $A\subseteq\mathfrak{A}$ and $\#(A)\le d(L^0)$. Then A is topologically dense in $\mathfrak{A}$. **P** If $c\in\mathfrak{A}$, $a\in\mathfrak{A}^f$ and $\epsilon>0$, there is a $u\in B$ such that $\int|u-\chi c|\wedge\chi a\le\frac{1}{2}\epsilon$. But in this case, setting $b=[\![u>\frac{1}{2}]\!]$, $|\chi(b\triangle c)|\le 2|u-\chi c|$, so

$$\bar\mu(a\cap(b\triangle c))\le 2\int|u-\chi c|\wedge\chi a\le\epsilon.$$

As c, a and ϵ are arbitrary, A is topologically dense in $\mathfrak{A}$. **Q**

Accordingly

$$\tau(\mathfrak{A})=d_{\mathfrak{T}}(\mathfrak{A})\le\#(A)\le d(L^0)$$

and $d(L^0)=\tau(\mathfrak{A})$, as claimed.

529C Theorem (Fremlin 91) Let U be an L-space. Then $U\equiv_{\mathrm{T}}\ell^1(\kappa)$, where $\kappa=\dim U$ if U is finite-dimensional, and otherwise is the topological density of U.

proof (a) The finite-dimensional case is trivial, since in this case U and $\ell^1(\kappa)$ are isomorphic as Banach lattices. So henceforth let us suppose that U is infinite-dimensional. Now $\vee:U\times U\to U$ is uniformly continuous. **P** We have only to observe that $u\vee v=\frac{1}{2}(u+v+|u-v|)$ in any Riesz space, that addition and subtraction are uniformly continuous in any linear topological space, and that $u\mapsto|u|$ is uniformly continuous just because $||u|-|v||\le|u-v|$ (see 354B). **Q** So 524C, with $Q=P=U$, tells us that $U\preccurlyeq_{\mathrm{T}}\ell^1(\kappa)$.

The rest of the proof will therefore be devoted to showing that $\ell^1(\kappa)\preccurlyeq_{\mathrm{T}}U$.

(b) By Kakutani's theorem (369E), there is a localizable measure algebra $(\mathfrak{A},\bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1(\mathfrak{A},\bar\mu)$. Let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that $0<\bar\mu a_i<\infty$ and the principal ideal $\mathfrak{A}_{a_i}$ is homogeneous for each i. Set $\kappa_i=\tau(\mathfrak{A}_{a_i})$, so that κ_i is either 0 or infinite for every i, and $\kappa=\max(\#(I),\sup_{i\in I}\kappa_i)$ by 529Ba.

It will simplify the calculations to follow if we arrange that all the a_i have measure 1. To do this, set $\bar\nu a=\sum_{i\in I}\frac{\bar\mu(a\cap a_i)}{\bar\mu a_i}$ for $a\in\mathfrak{A}$; that is, $\bar\nu a=\int_a w\,d\bar\mu$, where $w=\sup_{i\in I}\frac{1}{\bar\mu a_i}\chi a_i$ in $L^0(\mathfrak{A})$. In this case, $\int v\,d\bar\nu=\int v\times w\,d\bar\nu$ for every $v\in L^1(\mathfrak{A},\bar\nu)$, while $\int u\,d\bar\mu=\int\frac{1}{w}\times u\,d\bar\nu$ for every $u\in L^1(\mathfrak{A},\bar\mu)$ (365S[14]). But this means that $u\mapsto u\times\frac{1}{w}$ is a Banach lattice isomorphism between $L^1(\mathfrak{A},\bar\mu)$ and $L^1(\mathfrak{A},\bar\nu)$, and U is isomorphic, as L-space, to $L^1=L^1(\mathfrak{A},\bar\nu)$; while $\bar\nu a_i=1$ for every i.

(c) There are a set J, with cardinal κ, and a family $\langle u_j\rangle_{j\in J}$ in L^1 such that $\#(J)=\kappa$, $\|u_j\|\le 2$ for every $j\in J$ and $\|\sup_{j\in K}u_j\|\ge\frac{1}{2}\sqrt{\#(K)}$ for every finite $K\subseteq J$. **P** Set

$$J=\{(i,0):i\in I,\,\kappa_i=0\}\cup\{(i,\xi):i\in I,\,\xi<\kappa_i\}.$$

Then $\#(J)=\kappa$. If $i\in I$ and $\kappa_i=0$, set $u_{(i,0)}=\chi a_i$. If $i\in I$ and $\kappa_i>0$, then $(\mathfrak{A}_{a_i},\bar\nu\restriction\mathfrak{A}_{a_i})$ is a homogeneous probability algebra with Maharam type $\kappa_i\ge\omega$, so is isomorphic to the measure algebra $(\mathfrak{C}_i,\bar\lambda_i)$ of $]0,1]^{\kappa_i}$ with its usual measure λ_i, the product of Lebesgue measure on each copy of $]0,1]$ (334E). For $\xi<\kappa_i$, set

$$h_{i\xi}(t)=\frac{1}{\sqrt{t(\xi)}}\text{ for }t\in\,]0,1]^{\kappa_i},$$

and let $u_{(i,\xi)}\in L^1$ correspond to $h^\bullet_{i\xi}\in L^1(\lambda_i)\cong L^1(\mathfrak{C}_i,\bar\lambda_i)$ (365B). Of course

$$\|u_{(i,\xi)}\|=\int h_{i\xi}(t)\lambda_i(dt)=\int_0^1\frac{1}{\sqrt{\alpha}}d\alpha$$

(because the coordinate map $t\mapsto t(\xi)$ is inverse-measure-preserving)

$$=2.$$

If $L\subseteq\kappa_i$ is finite and not empty, then $\|\sup_{\xi\in L}u_{(i,\xi)}\|=\int g\,d\lambda_i$ where $g=\sup_{\xi\in L}h_{i\xi}$, that is, $g(t)=\sup_{\xi\in L}\frac{1}{\sqrt{t(\xi)}}$ for $t\in\,]0,1]^{\kappa_i}$. Now, for any $\alpha\ge1$,

[14] Formerly 365T.

$$\lambda_i\{t : g(t) \le \alpha\} = \lambda_i\{t : \alpha^2 t(\xi) \ge 1 \text{ for every } \xi \in L\}$$
$$= (1 - \frac{1}{\alpha^2})^{\#(L)} \le \max(\frac{1}{2}, 1 - \frac{1}{2\alpha^2}\#(L))$$

(induce on $\#(L)$)

$$= 1 - \frac{1}{2\alpha^2}\#(L) \text{ if } \alpha \ge \sqrt{\#(L)}.$$

So

$$\|\sup_{\xi\in L} u_{(i,\xi)}\| = \int g\,d\lambda_i = \int_0^\infty \lambda_i\{t : g(t) > \alpha\}d\alpha$$

(252O)

$$\ge \int_{\sqrt{\#(L)}}^\infty \lambda_i\{t : g(t) > \alpha\}d\alpha$$
$$\ge \int_{\sqrt{\#(L)}}^\infty \frac{1}{2\alpha^2}\#(L)d\alpha = \frac{1}{2}\sqrt{\#(L)}.$$

What this means is that if $K \subseteq J$ is finite and all the first coordinates of members of K are the same, then $\|\sup_{j\in K} u_j\| \ge \frac{1}{2}\sqrt{\#(K)}$. In general, if $K \subseteq J$ is finite, then for each $i \in I$ set $L_i = \{\xi : (i,\xi) \in K\}$. Set $v_i = 0$ if L_i is empty, $\sup_{\xi\in L_i} u_{(i,\xi)}$ otherwise, so that $\|v_i\| \ge \frac{1}{2}\sqrt{\#(L_i)}$; now $\sup_{j\in K} u_j = \sum_{i\in I} v_i$, so

$$\|\sup_{j\in K} u_j\| = \sum_{i\in I}\|v_i\| \ge \frac{1}{2}\sum_{i\in I}\sqrt{\#(L_i)} \ge \frac{1}{2}\sqrt{\sum_{i\in I}\#(L_i)} = \frac{1}{2}\sqrt{\#(K)},$$

as required. **Q**

(d) We can now apply the idea of the proof of 524C, as follows. The density of $\ell^1(\kappa)$ is of course κ, by 529Ba applied to counting measure on κ, or otherwise. Index a dense subset of $\ell^1(\kappa)$ as $\langle y_j\rangle_{j\in J}$. For each $x \in \ell^1$, choose a sequence $\langle k(x,n)\rangle_{n\in\mathbb{N}}$ in J such that

$$\|x - \sum_{m=0}^n y_{k(x,m)}\| \le 8^{-n}$$

for every n. Note that

$$\|y_{k(x,n)}\| \le \|x - \sum_{m=0}^n y_{k(x,m)}\| + \|x - \sum_{m=0}^{n-1} y_{k(x,m)}\| \le 9\cdot 8^{-n}$$

for each n. Choose $f(x) \in L^1$ such that $\|f(x)\| \ge \|x\|$ and $f(x) \ge \sum_{n=0}^\infty 2^{-n}u_{k(x,n)}$; this is possible because $\{u_{k(x,n)} : n \in \mathbb{N}\}$ is bounded.

