Version of 3.8.15

Appendix to Volume 2

Useful Facts

In the course of writing this volume, I have found that a considerable number of concepts and facts from various branches of mathematics are necessary to us. Nearly all of them are embedded in important and well-established theories for which many excellent textbooks are available and which I very much hope that you will one day study in depth. Nevertheless, I am reluctant to send you off immediately to courses in general topology, functional analysis and set theory, as if these were essential prerequisites for our work here, along with real analysis and basic linear algebra. For this reason I have written this Appendix, setting out those results which we actually need at some point in this volume. The great majority of them really are elementary – indeed, some are so elementary that they are not always spelt out in detail in orthodox treatments of their subjects.

While I do not put this book forward as the proper place to learn any of these topics, I have tried to set them out in a way that you will find easy to integrate into regular approaches. I do not expect anybody to read systematically through this work, and I hope that the references given in the main chapters of this volume will be adequate to guide you to the particular items you need.

Version of 20.1.13

2A1 Set theory

Especially for the examples in Chapter 21, we need some non-trivial set theory, which is best approached through the standard theory of cardinals and ordinals; and elsewhere in this volume I make use of Zorn's Lemma. Here I give a very brief outline of the results involved, largely omitting proofs. Most of this material should be in any sound introduction to set theory. The references I give are to books which happen to have come my way and which I can recommend as reasonably suitable for beginners.

I do not discuss axiom systems or logical foundations. The set theory I employ is 'naive' in the sense that I rely on my understanding of the collective experience of the last hundred years, rather than on any attempt at formal description, to distinguish legitimate from unsafe arguments. There are, however, points in Volume 5 at which such a relaxed philosophy becomes inappropriate, and I therefore use arguments which can, I believe, be translated into standard Zermelo-Fraenkel set theory without new ideas being invoked.

Although in this volume I use the axiom of choice without scruple whenever appropriate, I will divide this section into two parts, starting with ideas and results not dependent on the axiom of choice (2A1A-2A1I) and continuing with the remainder (2A1J-2A1P). I believe that even at this level it helps us to understand the nature of the arguments better if we maintain a degree of separation.

2A1A Ordered sets (a) Recall that a **partially ordered set** is a set P together with a relation $\le$ on P such that

if $p \le q$ and $q \le r$ then $p \le r$,
$p \le p$ for every $p \in P$,
if $p \le q$ and $q \le p$ then $p = q$.

In this context, I will write $p \ge q$ to mean $q \le p$, and $p < q$ or $q > p$ to mean '$p \le q$ and $p \ne q$'. $\le$ is a **partial order** on P.

(b) Let $(P, \le)$ be a partially ordered set, and $A \subseteq P$. A **maximal** element of A is a $p \in A$ such that $p \not< a$ for any $a \in A$. Note that A may have more than one maximal element. An **upper bound** for A is a $p \in P$ such that $a \le p$ for every $a \in A$; a **supremum** or **least upper bound** is an upper bound p such that $p \leq q$ for every upper bound q of A. There can be at most one such. Accordingly we may safely write $p = \sup A$ if p is the least upper bound of A.

Similarly, a **minimal** element of A is a $p \in A$ such that $p \not> a$ for every $a \in A$; a **lower bound** of A is a $p \in P$ such that $p \leq a$ for every $a \in A$; and $\inf A = p$ means that

$$\text{for every } q \in P,\ p \geq q \iff a \geq q\ \forall\, a \in A.$$

A subset A of P is **order-bounded** if it has both an upper bound and a lower bound.

A subset A of P is **upwards-directed** if for any p, $p' \in A$ there is a $q \in A$ such that $p \leq q$ and $p' \leq q$. Similarly, $A \subseteq P$ is **downwards-directed** if for any p, $p' \in A$ there is a $q \in A$ such that $q \leq p$ and $q \leq p'$.

$[p, q]$ will be $\{r : p \leq r \leq q\}$.

(c) A **totally ordered set** is a partially ordered set $(P, \leq)$ such that

for any p, $q \in P$, either $p \leq q$ or $q \leq p$.

$\leq$ is then a **total** or **linear** order on P.

In any totally ordered set we have a **median function**: for p, q, $r \in P$ set

$$\begin{aligned}\operatorname{med}(p, q, r) &= \max(\min(p, q), \min(p, r), \min(q, r))\\ &= \min(\max(p, q), \max(p, r), \max(q, r)).\end{aligned}$$

(d) A **lattice** is a partially ordered set $(P, \leq)$ such that

for any p, $q \in P$, $p \vee q = \sup\{p, q\}$ and $p \wedge q = \inf\{p, q\}$ are defined in P.

(e) A **well-ordered set** is a totally ordered set $(P, \leq)$ such that every non-empty subset of P has a least element. In this case $\leq$ is a **well-ordering** of P.

2A1B Transfinite Recursion: Theorem Let $(P, \leq)$ be a well-ordered set and X any class. For $p \in P$ write L_p for the set $\{q : q \in P, q < p\}$ and X^{L_p} for the class of all functions from L_p to X. Let $F : \bigcup_{p \in P} X^{L_p} \to X$ be any function. Then there is a unique function $f : P \to X$ such that $f(p) = F(f \restriction L_p)$ for every $p \in P$.

2A1C Ordinals An **ordinal** (sometimes called a 'von Neumann ordinal') is a set ξ such that

if $\eta \in \xi$ then η is a set and $\eta \notin \eta$,

if $\eta \in \zeta \in \xi$ then $\eta \in \xi$,

writing '$\eta \leq \zeta$' to mean '$\eta \in \zeta$ or $\eta = \zeta$', $(\xi, \leq)$ is well-ordered.

2A1D Basic facts about ordinals (a) If ξ is an ordinal, then every member of ξ is an ordinal.

(b) If ξ, η are ordinals then either $\xi \in \eta$ or $\xi = \eta$ or $\eta \in \xi$ (and no two of these can occur together). It is customary to write $\eta < \xi$ if $\eta \in \xi$ and $\eta \leq \xi$ if either $\eta \in \xi$ or $\eta = \xi$. Note that $\eta \leq \xi$ iff $\eta \subseteq \xi$.

(c) If A is any non-empty class of ordinals, then there is an $\alpha \in A$ such that $\alpha \leq \xi$ for every $\xi \in A$.

(d) If ξ is an ordinal, so is $\xi \cup \{\xi\}$; call it '$\xi + 1$'. $\xi + 1$ is the least ordinal greater than ξ. For any ordinal ξ, either there is a greatest ordinal $\eta < \xi$, in which case $\xi = \eta + 1$ and we call ξ a **successor ordinal**, or $\xi = \bigcup \xi$, in which case we call ξ a **limit ordinal**.

(e) The first few ordinals are $0 = \emptyset$, $1 = 0 + 1 = \{0\} = \{\emptyset\}$, $2 = 1 + 1 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, $3 = 2 + 1 = \{0, 1, 2\}$, ... The first infinite ordinal is $\omega = \{0, 1, 2, \dots\}$, which may be identified with $\mathbb{N}$.

