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Chapter 28

Fourier analysis

For the last chapter of this volume, I attempt a brief account of one of the most important topics in analysis. This is a bold enterprise, and I cannot hope to satisfy the reasonable demands of anyone who knows and loves the subject as it deserves. But I also cannot pass it by without being false to my own subject, since problems contributed by the study of Fourier series and transforms have led measure theory throughout its history. What I will try to do, therefore, is to give versions of those results which everyone ought to know in language unifying them with the rest of this treatise, aiming to open up a channel for the transfer of intuitions and techniques between the abstract general study of measure spaces, which is the centre of our work, and this particular family of applications of the theory of integration.

I have divided the material of this chapter, conventionally enough, into three parts: Fourier series, Fourier transforms and the characteristic functions of probability theory. While it will be obvious that many ideas are common to all three, I do not think it useful, at this stage, to try to formulate an explicit generalization to unify them; that belongs to a more general theory of harmonic analysis on groups, which must wait until Volume 4. I begin however with a section on the Stone-Weierstrass theorem (§281), which is one of the basic tools of functional analysis, as well as being useful for this chapter. The final section (§286), a proof of Carleson's theorem, is at a rather different level from the rest.

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281 The Stone-Weierstrass theorem

Before we begin work on the real subject of this chapter, it will be helpful to have a reasonably general statement of a fundamental theorem on the approximation of continuous functions. In fact I give a variety of forms (281A, 281E, 281F and 281G, together with 281Ya, 281Yd and 281Yg), all of which are sometimes useful. I end the section with a version of Weyl's Equidistribution Theorem (281M-281N).

281A Stone-Weierstrass theorem: first form Let X be a topological space and K a compact subset of X. Write $C_b(X)$ for the space of all bounded continuous real-valued functions on X, so that $C_b(X)$ is a linear space over $\mathbb{R}$. Let $A \subseteq C_b(X)$ be such that

A is a linear subspace of $C_b(X)$;

$|f| \in A$ for every $f \in A$;

$\chi X \in A$;

whenever x, y are distinct points of K there is an $f \in A$ such that $f(x) \neq f(y)$.

Then for every continuous $h : K \to \mathbb{R}$ and $\epsilon > 0$ there is an $f \in A$ such that

$|f(x) - h(x)| \leq \epsilon$ for every $x \in K$,

if $K \neq \emptyset$, $\inf_{x \in X} f(x) \geq \inf_{x \in K} h(x)$ and $\sup_{x \in X} f(x) \leq \sup_{x \in K} h(x)$.

Remark I have stated this theorem in its natural context, that of general topological spaces. But if these are unfamiliar to you, you do not in fact need to know what they are. If you read 'let X be a topological space' as 'let X be a subset of $\mathbb{R}^r$' and 'K is a compact subset of X' as 'K is a subset of X which is closed and bounded in $\mathbb{R}^r$', you will have enough for all the applications in this chapter. In order to follow the proof, of course, you will need to know a little about compactness in $\mathbb{R}^r$; I have written out the necessary facts in §2A2.

proof (a) If K is empty, then we can take $f = \mathbf{0}$ to be the constant function with value 0. So henceforth let us suppose that $K \neq \emptyset$.

(b) The first point to note is that if f, $g \in A$ then $f \wedge g$ and $f \vee g$ belong to A, where

$$(f \wedge g)(x) = \min(f(x), g(x)), \quad (f \vee g)(x) = \max(f(x), g(x))$$

for every $x \in X$; this is because

$$f \wedge g = \frac{1}{2}(f + g - |f - g|), \quad f \vee g = \frac{1}{2}(f + g + |f - g|).$$

It follows by induction on n that $f_0 \wedge \ldots \wedge f_n$ and $f_0 \vee \ldots \vee f_n$ belong to A for all $f_0, \ldots, f_n \in A$.

(c) If x, y are distinct points of K, and a, $b \in \mathbb{R}$, there is an $f \in A$ such that $f(x) = a$ and $f(y) = b$. **P** Start from $g \in A$ such that $g(x) \neq g(y)$; this is the point at which we use the last of the list of four hypotheses on A. Set

$$\alpha = \frac{a-b}{g(x)-g(y)}, \quad \beta = \frac{bg(x)-ag(y)}{g(x)-g(y)}, \quad f = \alpha g + \beta \chi X \in A. \ \mathbf{Q}$$

(d) (The heart of the proof lies in the next two paragraphs.) Let $h : K \to [0, \infty[$ be a continuous function and x any point of K. For any $\epsilon > 0$, there is an $f \in A$ such that $f(x) = h(x)$ and $f(y) \leq h(y) + \epsilon$ for every $y \in K$. **P** Let $\mathcal{G}_x$ be the family of those open sets $G \subseteq X$ for which there is some $f \in A$ such that $f(x) = h(x)$ and $f(w) \leq h(w) + \epsilon$ for every $w \in K \cap G$. I claim that $K \subseteq \bigcup \mathcal{G}_x$. To see this, take any $y \in K$. By (c), there is an $f \in A$ such that $f(x) = h(x)$ and $f(y) = h(y)$. Now $h - f \restriction K : K \to \mathbb{R}$ is a continuous function, taking the value 0 at y, so there is an open subset G of X, containing y, such that $(h - f \restriction K)(w) \geq -\epsilon$ for every $w \in G \cap K$, that is, $f(w) \leq h(w) + \epsilon$ for every $w \in G \cap K$. Thus $G \in \mathcal{G}_x$ and $y \in \bigcup \mathcal{G}_x$, as required.