(e) f is a Tukey function. **P** Take $v \in L^1$ and set

$$A = \{x : f(x) \le v\}, \quad K_n = \{k(x,n) : x \in A\}$$

for $n \in \mathbb{N}$. Fix n for the moment. If $j \in K_n$, then there is an $x \in A$ such that $j = k(x,n)$ and

$$u_j = u_{k(x,n)} \le 2^n f(x) \le 2^n v,$$

while $\|y_j\| = \|y_{k(x,n)}\| \le 9\cdot 8^{-n}$. If $K \subseteq K_n$ is finite, $\|2^n v\| \ge \frac{1}{2}\sqrt{\#(K)}$, by (c); so $\#(K_n) \le 2^{2n+2}\|v\|^2$. This means that if we set $z_n = \sum_{j\in K_n}|y_j|$ we shall have $\|z_n\| \le 9\cdot 8^{-n}\#(K_n) \le 36\cdot 2^{-n}\|v\|^2$, while $y_{k(x,n)} \le z_n$ for every $x \in A$.

Now $z = \sum_{n=0}^\infty z_n$ is defined in $\ell^1(\kappa)$, and if $x \in A$ then $\sum_{m=0}^n y_{k(x,m)} \le z$ for every $n \in \mathbb{N}$, so that $x \le z$. Thus A is bounded above in $\ell^1(\kappa)$. As v is arbitrary, f is a Tukey function. **Q**

(f) Accordingly $\ell^1(\kappa) \preccurlyeq_{\mathrm{T}} L^1 \cong U$, and the proof is complete.

529D Theorem (Fremlin 91) Let $\mathfrak{A}$ be a homogeneous measurable algebra with Maharam type $\kappa \ge \omega$. Then $L^0(\mathfrak{A}) \equiv_{\mathrm{T}} \ell^1(\kappa)$.

proof (a) Let $\bar{\mu}$ be such that $(\mathfrak{A},\bar{\mu})$ is a probability algebra. If we give $L^0=L^0(\mathfrak{A})$ its topology of convergence in measure, its density is κ, by 529Bb. Moreover, this topology is defined by the metric $(u,v)\mapsto\int|u-v|\wedge\chi 1$, under which the lattice operation $\vee$ is uniformly continuous. **P** Just as in part (a) of the proof of 529C, we have $u\vee v=\frac{1}{2}(u+v+|u-v|)$ for all u and v, addition and subtraction are uniformly continuous, and $u\mapsto|u|$ is uniformly continuous. **Q** So, just as in 529C, we can use 524C to see that $L^0\preccurlyeq_{\mathrm{T}}\ell^1(\kappa)$.

(b) For the reverse connection, I repeat ideas from the proof of 529C. $(\mathfrak{A},\bar{\mu})$ is isomorphic to the measure algebra $(\mathfrak{C},\bar{\lambda})$ of $]0,1]^\kappa$ with its usual measure λ. For $\xi<\kappa$ and $t\in\,]0,1]^\kappa$ set $h_\xi(t)=\dfrac{1}{\sqrt{t(\xi)}}$, and set $u_\xi=h_\xi^\bullet$ in $L^0(\lambda)\cong L^0(\mathfrak{C})$ (364Ic). This time, observe that if $x\in\ell^1(\kappa)^+$ and $\alpha\ge\sqrt{\|x\|}$ then

$$\begin{aligned}\bar{\lambda}[\![\sup_{\xi<\kappa}\sqrt{x(\xi)}u_\xi\le\alpha]\!]&=\bar{\lambda}(\inf_{\xi<\kappa}[\![\sqrt{x(\xi)}u_\xi\le\alpha]\!])=\prod_{\xi<\kappa}\bar{\lambda}[\![\sqrt{x(\xi)}u_\xi\le\alpha]\!]\\&=\prod_{\xi<\kappa}\lambda\{t:\sqrt{\frac{x(\xi)}{t(\xi)}}\le\alpha\}=\prod_{\xi<\kappa}(1-\frac{x(\xi)}{\alpha^2})\\&\ge 1-\frac{1}{\alpha^2}\sum_{\xi<\kappa}x(\xi)\to 0\end{aligned}$$

as $\alpha\to\infty$. This means that $\sup_{\xi<\kappa}\sqrt{x(\xi)}u_\xi$ is defined in $L^0(\lambda)$ (364La). So we can define $f:\ell^1(\kappa)\to L^0(\lambda)$ by saying that $f(x)=\sup_{\xi<\kappa}\sqrt{\max(0,x(\xi))}u_\xi$ for every $x\in\ell^1(\kappa)$.

(c) f is a Tukey function. **P** Take $v\in L^0(\lambda)^+$, and set $A=\{x:f(x)\le v\}$. Note that $f(x\vee x')=f(x)\vee f(x')$ for all x, $x'\in\ell^1(\kappa)$, so A is upwards-directed. Take $\alpha>0$ such that $\bar{\lambda}[\![v\le\alpha]\!]=\beta>\frac{1}{2}$. If $x\in A$ and $x\ge 0$ then $f(x)\ge\sqrt{x(\xi)}\chi 1$ so $x(\xi)\le\alpha$ for every ξ. Now the calculation in (b) tells us that

$$\begin{aligned}\beta\le\bar{\lambda}[\![\sup_{\xi<\kappa}\sqrt{x(\xi)}u_\xi\le\alpha]\!]&=\prod_{\xi<\kappa}(1-\frac{1}{\alpha^2}x(\xi))\\&\le\max(\frac{1}{2},1-\frac{1}{2\alpha^2}\sum_{\xi<\kappa}x(\xi))=\max(\frac{1}{2},1-\frac{1}{2\alpha^2}\|x\|),\end{aligned}$$

so $\|x\|\le 2\alpha^2(1-\beta)$. As A is upwards-directed and norm-bounded and contains 0, it is bounded above in $\ell^1(\kappa)$ (354N). As v is arbitrary, f is a Tukey function. **Q**

(d) Accordingly $\ell^1(\kappa)\preccurlyeq_{\mathrm{T}}L^0(\lambda)\cong L^0(\mathfrak{A})$ and $\ell^1(\kappa)$ and $L^0(\mathfrak{A})$ are Tukey equivalent.

529E Proposition Let $\mathcal{S}^\infty$ be the $(\mathbb{N},\infty)$-localization poset (528I). Then $\mathrm{RO}(\{0,1\}^{\mathfrak{c}})$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty)$.

proof (a) Let $\langle h_\xi\rangle_{\xi<\mathfrak{c}}$ be a family of eventually-different functions in $\mathbb{N}^\mathbb{N}$ (5A1Nc). Set

$$\begin{aligned}W_0=\bigcup_{n\in\mathbb{N}\text{ is even}}&\{(h,p):h\in\mathbb{N}^\mathbb{N},\,p\in\mathcal{S}^\infty,\,\#(p[\{n\}])=2^n,\\&\qquad(n,h(n))\notin p,\,(i,h(i))\in p\text{ for every }i>n\}\\&\cup\{(h,p):h\in\mathbb{N}^\mathbb{N},\,p\in\mathcal{S}^\infty,\,(i,h(i))\in p\text{ for every }i\in\mathbb{N}\},\\W_1=\bigcup_{n\in\mathbb{N}\text{ is odd}}&\{(h,p):h\in\mathbb{N}^\mathbb{N},\,p\in\mathcal{S}^\infty,\,\#(p[\{n\}])=2^n,\\&\qquad(n,h(n))\notin p,\,(i,h(i))\in p\text{ for every }i>n\}.\end{aligned}$$

Observe that (i) $W_0\cap W_1=\emptyset$ (ii) if $(h,p)\in W_j$, where $j=0$ or $j=1$, and $p\subseteq q\in\mathcal{S}^\infty$ then $(h,q)\in W_j$ (iii) if $p\in\mathcal{S}^\infty$ then

$$\#(\{\xi:(h_\xi,p)\in W_0\cup W_1\})\le\|p\|$$

is finite.