(f) The union of any set of ordinals is an ordinal.

(g) If $(P, \leq)$ is any well-ordered set, there is a unique ordinal ξ such that P is order-isomorphic to ξ, and the order-isomorphism is unique.

2A1E Initial ordinals An **initial ordinal** is an ordinal κ such that there is no bijection between κ and any member of κ.

2A1F Basic facts about initial ordinals (a) All finite ordinals, and the first infinite ordinal ω, are initial ordinals.

(b) For every well-ordered set P there is a unique initial ordinal κ such that there is a bijection between P and κ.

(c) For every ordinal ξ there is a least initial ordinal greater than ξ. If κ is an initial ordinal, write κ^+ for the least initial ordinal greater than κ. We write ω_1 for ω^+, ω_2 for ω_1^+, and so on.

(d) For any initial ordinal $\kappa \ge \omega$ there is a bijection between $\kappa\times\kappa$ and κ; consequently there are bijections between κ and κ^r for every $r \ge 1$.

2A1G Schröder-Bernstein theorem If X and Y are sets and there are injections $f : X \to Y$, $g : Y \to X$ then there is a bijection $h : X \to Y$.

2A1H Countable subsets of $\mathcal{P}\mathbb{N}$ (a) There is a bijection between $\mathcal{P}\mathbb{N}$ and $\mathbb{R}$.

(b) Suppose that X is any set such that there is an injection from X into $\mathcal{P}\mathbb{N}$. Let $\mathcal{C}$ be the set of countable subsets of X. Then there is a surjection from $\mathcal{P}\mathbb{N}$ onto $\mathcal{C}$.

(c) Again suppose that X is a set such that there is an injection from X to $\mathcal{P}\mathbb{N}$, and write H for the set of functions h such that $\operatorname{dom} h$ is a countable subset of X and h takes values in $\{0, 1\}$. Then there is a surjection from $\mathcal{P}\mathbb{N}$ onto H.

2A1I Filters (a) Let X be a non-empty set. A **filter** on X is a family $\mathcal{F}$ of subsets of X such that

$X \in \mathcal{F}$, $\quad \emptyset \notin \mathcal{F}$,

$E \cap F \in \mathcal{F}$ whenever $E, F \in \mathcal{F}$,

$E \in \mathcal{F}$ whenever $X \supseteq E \supseteq F \in \mathcal{F}$.

The second condition implies (inducing on n) that $F_0 \cap \ldots \cap F_n \in \mathcal{F}$ whenever $F_0, \ldots , F_n \in \mathcal{F}$.

(b) Let X, Y be non-empty sets, $\mathcal{F}$ a filter on X and $f : D \to Y$ a function, where $D \in \mathcal{F}$. Then

$$\{E : E \subseteq Y,\ f^{-1}[E] \in \mathcal{F}\}$$

is a filter on Y; I will call it $f[[\mathcal{F}]]$, the **image filter** of $\mathcal{F}$ under f.

2A1J The Axiom of Choice Let me remind you of the statement of this axiom:

(AC) 'whenever I is a set and $\langle X_i\rangle_{i\in I}$ is a family of non-empty sets indexed by I, there is a function f, with domain I, such that $f(i) \in X_i$ for every $i \in I$'.

The function f is a **choice function**.

2A1K Zermelo's Well-Ordering Theorem (a) The Axiom of Choice is equiveridical with each of the statements

'for every set X there is a well-ordering of X',

'for every set X there is a bijection between X and some ordinal',

'for every set X there is a unique initial ordinal κ such that there is a bijection between X and κ.'

(b) When assuming the axiom of choice, I write $\#(X)$ for that initial ordinal κ such that there is a bijection between κ and X; I call this the **cardinal** of X.

2A1L Fundamental consequences of the Axiom of Choice (a) For any two sets X and Y, there is a bijection between X and Y iff $\#(X) = \#(Y)$. More generally, there is an injection from X to Y iff $\#(X) \le \#(Y)$, and a surjection from X onto Y iff either $\#(X) \ge \#(Y) > 0$ or $\#(X) = \#(Y) = 0$.

(b) $\#(\mathcal{P}\mathbb{N}) = \#(\mathbb{R})$; write $\mathfrak{c}$ for this common value, the **cardinal of the continuum**. $\omega_1 \le \mathfrak{c}$.

(c) If X is any infinite set, and $r \ge 1$, then there is a bijection between X^r and X.

(d) Suppose that κ is an infinite cardinal. If I is a set with cardinal at most κ and $\langle A_i\rangle_{i\in I}$ is a family of sets with $\#(A_i) \le \kappa$ for every $i \in I$, then $\#(\bigcup_{i\in I} A_i) \le \kappa$. Consequently $\#(\bigcup\mathcal{A}) \le \kappa$ whenever $\mathcal{A}$ is a family of sets such that $\#(\mathcal{A}) \le \kappa$ and $\#(A) \le \kappa$ for every $A \in \mathcal{A}$. ω_1 cannot be expressed as a countable union of countable sets, and ω_2 cannot be expressed as a countable union of sets with cardinal at most ω_1.

(e) Now we can rephrase 2A1Hc as: if $\#(X) \le \mathfrak{c}$, then $\#(H) \le \mathfrak{c}$, where H is the set of functions from a countable subset of X to $\{0,1\}$.

(f) Any non-empty class of cardinals has a least member.

2A1M Zorn's Lemma I come now to another proposition which is equiveridical with the axiom of choice:

> 'Let $(P, \le)$ be a non-empty partially ordered set such that every non-empty totally ordered subset of P has an upper bound in P. Then P has a maximal element.'

This is **Zorn's Lemma**.

2A1N Ultrafilters A filter $\mathcal{F}$ on a set X is an **ultrafilter** if for every $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. If $\mathcal{F}$ is an ultrafilter on X and $f : D \to Y$ is a function, where $D \in \mathcal{F}$, then $f[[\mathcal{F}]]$ is an ultrafilter on Y.

One type of ultrafilter can be described easily: if x is any point of a set X, then $\mathcal{F} = \{F : x \in F \subseteq X\}$ is an ultrafilter on X. (Ultrafilters of this type are called **principal ultrafilters**.)

2A1O The Ultrafilter Theorem Let X be any non-empty set, and $\mathcal{F}$ a filter on X. Then there is an ultrafilter $\mathcal{H}$ on X such that $\mathcal{F} \subseteq \mathcal{H}$.

2A1P Theorem (a) Let $\langle K_\alpha\rangle_{\alpha\in A}$ be a family of countable sets, with $\#(A)$ strictly greater than $\mathfrak{c}$, the cardinal of the continuum. Then there are a set M, with cardinal at most $\mathfrak{c}$, and a set $B \subseteq A$, with cardinal strictly greater than $\mathfrak{c}$, such that $K_\alpha \cap K_\beta \subseteq M$ whenever α, β are distinct members of B.

(b) Let I be a set, and $\langle f_\alpha\rangle_{\alpha\in A}$ a family in $\{0,1\}^I$, the set of functions from I to $\{0,1\}$, with $\#(A) > \mathfrak{c}$. If $\langle K_\alpha\rangle_{\alpha\in A}$ is any family of countable subsets of I, then there is a set $B \subseteq A$, with cardinal greater than $\mathfrak{c}$, such that f_α and f_β agree on $K_\alpha \cap K_\beta$ for all α, $\beta \in B$.

(c) In particular, under the conditions of (b), there are distinct α, $\beta \in A$ such that f_α and f_β agree on $K_\alpha \cap K_\beta$.

Version of 30.11.09

2A2 The topology of Euclidean space

In the appendix to Volume 1 I discussed open and closed sets in $\mathbb{R}^r$; the chief aim there was to support the idea of 'Borel set', which is vital in the theory of Lebesgue measure, but of course they are also fundamental to the study of continuous functions, and indeed to all aspects of real analysis. I give here a very brief introduction to the further elementary facts about closed and compact sets and continuous functions which we need for this volume. Much of this material can be derived from the generalizations in §2A3, but nevertheless I sketch the proofs, since for the greater part of the volume (most of the exceptions are in Chapter 24) Euclidean space is sufficient for our needs.

2A2A Closures: Definition For any $r \ge 1$ and any $A \subseteq \mathbb{R}^r$, the **closure** of A, $\overline{A}$, is the intersection of all the closed subsets of $\mathbb{R}^r$ including A. This is the smallest closed set including A. A is closed iff $\overline{A} = A$.

2A2B Lemma Let $A \subseteq \mathbb{R}^r$ be any set. Then for $x \in \mathbb{R}^r$ the following are equiveridical:
(i) $x \in \overline{A}$;
(ii) $B(x, \delta) \cap A \ne \emptyset$ for every $\delta > 0$, where $B(x, \delta) = \{y : \|y - x\| \le \delta\}$;
(iii) there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in A such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

2A2C Continuous functions (a) If r, $s \ge 1$, $D \subseteq \mathbb{R}^r$ and $\phi : D \to \mathbb{R}^s$ is a function, we say that ϕ is **continuous** if for every $x \in D$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(y) - \phi(x)\| \le \epsilon$ whenever $y \in D$ and $\|y - x\| \le \delta$. ϕ is continuous iff for every open set $G \subseteq \mathbb{R}^s$ there is an open set $H \subseteq \mathbb{R}^r$ such that $\phi^{-1}[G] = D \cap H$.

(b) Using the ϵ-δ definition of continuity, it is easy to see that a function ϕ from a subset D of $\mathbb{R}^r$ to $\mathbb{R}^s$ is continuous iff all its components ϕ_i are continuous, writing $\phi(x) = (\phi_1(x), \dots, \phi_s(x))$ for $x \in D$.

(c) If r, $s \ge 1$, $D \subseteq \mathbb{R}^r$ and $\phi : D \to \mathbb{R}^s$ is a function, we say that ϕ is **uniformly continuous** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(y) - \phi(x)\| \le \epsilon$ whenever x, $y \in D$ and $\|y - x\| \le \delta$. A uniformly continuous function is continuous.

2A2D Compactness in $\mathbb{R}^r$: Definition A subset F of $\mathbb{R}^r$ is called **compact** if whenever $\mathcal{G}$ is a family of open sets covering F then there is a finite subset $\mathcal{G}_0$ of $\mathcal{G}$ still covering F.

2A2E Elementary properties of compact sets Take any $r \ge 1$, and subsets D, F, G and K of $\mathbb{R}^r$.

(a) If K is compact and F is closed, then $K \cap F$ is compact.

(b) If $s \ge 1$, $\phi : D \to \mathbb{R}^s$ is a continuous function, K is compact and $K \subseteq D$, then $\phi[K]$ is compact.

(c) If K is compact, it is closed.

(d) If K is compact and G is open and $K \subseteq G$, then there is a $\delta > 0$ such that $K + B(\mathbf{0}, \delta) \subseteq G$.

2A2F Theorem For any $r \ge 1$, a subset K of $\mathbb{R}^r$ is compact iff it is closed and bounded.

2A2G Corollary If $\phi : D \to \mathbb{R}$ is continuous, where $D \subseteq \mathbb{R}^r$, and $K \subseteq D$ is a non-empty compact set, then ϕ is bounded and attains its bounds on K.

2A2H Lim sup and lim inf revisited In §1A3 I briefly discussed $\limsup_{n\to\infty} a_n$, $\liminf_{n\to\infty} a_n$ for real sequences $\langle a_n \rangle_{n \in \mathbb{N}}$. In this volume we need the notion of $\limsup_{\delta\downarrow 0} f(\delta)$, $\liminf_{\delta\downarrow 0} f(\delta)$ for real functions f. I say that $\limsup_{\delta\downarrow 0} f(\delta) = u \in [-\infty, \infty]$ if (i) for every $v > u$ there is an $\eta > 0$ such that $f(\delta)$ is defined and less than or equal to v for every $\delta \in]0, \eta]$ (ii) for every $v < u$ and $\eta > 0$ there is a $\delta \in]0, \eta]$ such that $f(\delta)$ is defined and greater than or equal to v. Similarly, $\liminf_{\delta\downarrow 0} f(\delta) = u \in [-\infty, \infty]$ if (i) for every $v < u$ there is an $\eta > 0$ such that $f(\delta)$ is defined and greater than or equal to v for every $\delta \in]0, \eta]$ (ii) for every $v > u$ and $\eta > 0$ there is an $\delta \in]0, \eta]$ such that $f(\delta)$ is defined and less than or equal to v.

2A2I Proposition If $G \subseteq \mathbb{R}$ is any open set, it is expressible as the union of a countable disjoint family of open intervals.

Version of 25.7.07

2A3 General topology

At various points – principally §§245-247, but also for certain ideas in Chapter 27 – we need to know something about non-metrizable topologies. I must say that you should probably take the time to look at some book on elementary functional analysis which has the phrases 'weak compactness' or 'weakly compact' in the index. But I can list here the concepts actually used in this volume, in a good deal less space than any orthodox, complete treatment would employ.

2A3A Topologies If X is any set, a **topology** on X is a family $\mathfrak{T}$ of subsets of X such that (i) $\emptyset$, $X \in \mathfrak{T}$ (ii) if G, $H \in \mathfrak{T}$ then $G \cap H \in \mathfrak{T}$ (iii) if $\mathcal{G} \subseteq \mathfrak{T}$ then $\bigcup \mathcal{G} \in \mathfrak{T}$. $(X, \mathfrak{T})$ is now a **topological space**. In this context, members of $\mathfrak{T}$ are called **open** and their complements are called **closed**.