(b) Set $Q=\mathrm{Fn}_{<\omega}(\mathfrak{c},\{0,1\})$, the set of functions from finite subsets of $\mathfrak{c}$ to $\{0,1\}$, ordered by extension of functions, so that $(Q,\subseteq)$ is isomorphic to $(\mathcal{U},\supseteq)$ where $\mathcal{U}$ is the usual base of the topology of $\{0,1\}^{\mathfrak{c}}$, and $\mathrm{RO}^{\uparrow}(Q)\cong\mathrm{RO}^{\downarrow}(\mathcal{U})$ can be identified with the regular open algebra of $\{0,1\}^{\mathfrak{c}}$ (514Sd). Define $f:\mathcal{S}^\infty\to Q$ by setting $f(p)(\xi)=j$ if $(h_\xi,p)\in W_j$. Then f is order-preserving.

(c) For $p\in\mathcal{S}^\infty$, set

$$A_0(p)=\{\xi:\xi<\mathfrak{c},\,\{n:n\text{ is even},\,(n,h_\xi(n))\notin p\}\text{ is finite}\},$$

$$A_1(p)=\{\xi:\xi<\mathfrak{c},\,\{n:n\text{ is odd},\,(n,h_\xi(n))\notin p\}\text{ is finite}\},$$

$$A(p)=A_0(p)\cup A_1(p),$$

so that $A(p)$ is finite and $\mathrm{dom}\,f(p)\subseteq A(p)$. Now $P_1=\{p:p\in\mathcal{S}^\infty,\,A(p)=\mathrm{dom}\,f(p)\}$ is cofinal with $\mathcal{S}^\infty$. **P** Take $p\in\mathcal{S}^\infty$. Let m be such that

$$2^m\ge\|p\|+\#(A(p)),$$

$$(n,h_\xi(n))\in p\text{ whenever }\xi\in A_0(p)\text{ and }n>m\text{ is even},$$

$$(n,h_\xi(n))\in p\text{ whenever }\xi\in A_1(p)\text{ and }n>m\text{ is odd}.$$

Let $p'\in\mathcal{S}^\infty$ be such that

$$\text{for }n\le m,\ p'[\{n\}]\supseteq p[\{n\}]\text{ and }\#(p'[\{n\}])=2^n,$$

$$\text{for }n>m,\ p'[\{n\}]=p[\{n\}]\cup\{h_\xi(n):\xi\in A(p)\}.$$

Then $p\le p'$ and $A(p')=A(p)$. Also $A(p)=\mathrm{dom}\,f(p')$, because if $\xi\in A(p)$ then either $(n,h_\xi(n))\in p'$ for every n and $(h_\xi,p')\in W_0$, or there is a largest n such that $(n,h_\xi(n))\notin p'$, in which case $n\le m$ and $\#(p'[\{n\}])=2^n$, so (h_ξ,p') belongs to W_0 if n is even and W_1 otherwise. **Q**

(d) If $p\in P_1$ and $q\in Q$ extends $f(p)$, there is a $p_1\in\mathcal{S}^\infty$ such that $p_1\supseteq p$ and $f(p_1)=q$. **P** Let m be such that $2^m\ge\|p\|+\#(\mathrm{dom}\,q)$ and $h_\xi(n)\ne h_\eta(n)$ whenever $\xi,\eta\in\mathrm{dom}\,q$ are distinct and $n\ge m$. For each $\xi\in\mathrm{dom}\,q\setminus\mathrm{dom}\,f(p)=\mathrm{dom}\,q\setminus A(p)$, $\{n:n\text{ is even},\,(n,h_\xi(n))\notin p\}$ and $\{n:n\text{ is odd},\,(n,h_\xi(n))\notin p\}$ are both infinite. So we can find $m'\ge m$ such that all these sets meet $m'\setminus m$. Set

$$\begin{aligned}p'=p\cup\{(n,h_\xi(n)):n\in m'\setminus m\text{ is odd},\,q(\xi)=0\}\\ \cup\{(n,h_\xi(n)):n\in m'\setminus m\text{ is even},\,q(\xi)=1\}\\ \cup\{(n,h_\xi(n)):n\in\mathbb{N}\setminus m',\,\xi\in\mathrm{dom}\,q\},\end{aligned}$$

so that $p\subseteq p'\in\mathcal{S}^\infty$. Let $p_1\in\mathcal{S}^\infty$ be such that $p_1\supseteq p'$, $p_1\setminus p'$ is finite, $\#(p_1[\{n\}])=2^n$ for every $n<m'$ and $(n,h_\xi(n))\notin p_1\setminus p'$ whenever $n\in\mathbb{N}$ and $\xi\in\mathrm{dom}\,q$. Now $f(p_1)=q$, while $p\subseteq p_1$. **Q**

(e) Putting (c) and (d) together, we see that $f^{-1}[Q_0]$ must be cofinal with $\mathcal{S}^\infty$ for every cofinal $Q_0\subseteq Q$; moreover, since $\emptyset\in P_1$ and $f(\emptyset)$ is the empty function, $f[\mathcal{S}^\infty]=Q$. So f satisfies the conditions of 514O, and $\mathrm{RO}^{\uparrow}(Q)\cong\mathrm{RO}(\{0,1\}^{\mathfrak{c}})$ can be regularly embedded in $\mathrm{RO}^{\uparrow}(\mathcal{S}^\infty)$.

529F Corollary (Brendle 00, 2.3.10; Brendle 06, Theorem 1) $n(\{0,1\}^I)\ge\mathrm{add}\,\mathcal{N}$ for every set I.

proof If I is finite, this is trivial. Otherwise, write $\lambda=n(\{0,1\}^I)$. Then $\lambda\ge n(\{0,1\}^{\mathfrak{c}})$. **P** If $J\subseteq I$ is a countably infinite set, then $\{\{x:x\restriction J=z\}:z\in\{0,1\}^J\}$ is a cover of $\{0,1\}^I$ by continuum many nowhere dense sets, so $\lambda\le\mathfrak{c}$. Let $\langle E_\xi\rangle_{\xi<\lambda}$ be a cover of $\{0,1\}^\kappa$ by nowhere dense sets. Then each E_ξ is included in a nowhere dense closed set F_ξ determined by coordinates in a countable set $K_\xi\subseteq I$ (4A2E(b-iii)). Set $K=\bigcup_{\xi<\lambda}K_\xi$, so that $\#(K)\le\mathfrak{c}$. Then all the projections $F'_\xi=\{x\restriction K:x\in F_\xi\}$ are nowhere dense in $\{0,1\}^K$ (apply 4A2B(f-ii) to the continuous open surjections $x\mapsto x\restriction K_\xi:\{0,1\}^I\to\{0,1\}^{K_\xi}$ and $y\mapsto y\restriction K_\xi:\{0,1\}^K\to\{0,1\}^K_\xi$), and they cover $\{0,1\}^K$. Next, we have an injection $\phi:K\to\mathfrak{c}$, and the sets $F''_\xi=\{x:x\phi\in F'_\xi\}$ form a cover of $\{0,1\}^{\mathfrak{c}}$ by nowhere dense sets; so $n(\{0,1\}^{\mathfrak{c}})\le\lambda$. **Q**

Because every non-empty open set $\{0,1\}^{\mathfrak{c}}$ includes an open set homeomorphic to $\{0,1\}^{\mathfrak{c}}$,

$$n(\{0,1\}^{\mathfrak{c}}) = \min\{n(H) : H \subseteq \{0,1\}^{\mathfrak{c}} \text{ is open and not empty}\}$$
$$= \mathfrak{m}(\mathrm{RO}(\{0,1\}^{\mathfrak{c}}))$$

(517J)

$$\geq \mathfrak{m}(\mathrm{RO}^{\uparrow}(\mathcal{S}^{\infty}))$$

(where $\mathcal{S}^{\infty}$ is the $(\mathbb{N},\infty)$-localization poset, by 529E and 517Ia)

$$= \operatorname{add}\mathcal{N}$$

by 528N.

529G Reaping numbers (following Brendle 00) For cardinals $\theta \leq \lambda$ let $\mathfrak{r}(\theta,\lambda)$ be the smallest cardinal of any set $\mathcal{A} \subseteq [\lambda]^{\theta}$ such that for every $B \subseteq \lambda$ there is an $A \in \mathcal{A}$ such that either $A \subseteq B$ or $A \cap B = \emptyset$.

529H Proposition (Brendle 00, 2.7; Brendle 06, Theorem 5) $\mathfrak{r}(\omega_1,\lambda) \geq \operatorname{cov}\mathcal{N}$ for all uncountable λ.

proof Let $\langle A_\xi\rangle_{\xi<\kappa}$ be a family in $[\lambda]^{\omega_1}$, where $\kappa < \operatorname{cov}(\mathcal{N})$. I seek a $B \subseteq \lambda$ such that $A_\xi \cap B$ and $A_\xi \setminus B$ are non-empty for every $\xi < \kappa$.

(a) If $\kappa \leq \omega_1$, then choose $\langle\alpha_\xi\rangle_{\xi<\kappa}$ and $\langle\beta_\xi\rangle_{\xi<\kappa}$ inductively so that

$$\alpha_\xi \in A_\xi \setminus \{\beta_\eta : \eta < \xi\}, \quad \beta_\xi \in A_\xi \setminus \{\alpha_\eta : \eta \leq \xi\}$$

for every $\xi < \kappa$, and set $B = \{\beta_\xi : \xi < \kappa\}$; this serves. So henceforth let us suppose that $\kappa > \omega_1$.

(b) For each $\xi < \kappa$ let $A'_\xi \subseteq A_\xi$ be a set of order type ω_1. For each n, let X_n be a set with cardinal $n!$ with its discrete topology and the uniform probability measure which gives measure $\frac{1}{n!}$ to every singleton. Give $X = \prod_{n\in\mathbb{N}} X_n$ its product measure μ and its product topology. Because X is a compact metrizable space and μ is a Radon measure (416U), $\operatorname{cov}\mathcal{N}(\mu) = \operatorname{cov}\mathcal{N}$ (522Wa). We can therefore choose a family $\langle x_\xi\rangle_{\xi<\kappa}$ in X in such a way that each x_ζ is random over its predecessors in the sense that

$$\text{whenever } \xi < \kappa \text{ and } \overline{\{x_\eta : \eta \in A'_\xi \cap \zeta\}} \text{ is negligible, it does not contain } x_\zeta.$$

For distinct x, $y \in X$, set $\Delta(x,y) = \min\{i : x(i) \neq y(i)\}$. For $x \in X$, set $B(x) = \{\eta : x_\eta \neq x,\ \Delta(x_\eta,x)$ is even$\}$.

(c) For every $\xi < \kappa$, $\{x : x \in X,\ A_\xi \subseteq B(x)\}$ and $\{x : x \in X,\ A_\xi \cap B(x) = \emptyset\}$ are negligible. **P** There is a $\zeta < \kappa$ such that $\zeta \in A'_\xi$, $A'_\xi \cap \zeta$ is countable and $D = \{x_\eta : \eta \in A'_\xi \cap \zeta\}$ is dense in $\{x_\eta : \eta \in A'_\xi\}$. Since $x_\zeta \in \overline{D}$, $\overline{D}$ has measure greater than 0. By 275I, applied to the sequence $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$ where Σ_n is the finite algebra of subsets of X determined by coordinates less than n, $\overline{D}$ has a point w which is a density point in the sense that

$$\lim_{n\to\infty} \frac{\mu\{y : y\restriction n = w\restriction n,\, y \in \overline{D}\}}{\mu\{y : y\restriction n = w\restriction n\}} = 1.$$

Consequently, setting

$$J_n = \{y(n) : y \in D\} = \{y(n) : y \in \overline{D}\} \supseteq \{y(n) : y \in \overline{D},\ y\restriction n = w\restriction n\},$$

$$\frac{\#(J_n)}{n!} \geq \frac{\mu\{y : y\restriction n = w\restriction n,\, y \in \overline{D}\}}{\mu\{y : y\restriction n = w\restriction n\}} \to 1$$

as $n \to \infty$.

Next note that, for any $y \in X$ and $n \in \mathbb{N}$,

$$\mu\{x : \exists\, i > n,\ x(i) = y(i)\} \leq \sum_{i=n+1}^{\infty} \frac{1}{i!} \leq \frac{n+2}{(n+1)(n+1)!}.$$

So

$$\mu\{x : \exists\, y \in D, x(n) = y(n),\ x(i) \neq y(i) \text{ for every } i > n\}$$
$$\geq \tfrac{\#(J_n)}{n!}\bigl(1 - \tfrac{n+2}{(n+1)(n+1)!}\bigr) \to 1$$

as $n \to \infty$, and

$$\mu\{x : \exists\, y \in D \setminus \{x\},\ \Delta(x,y) \text{ is even}\} = \mu\{x : \exists\, y \in D \setminus \{x\},\ \Delta(x,y) \text{ is odd}\} = 1.$$

But if $y \in D \setminus \{x\}$ and $\Delta(x,y)$ is even, we have an $\eta \in A_\xi$ such that $\Delta(x,x_\eta)$ is even, and $\eta \in A_\xi \cap B(x)$; similarly, if there is a $y \in D \setminus \{x\}$ such that $\Delta(x,y)$ is odd, there is an $\eta \in A_\xi \setminus B(x)$. So $\{x : A_\xi \subseteq B(x)\}$ and $\{x : A_\xi \cap B(x) = \emptyset\}$ are both negligible. **Q**

(d) Since $\operatorname{cov}\mathcal{N}(\mu) = \operatorname{cov}\mathcal{N} > \kappa$, there is an $x \in X$ such that both $A_\xi \cap B(x)$ and $A_\xi \setminus B(x)$ are non-empty for every $\xi < \kappa$. So in this case also we have a suitable set B.

529X Basic exercises (a) Let (X,Σ,μ) be a measure space, and $p \in [1,\infty[$. Show that $L^p(\mu) \equiv_{\mathrm{T}} \ell^1(\kappa)$, where $\kappa = \dim L^p(\mu)$ if this is finite, $d(L^p(\mu))$ otherwise.

(b) Let U be an L-space. (i) Show that $\operatorname{add} U = \infty$ if $U = \{0\}$, ω otherwise. (ii) Show that $\operatorname{add}_\omega U = \infty$ if U is finite-dimensional, $\operatorname{add}\mathcal{N}$ if U is separable and infinite-dimensional, ω_1 otherwise. (iii) Show that $\operatorname{cf} U = 1$ if $U = \{0\}$, ω if $0 < \dim U < \omega$, $\max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[d(U)]^{\leq\omega})$ otherwise. (iv) Show that $\operatorname{link}^{\uparrow}_{<\kappa}(U) = 1$ if $\kappa \leq \omega$, $\operatorname{cf} U$ otherwise.

(c) Let U be a separable Banach lattice. Suppose that $\langle u_\xi\rangle_{\xi<\kappa}$ is a family in U, where $\kappa < \operatorname{add}\mathcal{N}$. Show that there is a family $\langle \epsilon_\xi\rangle_{\xi<\kappa}$ of strictly positive real numbers such that $\{\epsilon_\xi u_\xi : \xi < \kappa\}$ is order-bounded in U.

>(d) Let U be a separable Banach lattice, and $D \subseteq U$ a dense set. Let $A \subseteq U$ be a set with cardinal less than $\operatorname{add}\mathcal{N}$. Show that there is a $w \in U$ such that for every $u \in A$ and every $\epsilon > 0$ there is a $v \in D$ such that $|u - v| \leq \epsilon w$.

(e) Let $\mathcal{I}$ be the ideal of subsets I of $\mathbb{N}$ such that $\sum_{n\in I}\frac{1}{n+1}$ is finite. (See 419A.) Show that $\ell^1 \equiv_{\mathrm{T}} \mathcal{I}$, so that $\operatorname{add}_\omega \mathcal{I} = \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{I} = \operatorname{cf}\mathcal{N}$.

(f) Show that if $\theta \leq \theta' \leq \lambda' \leq \lambda$ are cardinals, then $\mathfrak{r}(\theta,\lambda) \leq \mathfrak{r}(\theta',\lambda')$.

(g)(i) Show that $\mathfrak{r}(\omega,\omega) \geq \operatorname{cov}\mathcal{E} \geq \max(\operatorname{cov}\mathcal{N}, \mathfrak{m}_{\text{countable}})$, where $\mathcal{E}$ is the ideal of subsets of $\mathbb{R}$ with Lebesgue negligible closures. (ii) Show that if λ is an infinite cardinal then $\mathfrak{r}(\omega,\lambda) \geq \max(\operatorname{add}\mathcal{N}, \operatorname{cov}\mathcal{N}_\lambda)$, where $\mathcal{N}_\lambda$ is the null ideal of the usual measure on $\{0,1\}^\lambda$. (*Hint*: 529F.)