2A3B Continuous functions (a) If $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ are topological spaces, a function $\phi : X \to Y$ is **continuous** if $\phi^{-1}[G] \in \mathfrak{T}$ for every $G \in \mathfrak{S}$.

(b) If $(X, \mathfrak{T})$, $(Y, \mathfrak{S})$ and $(Z, \mathfrak{U})$ are topological spaces and $\phi : X \to Y$ and $\psi : Y \to Z$ are continuous, then $\psi\phi : X \to Z$ is continuous.

(c) If $(X, \mathfrak{T})$ is a topological space, a function $f : X \to \mathbb{R}$ is continuous iff $\{x : a < f(x) < b\}$ is open whenever $a < b$ in $\mathbb{R}$.

(d) If $r \ge 1$, $(X, \mathfrak{T})$ is a topological space, and $\phi : X \to \mathbb{R}^r$ is a function, then ϕ is continuous iff $\phi_i : X \to \mathbb{R}$ is continuous for each $i \le r$, where $\phi(x) = (\phi_1(x), \dots, \phi_r(x))$ for each $x \in X$.

(e) If $(X, \mathfrak{T})$ is a topological space, $f_1, \dots, f_r$ are continuous functions from X to $\mathbb{R}$, and $h : \mathbb{R}^r \to \mathbb{R}$ is continuous, then $h(f_1, \dots, f_r) : X \to \mathbb{R}$ is continuous. In particular, $f + g$, $f \times g$ and $f - g$ are continuous for all continuous functions f, $g : X \to \mathbb{R}$.

(f) If $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ are topological spaces and $\phi : X \to Y$ is a continuous function, then $\phi^{-1}[F]$ is closed in X for every closed set $F \subseteq Y$.

2A3C Subspace topologies If $(X, \mathfrak{T})$ is a topological space and $D \subseteq X$, then $\mathfrak{T}_D = \{G \cap D : G \in \mathfrak{T}\}$ is a topology on D.

$\mathfrak{T}_D$ is called the **subspace topology** on D, or the topology on D **induced** by $\mathfrak{T}$. If $(Y, \mathfrak{S})$ is another topological space, and $\phi : X \to Y$ is $(\mathfrak{T}, \mathfrak{S})$-continuous, then $\phi \restriction D : D \to Y$ is $(\mathfrak{T}_D, \mathfrak{S})$-continuous.

2A3D Closures and interiors (a) Let $(X, \mathfrak{T})$ be any topological space and A any subset of X. Write

$$\operatorname{int} A = \bigcup \{G : G \in \mathfrak{T}, \, G \subseteq A\}.$$

Then $\operatorname{int} A$ is the largest open set included in A, and is called the **interior** of A.

(b) Because a set is closed iff its complement is open, we have a complementary notion:

$$\overline{A} = \bigcap \{F : F \text{ is closed, } A \subseteq F\} = X \setminus \operatorname{int}(X \setminus A).$$

$\overline{A}$ is the smallest closed set including A; it is called the **closure** of A. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all A, $B \subseteq X$.

(c)

$$x \in \overline{A} \iff \text{every open set containing } x \text{ meets } A.$$

2A3E Hausdorff topologies (a) A topological space X is **Hausdorff** if for all distinct x, $y \in X$ there are disjoint open sets G, $H \subseteq X$ such that $x \in G$ and $y \in H$.

(b) In a Hausdorff space X, finite sets are closed.

2A3F Pseudometrics(a) Let X be a set. A **pseudometric** on X is a function $\rho : X \times X \to [0, \infty[$ such that

$\rho(x, z) \le \rho(x, y) + \rho(y, z)$ for all x, y, $z \in X$

$\rho(x, y) = \rho(y, x)$ for all x, $y \in X$;

$\rho(x, x) = 0$ for all $x \in X$.

A **metric** is a pseudometric ρ satisfying the further condition

if $\rho(x, y) = 0$ then $x = y$.

(b) Examples (i) For x, $y \in \mathbb{R}$, set $\rho(x, y) = |x - y|$ (the 'usual metric' on $\mathbb{R}$).

(ii) For x, $y \in \mathbb{R}^r$, where $r \ge 1$, set $\rho(x, y) = \|x - y\|$, defining $\|z\| = \sqrt{\sum_{i=1}^{r} \zeta_i^2}$, as usual. Then ρ is the **Euclidean metric** on $\mathbb{R}^r$.

(c) Now let X be a set and P a non-empty family of pseudometrics on X. Let $\mathfrak{T}$ be the family of those subsets G of X such that for every $x \in G$ there are $\rho_0, \dots , \rho_n \in \mathrm{P}$ and $\delta > 0$ such that

$$U(x; \rho_0, \dots , \rho_n; \delta) = \{y : y \in X, \max_{i \le n} \rho_i(y, x) < \delta\} \subseteq G.$$

Then $\mathfrak{T}$ is a topology on X.

$\mathfrak{T}$ is the **topology defined by** P.

(d) You may wish to have a convention to deal with the case in which P is the empty set; in this case the topology on X defined by P is $\{\emptyset, X\}$.

(f) A topology $\mathfrak{T}$ is **metrizable** if it is the topology defined by a family P consisting of a single metric. Thus the **Euclidean topology** on $\mathbb{R}^r$ is the metrizable topology defined by $\{\rho\}$, where ρ is the metric of (b-ii) above.

2A3G Proposition Let X be a set with a topology defined by a non-empty set P of pseudometrics on X. Then $U(x; \rho_0, \dots , \rho_n; \epsilon)$ is open for all $x \in X$, $\rho_0, \dots , \rho_n \in \mathrm{P}$ and $\epsilon > 0$.

2A3H Proposition Let X and Y be sets; let P be a non-empty family of pseudometrics on X, and Θ a non-empty family of pseudometrics on Y; let $\mathfrak{T}$ and $\mathfrak{S}$ be the corresponding topologies. Then a function $\phi : X \to Y$ is continuous iff whenever $x \in X$, $\theta \in \Theta$ and $\epsilon > 0$, there are $\rho_0, \dots , \rho_n \in \mathrm{P}$ and $\delta > 0$ such that $\theta(\phi(y), \phi(x)) \le \epsilon$ whenever $y \in X$ and $\max_{i \le n} \rho_i(y, x) \le \delta$.

2A3J Subspaces: Proposition If X is a set, P a non-empty family of pseudometrics on X defining a topology $\mathfrak{T}$ on X, and $D \subseteq X$, then

(a) for every $\rho \in \mathrm{P}$, the restriction $\rho^{(D)}$ of ρ to $D \times D$ is a pseudometric on D;

(b) the topology defined by $\mathrm{P}_D = \{\rho^{(D)} : \rho \in \mathrm{P}\}$ on D is precisely the subspace topology $\mathfrak{T}_D$ described in 2A3C.