529Y Further exercises (a) Let X be a Polish space and $\mathcal{K}_\sigma$ the family of K_σ subsets of X. Show that, defining $\leq^*$ as in 522C, $(\mathcal{K}_\sigma, \subseteq) \preccurlyeq_{\mathrm{T}} (\mathbb{N}^{\mathbb{N}}, \leq^*)$.

(b) Let X be a topological space with a countable network, and $c : \mathcal{P}X \to [0,\infty]$ an outer regular submodular Choquet capacity (definitions: 432J). Show that if $\mathcal{A}$ is an upwards-directed family of subsets of X such that $\#(\mathcal{A}) < \mathfrak{m}_{\sigma\text{-linked}}$, then $c(\bigcup\mathcal{A}) = \sup_{A\in\mathcal{A}} c(A)$.

(c) Let $r \geq 3$ be an integer. (i) Let $c : \mathcal{P}\mathbb{R}^r \to [0,\infty]$ be Choquet-Newton capacity (§479). Show that if $\mathcal{A}$ is an upwards-directed family of subsets of $\mathbb{R}^r$ such that $\#(\mathcal{A}) < \operatorname{add}\mathcal{N}$, then $c(\bigcup\mathcal{A}) = \sup_{A\in\mathcal{A}} c(A)$. (*Hint*: 479Xi.) (ii) Let $\mathcal{I}$ be the ideal of polar sets in $\mathbb{R}^r$. Show that $\operatorname{add}\mathcal{I} = \operatorname{add}\mathcal{N}$.

(d) Show that, for any infinite set I, the regular open algebra $\mathrm{RO}(\{0,1\}^I)$ of $\{0,1\}^I$ is homogeneous, so that $\mathfrak{m}(\mathrm{RO}(\{0,1\}^I)) = n(\{0,1\})^I$.

(e) Show that $\mathfrak{b} \leq \mathfrak{r}(\omega,\omega) \leq \pi(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})$.

529 Notes and comments Many of the ideas of the last two chapters were first embodied in forcing arguments. In 529E this becomes particularly transparent. If we have an upwards-directed set $R \subseteq \mathcal{S}^\infty$ which is 'generic' in the sense that it meets all the cofinal subsets of $\mathcal{S}^\infty$ definable in a language $\mathcal{L}$ with terms for all the functions h_ξ, as well as such obvious ones as $\{p : \#(p[\{n\}]) = 2^n\}$ for each n, and we set $S = \bigcup \mathcal{R}$, then S will belong to the set $\mathcal{S} = \mathcal{S}_\mathbb{N}$ of 522K, and we shall have $h_\xi \subseteq^* S$ for every ξ; so that we have a corresponding function $\tilde{f}(S) = \bigcup_{p \in R} f(p) \in \{0,1\}^\mathfrak{c}$ defined by setting

$$\tilde{f}(S)(\xi) \equiv \sup\{i : (i, h_\xi(i)) \notin S\} \mod 2.$$

Next, if $G \subseteq \{0,1\}^\mathfrak{c}$ is a dense open set with a definition in $\mathcal{L}$, then $\tilde{f}(S) \in G$; for, setting $U_q = \{\phi : q \subseteq \phi \in \{0,1\}^\mathfrak{c}\}$ when $q \in Q = \mathrm{Fn}_{<\omega}(\mathfrak{c}, \{0,1\})$, $\{q : U_q \subseteq G\}$ is cofinal with Q, so $\{p : U_{f(p)} \subseteq G\}$ is cofinal with $\mathcal{S}^\infty$ (part (d) of the proof of 529E) and meets R. Thus $\tilde{f}(S)$ is 'generic' in the sense that it belongs to every dense open set with a name in $\mathcal{L}$; and it is a commonplace in the theory of forcing that a function which transforms generic objects in one forcing extension into generic objects in another extension corresponds to a regular embedding of the corresponding regular open algebras.

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References for Volume 5

Argyros S.A. & Kalamidas N.D. [82] 'The $K_{\alpha n}$ property on spaces with strictly positive measure', Canadian J. Math. 34 (1982) 1047-1058. [525T.]

Argyros S.A. & Tsarpalias A. [82] 'Calibers of compact spaces', Trans. Amer. Math. Soc. 270 (1982) 149-162. [525N.]

Babiker A.G. [76] 'On uniformly regular topological measure spaces', Duke Math. J. 43 (1976) 775-789. [533Xd.]

Balcar B. & Franěk F. [82] 'Independent families in complete Boolean algebras', Trans. Amer. Math. Soc. 274 (1982) 607-618. [515H.]

Balcar B., Jech T. & Pazák T. [05] 'Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann', Bull. London Math. Soc. 37 (2005) 885-898. [§539 *intro.*, 539E, 539L, 539N, §539 *notes*, 555K.]

Balcar B., Jech T. & Zapletal J. [97] 'Semi-Cohen Boolean algebras', Ann. Pure & Applied Logic 87 (1997) 187-208. [§515 *notes*, §547 *notes*.]

Balcar B. & Vojtáš P. [77] 'Refining systems on Boolean algebras', pp. 45-58 in Lachlan Srebny & Zarach 77. [515E.]

Bar-Hillel Y. [70] *Mathematical Logic and Foundations of Set Theory.* North-Holland, 1970.

Bartoszyński T. [84] 'Additivity of measure implies additivity of category', Trans. Amer. Math. Soc. 281 (1984) 209-213. [Chap. 52 *intro.*, 522Q, §524 *notes*.]

Bartoszyński T. [87] 'Combinatorial aspects of measure and category', Fund. Math. 127 (1987) 225-239. [522S.]

Bartoszyński T. & Judah H. [89] 'On the cofinality of the smallest covering of the real line by meager sets', J. Symbolic Logic 54 (1989) 828-832. [522V.]

Bartoszyński T. & Judah H. [95] *Set Theory: on the structure of the real line.* A.K.Peters, 1995. [522R, §522 *notes*, 526M, §528 *notes*, 534Q, §534 *notes*, 546I, 553C, 554Ya, 5A1Q.]

Bartoszyński T. & Shelah S. [92] 'Closed measure zero sets', Ann. Pure & Applied Logic 58 (1992) 93-110. [526M.]

Baumgartner J.E., Taylor A.D. & Wagon S. [77] 'On splitting stationary subsets of large cardinals', J. Symbolic Logic 42 (1977) 203-214. [§541 *notes*, §544 *notes*.]

Bell M.G. [81] 'On the combinatorial principle $P(c)$', Fundamenta Math. 114 (1981) 149-157. [§517 *notes*.]

Benedikt M. [98] 'Ultrafilters which extend measures', J. Symbolic Logic 63 (1998) 638-662. [538M, 538Xm, §538 *notes*.]

Benedikt M. [99] 'Hierarchies of measure-theoretic ultrafilters', Annals of Pure and Applied Logic 97 (1999) 203-219. [538H.]

Blair C.E. [77] 'The Baire category theorem implies the principle of dependent choice', Bull. Acad. Polon. Sci. (Math. Astron. Phys.) 25 (1977) 933-934. [§566 *notes*.]

Blass A. & Laflamme C. [89] 'Consistency results about filters and the number of inequivalent growth types', J. Symbolic Logic 54 (1989) 50-56. [5A6Ib, 5A6J.]

Booth D. [70] 'Ultrafilters on a countable set', Ann. Math. Logic 2 (1970) 1-24. [517R.]

Borel E. [1919] 'Sur la classification des ensembles de mesure nulle', Bull. Soc. Math. France 47 (1919) 97-125. [§534 *intro*.]

Brendle J. [00] *Inner model theory and large cardinals.* Research report, March 2000. [528N, 529F-529H.]

Brendle J. [06] 'Cardinal invariants of the continuum and combinatorics on uncountable cardinals', Annals of Pure and Applied Logic 144 (2006) 43-72. [529F, 529H.]

Burke M.R. [93] 'Liftings for Lebesgue measure', pp. 119-150 in Judah 93. [§535 *notes*.]

Burke M.R. [n05] 'Non-null sets of minimal cofinality', note of 2.3.05. [523K.]

Burke M.R. & Magidor M. [90] 'Shelah's pcf theory and its applications', Ann. Pure and Applied Logic 50 (1990) 207-254. [513J, §5A2 *intro*..]

Burke M.R. & Shelah S. [92] 'Linear liftings for non-complete probability spaces', Israel J. Math. 79 (1992) 289-296. [§535 *notes*.]

Carlson T.J. [84] 'Extending Lebesgue measure to infinitely many sets', Pacific J. Math. 115 (1984) 33-45. [552N, §552 *notes*.]

Carlson T.J. [93] 'Strong measure zero and strongly meager sets', Proc. Amer. Math. Soc. 118 (1993) 577-586. [534R.]

Carlson T., Frankiewicz R. & Zbierski P. [94] 'Borel liftings of the measure algebra and the failure of the continuum hypothesis', Proc. Amer. Math. Soc. 120 (1994) 1247-1250. [§535 *notes*, 554I.]

Choksi J.R. & Fremlin D.H. [79] 'Completion regular measures on product spaces', Math. Ann. 241 (1979) 113-128. [532I.]

Cichoń J. & Pawlikowski J. [86] 'On ideals of subsets of the plane and on Cohen reals', J. Symbolic Logic 51 (1986) 560-569. [527F.]

Comfort W.W. & Negrepontis S. [74] *The Theory of Ultrafilters.* Springer, 1974. [538F, 538Yb.]

Comfort W.W. & Negrepontis S. [82] *Chain Conditions in Topology.* Cambridge U.P., 1982. [516 *notes*.]

Cummings J. [92] 'A model in which GCH holds at successors but fails at limits', Trans. Amer. Math. Soc. 329 (1992) 115-142. [525Z.]

Dellacherie C., Meyer P.A. & Weil M. (eds.) [73] *Séminaire de Probabilités VII.* Springer, 1973 (Springer Lecture Notes in Mathematics 321).

Devlin K.J. [84] *Constructibility.* Springer, 1984. [5A6D.]

Dow A. & Steprāns J. [94] 'The σ-linkedness of the measure algebra', Canad. Math. Bulletin 37 (1994) 42-45. [524L.]

Džamonja M. & Kunen K. [93] 'Measures on compact HS spaces', Fundamenta Math. 143 (1993) 41-54. [§531 *notes*.]

Džamonja M. & Plebanek G. [04] 'Precalibre pairs of measure algebras', Topology Appl. 144 (2004) 67-94. [525L.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [§5A4.]

Erdős P., Hajnal A., Máté A. & Rado R. [84] *Combinatorial Set Theory: Partition Relations for Cardinals.* Akadémiai Kiadó, 1984 (Disquisitiones Math. Hung. 13). [5A1F, 5A1H, 5A6F.]

Farah I. [00] *Analytic Quotients.* Mem. Amer. Math. Soc. 148 (2000). [5A6H.]

Farah I. [03] 'How many algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?', Illinois J. Math. 46 (2003) 999-1033. [§556 *notes*.]

Farah I. [06] 'Analytic Hausdorff gaps II: the density zero ideal', Israel J. Math. 154 (2006) 235-246. [556S.]

Farah I. & Veličković B. [06] 'Von Neumann's problem and large cardinals', Bull. London Math. Soc. 38 (2006) 907-912. [539Q.]

Feferman S. & Lévy A. [63] 'Independence results in set theory by Cohen's method', Notices Amer. Math. Soc. 10 (1963) 53. [561A.]

Fleissner W.G. [91] 'Normal measure axiom and Balogh's theorems', Topology and its Appl. 39 (1991) 123-143. [555N.]

Foreman M. [10] 'Ideals and generic elementary embeddings', pp. 885-1147 in Foreman & Kanamori 10, vol. 2. [547Z.]

Foreman M. & Kanamori A. [10] *Handbook of Set Theory.* Springer, 2010.

Foreman M. & Wehrung F. [91] 'The Hahn-Banach theorem implies the existence of a non-Lebesgue measurable set', Fundamenta Math. 138 (1991) 13-19. [563A.]

Foreman M. & Woodin W. [91] 'The generalized continuum hypothesis can fail everywhere', Annals of Math. 133 (1991) 1-35. [525Z.]

Fossy J. & Morillon M. [98] 'The Baire category property and some notions of compactness', J. London Math. Soc. (2) 57 (1998) 1-19. [§566 *notes*.]

Freese R. & Nation J.B. [78] 'Projective lattices', Pacific J. Math. 75 (1978) 93-106. [§518 *notes*.]

Freiling C. [86] 'Axioms of symmetry; throwing darts at the real number line', J. Symbolic Logic 51 (1986) 190-200. [537K.]

Fremlin D.H. [74] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [564Xc.]

Fremlin D.H. [77] 'Uncountable powers of **R** can be almost Lindelöf', Manuscripta Math. 22 (1977) 77-85. [533J.]

Fremlin D.H. [84a] *Consequences of Martin's Axiom.* Cambridge U.P., 1984. [§511 *notes*, §517 *notes*, §531 *notes*.]

Fremlin D.H. [84b] 'On the additivity and cofinality of Radon measures', Mathematika 31 (1984) 323-335. [§524 *notes*.]

Fremlin D.H. [87] *Measure-additive Coverings and Measurable Selectors.* Dissertationes Math. 260 (1987). [551A.]

Fremlin D.H. [88] 'Large correlated families of positive random variables', Math. Proc. Cambridge Phil. Soc. 103 (1988) 147-162. [525S.]

Fremlin D.H. [91] 'The partially ordered sets of measure theory and Tukey's ordering,' Note di Matematica 11 (1991) 177-214. [§524 *notes*, 526B, 526I, 527J, 529C, 529D, 534L.]

Fremlin D.H. [93] 'Real-valued-measurable cardinals', pp. 151-304 in Judah 93. [§537 *notes*, §541 *notes*, §544 *notes*, §545 *notes*, §5A2 *intro.*.]

Fremlin D.H. [97] 'On compact spaces carrying Radon measures of uncountable Maharam type', Fundamenta Math. 154 (1997) 295-304. [531T.]

Fremlin D.H. [03] 'Skew products of ideals', J. Applied Analysis 9 (2003) 1-18. [§527 *notes*.]

Fremlin D.H. & Grekas S. [95] 'Products of completion regular measures', Fundamenta Math. 147 (1995) 27-37. [532D.]

Fremlin D.H. & Miller A.W. [88] 'On some properties of Hurewicz, Menger and Rothberger', Fund. Math. 129 (1988) 17-33. [534Q.]

Fremlin D.H., Natkaniec T. & Recław I. [00] 'Universally Kuratowski-Ulam spaces', Fundamenta Math. 165 (2000) 239-247. [§527 *notes*.]

Friedman S.D. & Koepke P. [97] 'An elementary approach to the fine structure of L', Bull. Symbolic Logic 3 (1997) 453-468. [5A6D.]

Fuchino S., Geschke S., Shelah S. & Soukup L. [01] 'On the weak Freese-Nation property of complete Boolean algebras', Ann. Pure Appl. Logic 110 (2001) 89-105. [518K.]

Fuchino S., Geschke S. & Soukup L. [01] 'On the weak Freese-Nation property of $\mathcal{P}(\omega)$', Arch. Math. Logic 40 (2001) 425-435. [522U, §522 *notes*.]

Fuchino S., Koppelberg S. & Shelah S. [96] 'Partial orderings with the weak Freese-Nation property', Ann. Pure and Applied Logic 80 (1996) 35-54. [518A, 518G, 518Yb, §518 *notes*, 522U.]

Fuchino S. & Soukup L. [97] 'More set theory around the weak Freese-Nation property', Fundamenta Math. 154 (1997) 159-176. [518I, 518K.]

Galvin F. [80] 'Chain conditions and products', Fundamenta Math. 108 (1980) 33-48. [§553 *notes*.]

Galvin F., Mycielski J. & Solovay R.M. [79] 'Strong measure zero sets', Notices Amer. Math. Soc. 26 (1979) A280. [534K.]

Gandy R.O. & Hyland J.M.E. [77](eds.) *Logic Colloquium '76.* North-Holland, 1977.

Geschke S. [02] 'On tightly κ-filtered Boolean algebras', Algebra Universalis 47 (2002) 69-93. [518P, 518S, §535 *notes*.]

Gitik M. & Shelah S. [89] 'Forcings with ideals and simple forcing notions', Israel J. Math. 68 (1989) 129-160. [543E, 547F.]

Gitik M. & Shelah S. [93] 'More on simple forcing notions and forcings with ideals', Annals of Pure and Applied Logic 59 (1993) 219-238. [542E, 543E, 547F.]

Gitik M. & Shelah S. [01] 'More on real-valued measurable cardinals and forcing with ideals', Israel J. Math. 124 (2001) 221-242. [548E.]

Główczyński W. [91] 'Measures on Boolean algebras', Proc. Amer. Math. Soc. 111 (1991) 845-849. [555K.]