2A3K Closures and interiors Let X be a set, P a non-empty family of pseudometrics on X and $\mathfrak{T}$ the topology defined by P.

(a) For any $A \subseteq X$ and $x \in X$,

$$x \in \operatorname{int} A \iff \text{there are } \rho_0, \dots , \rho_n \in \mathrm{P},\ \delta > 0 \text{ such that } U(x; \rho_0, \dots , \rho_n; \delta) \subseteq A.$$

(b) For any $A \subseteq X$ and $x \in X$, $x \in \overline{A}$ iff $U(x; \rho_0, \dots, \rho_n; \delta) \cap A \neq \emptyset$ for every $\rho_0, \dots, \rho_n \in \mathrm{P}$ and $\delta > 0$.

2A3L Hausdorff topologies Now a topology defined on a set X by a non-empty family P of pseudometrics is Hausdorff iff for any two different points x, y of X there is a $\rho \in \mathrm{P}$ such that $\rho(x, y) > 0$.

In particular, metrizable topologies are Hausdorff.

2A3M Convergence of sequences (a) If $(X, \mathfrak{T})$ is any topological space, and $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in X, we say that $\langle x_n \rangle_{n \in \mathbb{N}}$ **converges** to $x \in X$, or that x is a **limit** of $\langle x_n \rangle_{n \in \mathbb{N}}$, or $\langle x_n \rangle_{n \in \mathbb{N}} \to x$, if for every open set G containing x there is an $n_0 \in \mathbb{N}$ such that $x_n \in G$ for every $n \geq n_0$.

(b) Warning In general topological spaces, it is possible for a sequence to have more than one limit. But in Hausdorff spaces, this does not occur. In particular, a sequence in a metrizable space can have at most one limit.

(c) Let X be a set, and P a non-empty family of pseudometrics on X, generating a topology $\mathfrak{T}$; let $\langle x_n \rangle_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$. Then $\langle x_n \rangle_{n \in \mathbb{N}}$ converges to x iff $\lim_{n \to \infty} \rho(x_n, x) = 0$ for every $\rho \in \mathrm{P}$.

(d) Let (X, ρ) be a metric space, A a subset of X and $x \in X$. Then $x \in \overline{A}$ iff there is a sequence in A converging to x.

2A3N Compactness (a) If $(X, \mathfrak{T})$ is any topological space, a subset K of X is **compact** if whenever $\mathcal{G}$ is a family in $\mathfrak{T}$ covering K, then there is a finite $\mathcal{G}_0 \subseteq \mathcal{G}$ covering K. A set $A \subseteq X$ is **relatively compact** in X if there is a compact subset of X including A.

(b)(i) If K is compact and E is closed, then $K \cap E$ is compact.

(ii) If $K \subseteq X$ is compact and $\phi : K \to Y$ is continuous, where $(Y, \mathfrak{S})$ is another topological space, then $\phi[K]$ is a compact subset of Y.

(iii) If $K \subseteq X$ is compact and $\phi : K \to \mathbb{R}$ is continuous, then ϕ is bounded and attains its bounds.

2A3O Cluster points (a) If $(X, \mathfrak{T})$ is a topological space, and $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in X, then a **cluster point** of $\langle x_n \rangle_{n \in \mathbb{N}}$ is an $x \in X$ such that whenever G is an open set containing x and $n \in \mathbb{N}$ then there is a $k \geq n$ such that $x_k \in G$.

(b) Now if $(X, \mathfrak{T})$ is a topological space and $A \subseteq X$ is relatively compact, every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in A has a cluster point in X.

2A3Q Convergent filters (a) Let $(X, \mathfrak{T})$ be a topological space, $\mathcal{F}$ a filter on X and x a point of X. We say that $\mathcal{F}$ is **convergent** to x, or that x is a **limit** of $\mathcal{F}$, and write $\mathcal{F} \to x$, if every open set containing x belongs to $\mathcal{F}$.

(b) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces, $\phi : X \to Y$ a continuous function, $x \in X$ and $\mathcal{F}$ a filter on X converging to x. Then $\phi[[\mathcal{F}]]$ converges to $\phi(x)$.

2A3R Theorem Let X be a topological space, and K a subset of X. Then K is compact iff every ultrafilter on X containing K has a limit in K.

2A3S Further calculations with filters (a) It is possible for a filter to have more than one limit; but in Hausdorff spaces this does not occur.

Accordingly we can safely write $x = \lim \mathcal{F}$ when $\mathcal{F} \to x$ in a Hausdorff space.

(b) Now suppose that X is a set, $\mathcal{F}$ is a filter on X, $(Y, \mathfrak{S})$ is a Hausdorff space, $D \in \mathcal{F}$ and $\phi : D \to Y$ is a function. Then we write $\lim_{x \to \mathcal{F}} \phi(x)$ for $\lim \phi[[\mathcal{F}]]$ if this is defined in Y; that is, $\lim_{x \to \mathcal{F}} \phi(x) = y$ iff $\phi^{-1}[H] \in \mathcal{F}$ for every open set H containing y.

If Z is another set, $\mathcal{G}$ is a filter on Z, and $\psi : Z \to X$ is such that $\mathcal{F} = \psi[[\mathcal{G}]]$, then the composition $\phi\psi$ is defined on $\psi^{-1}[D] \in \mathcal{G}$, and if one of the limits $\lim_{x \to \mathcal{F}} \phi(x)$, $\lim_{z \to \mathcal{G}} \phi\psi(z)$ is defined in Y so is the other, and they are then equal.

In the special case $Y = \mathbb{R}$, $\lim_{x \to \mathcal{F}} \phi(x) = a$ iff $\{x : |\phi(x) - a| \leq \epsilon\} \in \mathcal{F}$ for every $\epsilon > 0$.

(c) Suppose that X and Y are sets, $\mathcal{F}$ is a filter on X, Θ is a non-empty family of pseudometrics on Y defining a topology $\mathfrak{S}$ on Y, and $\phi : X \to Y$ is a function. Then the image filter $\phi[[\mathcal{F}]]$ converges to $y \in Y$ iff $\lim_{x\to\mathcal{F}} \theta(\phi(x), y) = 0$ in $\mathbb{R}$ for every $\theta \in \Theta$.

(d) In particular, if X has a topology $\mathfrak{T}$ defined by a non-empty family P of pseudometrics, then a filter $\mathcal{F}$ on X converges to $x \in X$ iff $\lim_{y\to\mathcal{F}} \rho(y, x) = 0$ for every $\rho \in \mathrm{P}$.

(e)(i) If X is any set, $\mathcal{F}$ is an ultrafilter on X, $(Y, \mathfrak{S})$ is a Hausdorff space, and $h : X \to Y$ is a function such that $h[F]$ is relatively compact in Y for some $F \in \mathcal{F}$, then $\lim_{x\to\mathcal{F}} h(x)$ is defined in Y.

(ii) If X is any set, $\mathcal{F}$ is an ultrafilter on X, and $h : X \to \mathbb{R}$ is a function such that $h[F]$ is bounded in $\mathbb{R}$ for some set $F \in \mathcal{F}$, then $\lim_{x\to\mathcal{F}} h(x)$ exists in $\mathbb{R}$.

(f) Suppose that $\mathcal{F}$ is a filter on a set X, and that $f : X \to [-\infty, \infty]$ is any function. Then

$$\limsup_{x\to\mathcal{F}} f(x) = \inf_{F\in\mathcal{F}} \sup_{x\in F} f(x) \in [-\infty, \infty],$$

$$\liminf_{x\to\mathcal{F}} f(x) = \sup_{F\in\mathcal{F}} \inf_{x\in F} f(x).$$

For any two functions f, $g : X \to \mathbb{R}$,

$$\lim_{x\to\mathcal{F}} f(x) = a \quad \text{iff} \quad a = \limsup_{x\to\mathcal{F}} f(x) = \liminf_{x\to\mathcal{F}} f(x),$$

and

$$\limsup_{x\to\mathcal{F}} f(x) + g(x) \le \limsup_{x\to\mathcal{F}} f(x) + \limsup_{x\to\mathcal{F}} g(x),$$

$$\liminf_{x\to\mathcal{F}} f(x) + g(x) \ge \liminf_{x\to\mathcal{F}} f(x) + \liminf_{x\to\mathcal{F}} g(x),$$

$$\liminf_{x\to\mathcal{F}}(-f(x)) = -\limsup_{x\to\mathcal{F}} f(x), \quad \limsup_{x\to\mathcal{F}}(-f(x)) = -\liminf_{x\to\mathcal{F}} f(x),$$

$$\liminf_{x\to\mathcal{F}} cf(x) = c \liminf_{x\to\mathcal{F}} f(x), \quad \limsup_{x\to\mathcal{F}} cf(x) = c \limsup_{x\to\mathcal{F}} f(x)$$

whenever the right-hand-sides are defined in $[-\infty, \infty]$ and $c \ge 0$. So if $a = \lim_{x\to\mathcal{F}} f(x)$ and $b = \lim_{x\to\mathcal{F}}(x)$ exist in $\mathbb{R}$, $\lim_{x\to\mathcal{F}} f(x) + g(x)$ exists and is equal to $a + b$ and $\lim_{x\to\mathcal{F}} cf(x)$ exists and is equal to $c \lim_{x\to\mathcal{F}} f(x)$ for every $c \in \mathbb{R}$.

If $f : X \to \mathbb{R}$ is such that

for every $\epsilon > 0$ there is an $F \in \mathcal{F}$ such that $\sup_{x\in F} f(x) \le \epsilon + \inf_{x\in F} f(x)$,

then $\lim_{x\to\mathcal{F}} f(x)$ is defined in $[-\infty, \infty]$.

(g) $\lim_{n\to\infty}$, $\limsup_{n\to\infty}$, $\liminf_{n\to\infty}$ correspond to $\lim_{n\to\mathcal{F}_{\mathrm{Fr}}}$, $\limsup_{n\to\mathcal{F}_{\mathrm{Fr}}}$, $\liminf_{n\to\mathcal{F}_{\mathrm{Fr}}}$ where $\mathcal{F}_{\mathrm{Fr}}$ is the **Fréchet filter** on $\mathbb{N}$, the filter $\{\mathbb{N} \setminus A : A \subseteq \mathbb{N}$ is finite$\}$ of cofinite subsets of $\mathbb{N}$. Similarly, $\lim_{\delta\downarrow a}$, $\limsup_{\delta\downarrow a}$, $\liminf_{\delta\downarrow a}$ correspond to $\lim_{\delta\to\mathcal{F}}$, $\limsup_{\delta\to\mathcal{F}}$, $\liminf_{\delta\to\mathcal{F}}$ where

$$\mathcal{F} = \{A : A \subseteq \mathbb{R}, \exists\, h > 0 \text{ such that }]a, a + h] \subseteq A\}.$$

2A3T Product topologies (a) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces. Let $\mathfrak{U}$ be the set of subsets U of $X \times Y$ such that for every $(x, y) \in U$ there are $G \in \mathfrak{T}$, $H \in \mathfrak{S}$ such that $(x, y) \in G \times H \subseteq U$. Then $\mathfrak{U}$ is a topology on $X \times Y$.

$\mathfrak{U}$ is called the **product topology** on $X \times Y$.

(b) Suppose, in (a), that $\mathfrak{T}$ and $\mathfrak{S}$ are defined by non-empty families P, Θ of pseudometrics. Then $\mathfrak{U}$ is defined by the family $\Upsilon = \{\tilde\rho : \rho \in \mathrm{P}\} \cup \{\bar\theta : \theta \in \Theta\}$ of pseudometrics on $X \times Y$, where

$$\tilde\rho((x, y), (x', y')) = \rho(x, x'), \quad \bar\theta((x, y), (x', y')) = \theta(y, y')$$

whenever x, $x' \in X$, y, $y' \in Y$, $\rho \in \mathrm{P}$ and $\theta \in \Theta$.

(c) In particular, the product topology on $\mathbb{R}^r \times \mathbb{R}^s$ is the Euclidean topology if we identify $\mathbb{R}^r \times \mathbb{R}^s$ with $\mathbb{R}^{r+s}$.

2A3U Dense sets (a) If X is a topological space, a set $D \subseteq X$ is **dense** in X if $\overline{D} = X$. More generally, if $D \subseteq A \subseteq X$, then D is dense in A if it is dense for the subspace topology of A.

(b) If $\mathfrak{T}$ is defined by a non-empty family P of pseudometrics on X, then $D \subseteq X$ is dense iff

$$U(x; \rho_0, \dots, \rho_n; \delta) \cap D \neq \emptyset$$

whenever $x \in X$, $\rho_0, \dots, \rho_n \in \mathrm{P}$ and $\delta > 0$.

(c) If $(X, \mathfrak{T})$, $(Y, \mathfrak{S})$ are topological spaces, of which Y is Hausdorff (in particular, if (X, ρ) and (Y, θ) are metric spaces), and f, $g : X \to Y$ are continuous functions which agree on some dense subset D of X, then $f = g$.

(d) A topological space is called **separable** if it has a countable dense subset.

Version of 4.3.14

2A4 Normed spaces

In Chapter 24 I discuss the spaces L^p, for $1 \le p \le \infty$, and describe their most basic properties. These spaces form a cluster of fundamental examples for the general theory of 'normed spaces', the basis of functional analysis. This is not the book from which you should learn that theory, but once again it may save you trouble if I briefly outline those parts of the general theory which are essential if you are to make sense of the ideas here.