Główczyński W. [08] 'Outer measure on Boolean algebras', Acta Univ. Carolinae (Math. et Phys.) 49 (2008) 3-8. [555K.]

Goldstern M., Judah H. & Shelah S. [93] 'Strong measure zero sets without Cohen reals', J. Symbolic Logic 58 (1993) 1323-1341. [§534 *notes*.]

Haydon R.G. [77] 'On Banach spaces which contain $\ell^1(\tau)$ and types of measures on compact spaces', Israel J. Math. 28 (1977) 313-324. [531E, 531L.]

Hodges W., Hyland M., Steinhorn C. & Truss J. [96] (eds.) *Logic: from Foundations to Applications, European Logic Colloquium, 1993.* Clarendon, 1996.

Humke P.D. & Laczkovich M. [05] 'Symmetrically approximately continuous functions, consistent density theorems, and Fubini type inequalities', Trans. Amer. Math. Soc. 357 (2005) 31-44. [537Q, §537 *notes*.]

Ihoda J.I. [88] 'Strong measure zero sets and rapid filters', J. Symbolic Logic 53 (1988) 393-402. [534Q.]

Jech T. [73] *The Axiom of Choice.* North-Holland, 1973. [561A, 561Xc, 561Yc, 561Yi, 564Yd.]

Jech T. [78] *Set Theory.* Academic, 1978 (ISBN 0123819504). [*Intro.*, 553H, 555O, §555 *notes*, §562 *notes*, §5A1, 5A6B.]

Jech T. [03] *Set Theory, Millennium Edition.* Springer, 2003 (ISBN 3540440852). [§521 *notes*, §555 *notes*, §562 *notes*, §567 *notes*, §5A1, 5A3N, 5A3P, 5A6B.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Judah H. & Repický M. [95] 'Amoeba reals', J. Symbolic Logic 60 (1995) 1168-1185. [528N.]

Just W. [92] 'A modification of Shelah's oracle-c.c., with applications', Trans. Amer. Math. Soc. 329 (1992) 325-356. [527M.]

Just W. & Weese M. [96] *Discovering Modern Set Theory I.* Amer. Math. Soc., 1996 (Graduate Studies in Mathematics 8). [*Intro.*.]

Just W. & Weese M. [97] *Discovering Modern Set Theory II.* Amer. Math. Soc., 1997 (Graduate Studies in Mathematics 18). [5A1B, 5A1F, 5A1H.]

Kanamori A. [03] *The Higher Infinite.* Springer, 2003. [§541 *notes*, §555 *notes*, §567 *notes*, 5A1H, 5A6B, 5A6F.]

Kechris A.S. [95] *Classical Descriptive Set Theory.* Springer, 1995. [5A1D, §567 *notes*.]

Keisler H.J. & Tarski A. [64] 'From accessible to inaccessible cardinals', Fundamenta Math. 53 (1964) 225-308; errata Fundamenta Math. 57 (1965) 119. [§541 *intro.*, §541 *notes*.]

Kelley J.L. [50] 'The Tychonoff product theorem implies the axiom of choice', Fundamenta Math. 37 (1950) 75-76. [561D.]

Koppelberg S. [75] 'Homomorphic images of σ-complete Boolean algebras', Proc. Amer. Math. Soc. 51 (1975) 171-175. [515L.]

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of Monk 89. [§515 *notes*.]

Koppelberg S. & Shelah S. [96] 'Subalgebras of Cohen algebras need not be Cohen', pp. 261-275 in Hodges Hyland Steinhorn & Truss 96. [§515 *notes*, §547 *notes*.]

Kraszewski J. [01] 'Properties of ideals on the generalized Cantor spaces', J. Symbolic Logic 66 (2001) 1303-1320. [523G, 523H, 523J, §523 *notes*.]

Kumar A. [13] 'Avoiding rational distances', Real Analysis Exchange 38 (2012/13), 493-498. [548Xb.]

Kumar A. & Shelah S. [17] 'A transversal of full outer measure', Advances in Math. 321 (2017) 475-485. [547P, 547Q, §548 *intro.*, 548C.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [*Intro.*, 511A, §521 *notes*, §522 *notes*, §541 *notes*, 551Q, §551 *notes*, 556F, §562 *notes*, Appendix *intro.*, §5A1, §5A3, 5A6B.]

Kunen K. [81] 'A compact L-space under CH', Topology and its Appl. 12 (1981) 283-287. [§531 *notes*.]

Kunen K. [84] 'Random and Cohen reals', pp. 887-911 in Kunen & Vaughan 84. [§554 *notes*.]

Kunen K. [n70] 'Π^1_1 reflection at the continuum', note of January 1970. [543C, 544C, 544E, 544F.]

Kunen K. & Mill J.van [95] 'Measures on Corson compact spaces', Fundamenta Math. 147 (1995) 61-72. [531O.]

Kunen K. & Vaughan J.E. [84] (eds.) *Handbook of Set-Theoretic Topology.* North-Holland, 1984.

Kuratowski K. [66] *Topology,* vol. I. Academic, 1966. [562G.]

Lachlan A., Srebny M. & Zarach A. [77] (eds.) *Set Theory and Hierarchy Theory.* Springer, 1977 (Lecture Notes in Math. 619).

Larson P. [09] 'The filter dichotomy and medial limits', J. Math. Logic 9 (2009) 159-165. [538S.]

Larson P., Neeman I. & Shelah S. [10] 'Universally measurable sets in generic extensions', Fundamenta Math. 208 (2010) 173-192. [553O.]

Laver R. [76] 'On the consistency of Borel's conjecture', Acta Math. 137 (1976) 151-169. [534Q.]

Laver R. [87] 'Random reals and Souslin trees', Proc. Amer. Math. Soc. 100 (1987) 531-534. [553M.]

Levinski J.-P., Magidor M. & Shelah S. [90] 'Chang's conjecture for $\aleph_\omega$', Israel J. Math. 69 (1990) 161-172. [5A6F.]

Levy A. [71] 'The sizes of the indescribable cardinals', pp. 205-218 in Scott 71. [§541 *notes*, §544 *notes*.]

Lipecki Z. [09] 'Semivariations of an additive function on a Boolean ring', Colloquium Math. 117 (2009) 267-279. [552Xe.]

Maharam D. [1947] 'An algebraic characterization of measure algebras', Ann. Math. 48 (1947) 154-167. [539P.]

Makowsky J.A. & Ravve E.V. (eds.) [98] *Logic Colloquium '95.* Springer, 1998 (Lecture Notes in Logic 11).

Martin D.A. [70] 'Measurable cardinals and analytic games', Fundamenta Math. 66 (1970) 287-291. [567N.]

Martin D.A. [75] 'Borel determinacy', Ann. of Math. (2) 102 (1975) 363-371. [§567 *notes*.]

Martin D.A. & Solovay R.M. [70] 'Internal Cohen extensions', Ann. Math. Logic 2 (1970) 143-178. [Chap. 52 *intro*., §528 *intro*., §528 *notes*.]

Mátrai T. [p09] 'More cofinal types of definable directed orders', 2009 (`https://citeseerx.ist.psu.edu/viewdoc/sum`
[526L, §526 *notes*.]

Meyer P.-A. [73] 'Limites médiales d'après Mokobodzki', pp. 198-204 in Dellacherie Meyer & Weil 73. [538Yl.]

Meyer P.A. [75] (ed.) *Séminaire de Probabilités IX.* Springer, 1975 (Lecture Notes in Mathematics 465).

Miller A.W. [80] 'There are no Q-points in Laver's model for the Borel conjecture', Proc. Amer. Math. Soc. 78 (1980) 103-106. [§538 *notes*.]

Miller A.W. [81] 'Some properties of measure and category', Trans. Amer. Math. Soc. 266 (1981) 93-114. [522J.]

Miller A.W. [82] 'The Baire category theorem and cardinals of countable cofinality', J. Symbolic Logic 47 (1982) 275-288. [552G, 552Xb.]

Mokobodzki G. [75] 'Rélèvement borélien compatible avec une classe d'ensembles négligeables. Application à la désintégration des mesures', pp. 437-442 in Meyer 75. [535I.]

Mokobodzki G. [7?] 'Désintegration des mesures et relèvements Boreliens de sous-espaces de $L^\infty(X,\mathcal{B},\mu)$', Lab. d'Analyse Fonctionnelle, Univ. Paris VI (?). [535E, §535 *notes*.][15]

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Moore J.T. [05] 'Set mapping reflection', J. Math. Logic 5 (2005) 87-98. [517O.]

Moschovakis Y.N. [70] 'Determinacy and prewellorderings of the continuum', pp. 24-62 in Bar-Hillel 70. [567M.]

Mycielski J. [64] 'On the axiom of determinateness', Fund. Math. 53 (1964) 205-224. [567D.]

Mycielski J. & Świerczkowski S. [64] 'On the Lebesgue measurability and the axiom of determinateness', Fund. Math. 54 (1964) 67-71. [567F.]

Naimark M.A. [70] *Normed Rings.* Wolters-Noordhoff, 1970. [§561 *notes*.]

[15] I have not been able to locate this paper; I believe it was a seminar report. I took notes from it in 1977.

Neumann J.von [1931] 'Algebraische Repräsentanten der Funktionen "bis auf eine Menge vom Masse Null"', Crelle's J. Math. 165 (1931) 109-115. [535G.]

Neumann J.von & Stone M.H. [1935] 'The determination of representative elements in the residual classes of a Boolean algebra', Fundamenta Math. 25 (1935) 353-378. [§535 *notes*.]

Pawlikowski J. [86] 'Why Solovay real produces Cohen real', J. Symbolic Logic 51 (1986) 957-968. [552G, 552H.]

Pełczy'nski A. [68] 'On Banach spaces containing $L^1(\mu)$, Studia Math. 30 (1968) 231-246. [531Ye.]

Perović Ž. & Veličković B. [18] 'Ranks of Maharam algebras', Advances in Math. 330 (2018) 253-279. [539V.]

Plebanek G. [95] 'On Radon measures on first-countable compact spaces', Fundamenta Math. 148 (1995) 159-164. [531O.]

Plebanek G. [97] 'Non-separable Radon measures and small compact spaces', Fundamenta Math. 153 (1997) 25-40. [531L, 531M, 531V.]

Plebanek G. [00] 'Approximating Radon measures on first-countable compact spaces', Colloquium Math. 86 (2000) 15-23. [533H, 533Yc.]

Plebanek G. [02] 'On compact spaces carrying Radon measures of large Maharam type', Acta Univ. Carolinae 43 (2002) 87-99. [531U.]

Plebanek G. & Sobota D. [15] 'Countable tightness in the spaces of regular probability measures', Fund. Math. 229 (2015) 159-169. [531U.]

Pol R. [82] 'Note on the spaces $P(S)$ of regular probability measures whose topology is determined by countable subsets', Pacific J. Math. 100 (1982) 185-201. [533Ya.]

Prikry K. [75] 'Ideals and powers of cardinals', Bull. Amer. Math. Soc. 81 (1975) 907-909. [555N.]

Quickert S. [02] 'CH and the Saks property', Fundamenta Math. 171 (2002) 93-100. [539L.]

Raisonnier J. & Stern J. [85] 'The strength of measurability hypotheses', Israel J. Math. 50 (1985) 337-349. [Chap. 52 *intro.*, 522Q.]

Rothberger F. [1938a] 'Eine Äquivalenz zwischen der kontinuumhypothese unter der Existenz der Lusinschen und Sierpinschischen Mengen', Fundamenta Math. 30 (1938) 215-217. [522G.]

Rothberger F. [1938b] 'Eine Verschärfung der Eigenschaft C', Fundamenta Math. 30 (1938) 50-55. [§534 *notes*.]

Rothberger F. [1941] 'Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété **C**', Proc. Cambridge Phil. Soc. 37 (1941) 109-126. [534R.]

Scott D.S. [71] (ed.) *Axiomatic Set Theory.* Amer. Math. Soc., 1971 (Proceedings of Symposia in Pure Mathematics XIII, vol. 1).

Shelah S. [82] *Proper Forcing.* Springer, 1982 (Lecture Notes in Mathematics 940). [§538 *notes*.]

Shelah S. [83] 'Lifting problem of the measure algebra', Israel J. Math. 45 (1983) 90-96. [§535 *notes*.]

Shelah S. [84] 'Can you take Solovay inaccessible away?', Israel J. Math. 48 (1984) 1-47. [§522 *notes*.]

Shelah S. [92] 'Cardinal arithmetic for skeptics', Bull. Amer. Math. Soc. 26 (1992) 197-210. [5A2D.]

Shelah S. [94] *Cardinal Arithmetic.* Oxford U.P., 1994. [5A2D, 5A2G.]

Shelah S. [96] 'Further cardinal arithmetic', Israel J. Math. 95 (1996) 61-114. [542I.]

Shelah S. [98a] *Proper and Improper Forcing.* Springer, 1998. [§538 *notes*.]

Shelah S. [98b] 'There may be no nowhere dense ultrafilter', pp. 305-324 in Makovsky & Ravve 98. [538H, §538 *notes*.]

Shelah S. [00] 'Covering of the null ideal may have countable cofinality', Fundamenta Math. 166 (2000) 109-136. [§522 *notes*.]

Shelah S. [03] 'The null ideal restricted to some non-null set may be $\aleph_1$-saturated', Fundamenta Math. 179 (2003) 97-129. [§548 *notes*.]

Shelah S. & Steprāns J. [05] 'Comparing the uniformity invariants of null sets for different measures', Advances in Math. 192 (2005) 403-426. [534Za.]

Shipman J. [90] 'Cardinal conditions for strong Fubini theorems', Trans. Amer. Math. Soc. 321 (1990) 465-481. [537K.]

Solecki S. & Todorčević S. [04] 'Cofinal types of topological directed orders', Ann. Inst. Fourier 54 (2004) 1877-1911. [513K, 513O, 513Yi.]

Solecki S. & Todorčević S. [10] 'Avoiding families and Tukey functions on the nowhere-dense ideal', J. Inst. Math. Jussieu 9 (2010) 1-31. [526L.]

Solovay R.M. [66] 'New proof of a theorem of Gaifman and Hales', Bull. Amer. Math. Soc. 72 (1966) 282-284. [514Xi.]

Solovay R.M. [70] 'A model of set theory in which every set of reals is Lebesgue measurable', Annals of Math. 92 (1970) 1-56. [§522 *notes*.]

Solovay R.M. [71] 'Real-valued measurable cardinals', pp. 397-428 in SCOTT 71. [541J, 541P, 541Ya, §541 *notes*, 555D, 555O.]

Talagrand M. [84] *Pettis Integral and Measure Theory.* Mem. Amer. Math. Soc. 307 (1984). [536C.]

Talagrand M. [08] 'Maharam's problem', Annals of Math. 168 (2008) 981-1009. [539A.]

Tarski A. [1945] 'Ideale in volständigen Mengenkörpen II', Fundamenta Math. 33 (1945) 51-65. [541P.]

Todorčević S. [85] 'Remarks on chain conditions in products', Compositio Math. 55 (1985) 295-302. [537G.]

Todorčević S. [87] 'Partitioning pairs of countable ordinals', Acta Math. 159 (1987) 261-294. [554Yc.]

Todorčević S. [00] 'A dichotomy for P-ideals of countable sets', Fundamenta Math. 166 (2000) 251-267. [5A6G.]

Törnquist A. [11] 'On the pointwise implementation of near-actions', Trans. Amer. Math. Soc. 363 (2011) 4929-4944. [535Yc.]

Truss J.K. [77] 'Sets having calibre $\aleph_1$', pp. 595-612 in GANDY & HYLAND 77. [522J.]

Truss J.K. [88] 'Connections between different amoeba algebras', Fund. Math. 130 (1988) 137-155. [Chap. 52 *intro.*, 528A, 528Da, 528K, §528 *notes*.]

Tukey J.W. [1940] *Convergence and Uniformity in Topology.* Princeton U.P., 1940 (Ann. Math. Studies 1). [513F, §513 *notes*.]

Ulam S. [1930] 'Zur Masstheorie in der allgemeinen Mengenlehre', Fundamenta Math. 16 (1930) 140-150. [§541 *intro.*, §541 *notes*.]

Veličković B. [92] 'Forcing axioms and stationary sets', Advances in Math. 94 (1992) 256-284. [517O.]

Veličković B. [05] 'CCC forcing and splitting reals', Israel J. Math. 147 (2005) 209-221. [§539 *intro.*, 539E, 539N.]

Vojtáš P. [93] 'Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis', pp. 619-643 in JUDAH 93. [512A.]

Wimmers E. [82] 'The Shelah P-point independence theorem', Israel J. Math. 43 (1982) 28-48. [§538 *notes*.]

Zakrzewski P. [92] 'Strong Fubini theorems from measure extension axioms', Comm. Math. Univ. Carolinae 33 (1992) 291-297. [544J.]