2A4A The real and complex fields While the most important parts of the theory, from the point of view of measure theory, are most effectively dealt with in terms of *real* linear spaces, there are many applications in which *complex* linear spaces are essential. I will therefore use the phrase

'U is a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$'

to mean that U is either a linear space over the field $\mathbb{R}$ or a linear space over the field $\mathbb{C}$; it being understood that in any particular context all linear spaces considered will be over the same field. In the same way, I will write '$\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$' to mean that α belongs to whichever is the current underlying field.

2A4B Definitions (a) A **normed space** is a linear space U over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ together with a **norm**, that is, a functional $\| \; \| : U \to [0, \infty[$ such that

$\|u + v\| \le \|u\| + \|v\|$ for all u, $v \in U$,
$\|\alpha u\| = |\alpha| \|u\|$ for $u \in U$, $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$,
$\|u\| = 0$ only when $u = 0$, the zero vector of U.

(b) If U is a normed space, then we have a metric ρ on U defined by saying that $\rho(u, v) = \|u - v\|$ for u, $v \in U$.

(c) If U is a normed space, a set $A \subseteq U$ is **bounded** if $\{\|u\| : u \in A\}$ is bounded in $\mathbb{R}$.

2A4C Linear subspaces (a) If U is any normed space and V is a linear subspace of U, then V is also a normed space, if we take the norm of V to be just the restriction to V of the norm of U.

(b) If V is a linear subspace of U, so is its closure.

2A4D Banach spaces (a) If U is a normed space, a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U is **Cauchy** if $\|u_m - u_n\| \to 0$ as m, $n \to \infty$.

(b) A normed space U is **complete** if every Cauchy sequence has a limit; a complete normed space is called a **Banach space**.

2A4E Lemma Let U be a normed space such that $\langle u_n\rangle_{n\in\mathbb{N}}$ is convergent in U whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\|u_{n+1}-u_n\|\le 4^{-n}$ for every $n\in\mathbb{N}$. Then U is complete.

2A4F Bounded linear operators (a) Let U, V be two normed spaces. A linear operator $T:U\to V$ is **bounded** if $\{\|Tu\|:u\in U,\ \|u\|\le 1\}$ is bounded. Write $\mathsf{B}(U;V)$ for the space of all bounded linear operators from U to V, and for $T\in\mathsf{B}(U;V)$ write $\|T\|=\sup\{\|Tu\|:u\in U,\ \|u\|\le 1\}$.

(b) $\|Tu\|\le\|T\|\|u\|$ whenever $T\in\mathsf{B}(U;V)$ and $u\in U$.

(c) A linear operator $T:U\to V$ is bounded iff it is continuous for the norm topologies on U and V.

(d) If U, V and W are normed spaces, $S\in\mathsf{B}(U;V)$ and $T\in\mathsf{B}(V;W)$ then $TS\in\mathsf{B}(U;W)$ and $\|TS\|\le\|T\|\|S\|$.

2A4G Theorem $\mathsf{B}(U;V)$ is a linear space over ${}^{\mathbb{R}}_{\mathbb{C}}$, and $\|\ \|$ is a norm on $\mathsf{B}(U;V)$.

2A4H Dual spaces The most important case of $\mathsf{B}(U;V)$ is when V is the scalar field ${}^{\mathbb{R}}_{\mathbb{C}}$ itself. In this case we call $\mathsf{B}(U;{}^{\mathbb{R}}_{\mathbb{C}})$ the **dual** of U; it is commonly denoted U' or U^*; I use the latter.

2A4I Extensions of bounded operators: Theorem Let U be a normed space and $V\subseteq U$ a dense linear subspace. Let W be a Banach space and $T_0:V\to W$ a bounded linear operator; then there is a unique bounded linear operator $T:U\to W$ extending T_0, and $\|T\|=\|T_0\|$.

2A4J Normed algebras (a) A **normed algebra** is a normed space $(U,\|\ \|)$ together with a multiplication, a binary operator $\times$ on U, such that

$$u\times(v\times w)=(u\times v)\times w,$$

$$u\times(v+w)=(u\times v)+(u\times w),\quad (u+v)\times w=(u\times w)+(v\times w),$$

$$(\alpha u)\times v=u\times(\alpha v)=\alpha(u\times v),$$

$$\|u\times v\|\le\|u\|\|v\|$$

for all u, v, $w\in U$ and $\alpha\in{}^{\mathbb{R}}_{\mathbb{C}}$.

(b) A **Banach algebra** is a normed algebra which is a Banach space. A normed algebra is **commutative** if its multiplication is commutative.

***2A4K Definition** A normed space U is **uniformly convex** if for every $\epsilon>0$ there is a $\delta>0$ such that $\|u+v\|\le 2-\delta$ whenever u, $v\in U$, $\|u\|=\|v\|=1$ and $\|u-v\|\ge\epsilon$.

Version of 13.11.07

2A5 Linear topological spaces

The principal objective of §2A3 is in fact the study of certain topologies on the linear spaces of Chapter 24. I give some fragments of the general theory.

2A5A Linear space topologies: Definition A **linear topological space** or **topological vector space** over ${}^{\mathbb{R}}_{\mathbb{C}}$ is a linear space U over ${}^{\mathbb{R}}_{\mathbb{C}}$ together with a topology $\mathfrak{T}$ such that the maps

$$(u,v)\mapsto u+v:U\times U\to U,$$

$$(\alpha, u) \mapsto \alpha u : \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}} \times U \to U$$

are both continuous, where the product spaces $U \times U$ and $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}} \times U$ are given their product topologies. Given a linear space U, a topology on U satisfying the conditions above is a **linear space topology**. Note that

$$(u, v) \mapsto u - v = u + (-1)v : U \times U \to U$$

will also be continuous.

2A5B Proposition Suppose that U is a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and T is a family of functionals $\tau : U \to [0, \infty[$ such that

(i) $\tau(u + v) \le \tau(u) + \tau(v)$ for all u, $v \in U$ and $\tau \in \mathrm{T}$;

(ii) $\tau(\alpha u) \le \tau(u)$ if $u \in U$, $|\alpha| \le 1$ and $\tau \in \mathrm{T}$;

(iii) $\lim_{\alpha \to 0} \tau(\alpha u) = 0$ whenever $u \in U$ and $\tau \in \mathrm{T}$.

For $\tau \in \mathrm{T}$, define $\rho_\tau : U \times U \to [0, \infty[$ by setting $\rho_\tau(u, v) = \tau(u - v)$ for all u, $v \in U$. Then each ρ_τ is a pseudometric on U, and the topology defined by $\mathrm{P} = \{\rho_\tau : \tau \in \mathrm{T}\}$ renders U a linear topological space.

Remark Functionals satisfying the conditions (i)-(iii) above are called **F-seminorms**; an F-seminorm τ such that $\tau(u) \ne 0$ for every non-zero u is an **F-norm**.

***2A5C Theorem** Let U be a linear space and $\mathfrak{T}$ a linear space topology on U.

(a) There is a family T of F-seminorms defining $\mathfrak{T}$ as in 2A5B.

(b) If $\mathfrak{T}$ is metrizable, we can take T to consist of a single functional.

2A5D Definition Let U be a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. Then a **seminorm** on U is a functional $\tau : U \to [0, \infty[$ such that

(i) $\tau(u + v) \le \tau(u) + \tau(v)$ for all u, $v \in U$;

(ii) $\tau(\alpha u) = |\alpha|\tau(u)$ if $u \in U$, $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

2A5E Convex sets (a) Let U be a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. A subset C of U is **convex** if $\alpha u + (1 - \alpha)v \in C$ whenever u, $v \in C$ and $\alpha \in [0, 1]$. The intersection of any family of convex sets is convex, so for every set $A \subseteq U$ there is a smallest convex set including A; this is the set of vectors expressible as $\sum_{i=0}^n \alpha_i u_i$ where $u_0, \dots, u_n \in A$, $\alpha_0, \dots, \alpha_n \in [0, 1]$ and $\sum_{i=0}^n \alpha_i = 1$; it is the **convex hull** of A. If C, $C' \subseteq U$ are convex, and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, then αC and $C + C'$ are convex. If $C \subseteq U$ is convex, V is another linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $T : U \to V$ is a linear operator, then $T[C] \subseteq V$ is convex.

(b) If U is a linear topological space, the closure of any convex set is convex. It follows that, for any $A \subseteq U$, the closure of the convex hull of A is the smallest closed convex set including A; this is the **closed convex hull** of A.

(c) I note for future reference that in a linear topological space, the closure of any linear subspace is a linear subspace.

2A5F Completeness in linear topological spaces: Definitions Let U be a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $\mathfrak{T}$ a linear space topology on U. A filter $\mathcal{F}$ on U is **Cauchy** if for every open set G in U containing 0 there is an $F \in \mathcal{F}$ such that $F - F = \{u - v : u, v \in F\}$ is included in G. U is **complete** if every Cauchy filter on U is convergent.

2A5G Lemma Let U be a linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and let T be a family of F-seminorms defining a linear space topology on U, as in 2A5B. Then a filter $\mathcal{F}$ on U is Cauchy iff for every $\tau \in \mathrm{T}$ and $\epsilon > 0$ there is an $F \in \mathcal{F}$ such that $\tau(u - v) \le \epsilon$ for all u, $v \in F$.

2A5H Normed spaces and sequential completeness: Proposition Let $(U, \| \, \|)$ be a normed space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and let $\mathfrak{T}$ be the linear space topology on U defined by the method of 2A5B from the set $\mathrm{T} = \{\| \, \|\}$. Then U is complete in the sense of 2A5F iff it is complete in the sense of 2A4D.

2A5I Weak topologies Let U be a normed linear space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

(a) Write U^* for its dual $\mathrm{B}(U;\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}})$. $\mathrm{T} = \{|h| : h \in U^*\}$ defines a linear space topology on U; this is the **weak topology** of U.

(b) A filter $\mathcal{F}$ on U converges to $u \in U$ for the weak topology of U iff $\lim_{v\to\mathcal{F}} h(v) = h(u)$ for every $h \in U^*$.

(c) A set $C \subseteq U$ is called **weakly compact** if it is compact for the weak topology of U. So a set $C \subseteq U$ is weakly compact iff for every ultrafilter $\mathcal{F}$ on U containing C there is a $u \in C$ such that $\lim_{v\to\mathcal{F}} h(v) = h(u)$ for every $h \in U^*$.

(d) A subset A of U is called **relatively weakly compact** if it is a subset of some weakly compact subset of U.

(e) If $h \in U^*$, then $h : U \to \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ is continuous for the weak topology on U and the usual topology of $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. So if $A \subseteq U$ is relatively weakly compact, $h[A]$ must be bounded in $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

(f) If V is another normed space and $T : U \to V$ is a bounded linear operator, then T is continuous for the respective weak topologies.

(g) Corresponding to the weak topology on a normed space U, we have the **weak*** or **w*-**topology on its dual U^*, defined by the set $\mathrm{T} = \{|\hat{u}| : u \in U\}$, where I write $\hat{u}(f) = f(u)$ for every $f \in U^*$, $u \in U$. As in (a), this is a linear space topology on U^*.

***2A5J Angelic spaces** First, a topological space X is **regular** if whenever $G \subseteq X$ is open and $x \in G$ then there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$. Next, a regular Hausdorff space X is **angelic** if whenever $A \subseteq X$ is such that every sequence in A has a cluster point in X, then $\overline{A}$ is compact and every point of $\overline{A}$ is the limit of a sequence in A. Now any normed space is angelic in its weak topology.

Version of 10.11.14

2A6 Factorization of matrices

I spend a couple of pages on the linear algebra of $\mathbb{R}^r$ required for Chapter 26. I give only one proof, because this is material which can be found in any textbook of elementary linear algebra; but I think it may be helpful to run through the basic ideas in the language which I use for this treatise.

2A6A Determinants

(i) Every $r \times r$ real matrix T has a real determinant $\det T$.

(ii) For any $r \times r$ matrices S and T, $\det ST = \det S \det T$.

(iii) If T is a diagonal matrix, its determinant is just the product of its diagonal entries.

(iv) For any $r \times r$ matrix T, $\det T^\top = \det T$, where $T^\top$ is the transpose of T.

(v) $\det T$ is a continuous function of the coefficients of T.

2A6B Orthonormal families For $x = (\xi_1, \dots, \xi_r)$, $y = (\eta_1, \dots, \eta_r) \in \mathbb{R}^r$, write $x\,.\,y = \sum_{i=1}^r \xi_i\eta_i$; $\|x\|$ is $\sqrt{x\,.\,x}$. $x_1, \dots, x_k$ are **orthonormal** if $x_i\,.\,x_j = 0$ for $i \ne j$, 1 for $i = j$.

(i) If $x_1, \dots, x_k$ are orthonormal vectors in $\mathbb{R}^r$, where $k < r$, then there are vectors $x_{k+1}, \dots, x_r$ in $\mathbb{R}^r$ such that $x_1, \dots, x_r$ are orthonormal.

(ii) An $r \times r$ matrix P is **orthogonal** if $P^\top P$ is the identity matrix; equivalently, if the columns of P are orthonormal.

(iii) For an orthogonal matrix P, $\det P$ must be ± 1.

(iv) If P is orthogonal, then $Px\,.\,Py = x\,.\,y$ for all x, $y \in \mathbb{R}^r$.

(v) If P is orthogonal, so is $P^\top = P^{-1}$.

(vi) If P and Q are orthogonal, so is PQ.

2A6C Proposition Let T be any real $r \times r$ matrix. Then T is expressible as PDQ where P and Q are orthogonal matrices and D is a diagonal matrix with non-negative coefficients.