Because K is compact, $\mathcal{G}_x$ has a finite subcover $G_0, \ldots, G_n$ say. For each $i \leq n$, take $f_i \in A$ such that $f_i(x) = h(x)$ and $f_i(w) \leq h(w) + \epsilon$ for every $w \in G_i \cap K$. Then

$$f = f_0 \wedge f_1 \wedge \ldots \wedge f_n \in A,$$

by (b), and evidently $f(x) = h(x)$, while if $y \in K$ there is some $i \leq n$ such that $y \in G_i$, so that

$$f(y) \leq f_i(y) \leq h(y) + \epsilon. \ \mathbf{Q}$$

(e) If $h : K \to \mathbb{R}$ is any continuous function and $\epsilon > 0$, there is an $f \in A$ such that $|f(y) - h(y)| \leq \epsilon$ for every $y \in K$. **P** This time, let $\mathcal{G}$ be the set of those open subsets G of X for which there is some $f \in A$ such that $f(y) \leq h(y) + \epsilon$ for every $y \in K$ and $f(x) \geq h(x) - \epsilon$ for every $x \in G \cap K$. Once again, $\mathcal{G}$ is an open cover of K. To see this, take any $x \in K$. By (d), there is an $f \in A$ such that $f(x) = h(x)$ and $f(y) \leq h(y) + \epsilon$ for every $y \in K$. Now $h - f \restriction K : K \to \mathbb{R}$ is a continuous function which is zero at x, so there is an open subset G of X, containing x, such that $(h - f \restriction K)(w) \leq \epsilon$ for every $w \in G \cap K$, that is, $f(w) \geq h(w) - \epsilon$ for every $w \in G \cap K$. Thus $G \in \mathcal{G}$ and $x \in \bigcup \mathcal{G}$, as required.

Because K is compact, $\mathcal{G}$ has a finite subcover $G_0, \ldots, G_m$ say. For each $j \leq m$, take $f_j \in A$ such that $f_j(y) \leq h(y) + \epsilon$ for every $y \in K$ and $f_j(w) \geq h(w) - \epsilon$ for every $w \in G_j \cap K$. Then

$$f = f_0 \vee f_1 \vee \ldots \vee f_m \in A,$$

by (b), and evidently $f(y) \leq h(y) + \epsilon$ for every $y \in K$, while if $x \in K$ there is some $j \leq m$ such that $x \in G_j$, so that

$$f(x) \geq f_j(x) \geq h(x) - \epsilon.$$

Thus $|f(x) - h(x)| \leq \epsilon$ for every $x \in K$, as required. **Q**

(f) Thus we have an f satisfying the first of the two requirements of the theorem. But for the second, set $M_0 = \inf_{x \in K} h(x)$ and $M_1 = \sup_{x \in K} h(x)$, and

$$f_1 = \operatorname{med}(M_0 \chi X, f, M_1 \chi X) = (M_0 \chi X) \vee (f \wedge M_1 \chi X);$$

f_1 satisfies the second condition as well as the first. (I am tacitly assuming here what is in fact the case, that M_0 and M_1 are finite; this is because K is compact – see 2A2G or 2A3N.)

281B We need some simple tools, belonging to the basic theory of normed spaces; but I hope they will be accessible even if you have not encountered 'normed spaces' before, if you keep a finger at the beginning of §2A4 as you read the next lemma.

Lemma Let X be any set. Write $\ell^\infty(X)$ for the set of bounded functions from X to $\mathbb{R}$. For $f \in \ell^\infty(X)$, set

$$\|f\|_\infty = \sup_{x\in X} |f(x)|,$$

counting the supremum as 0 if X is empty. Then

(a) $\ell^\infty(X)$ is a normed space.

(b) Let $A \subseteq \ell^\infty(X)$ be a subset and $\overline{A}$ its closure (2A3D).

(i) If A is a linear subspace of $\ell^\infty(X)$, so is $\overline{A}$.

(ii) If $f \times g \in A$ whenever f, $g \in A$, then $f \times g \in \overline{A}$ whenever f, $g \in \overline{A}$.

(iii) If $|f| \in A$ whenever $f \in A$, then $|f| \in \overline{A}$ whenever $f \in \overline{A}$.

proof (a) This is a routine verification. To confirm that $\ell^\infty(X)$ is a linear space over $\mathbb{R}$, we have to check that $f+g$, cf belong to $\ell^\infty(X)$ whenever f, $g \in \ell^\infty(X)$ and $c \in \mathbb{R}$; simultaneously we can confirm that $\| \|_\infty$ is a norm on $\ell^\infty(X)$ by observing that

$$|(f+g)(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty,$$

$$|cf(x)| = |c||f(x)| \le |c|\|f\|_\infty$$

whenever f, $g \in \ell^\infty(X)$ and $c \in \mathbb{R}$. It is worth noting at the same time that if f, $g \in \ell^\infty(X)$, then

$$|(f \times g)(x)| = |f(x)||g(x)| \le \|f\|_\infty \|g\|_\infty$$

for every $x \in X$, so that $\|f \times g\|_\infty \le \|f\|_\infty \|g\|_\infty$.

(Of course all these remarks are very elementary special cases of parts of §243; see 243Xl.)

(b) Recall that

$$\overline{A} = \{f : f \in \ell^\infty(X), \forall\, \epsilon > 0\ \exists\, f_1 \in A,\ \|f - f_1\|_\infty \le \epsilon\}$$

(2A3Kb). Take f, $g \in \overline{A}$ and $c \in \mathbb{R}$, and let $\epsilon > 0$. Set

$$\eta = \min(1, \tfrac{\epsilon}{2+|c|+\|f\|_\infty+\|g\|_\infty}) > 0.$$

Then there are f_1, $g_1 \in A$ such that $\|f - f_1\|_\infty \le \eta$ and $\|g - g_1\|_\infty \le \eta$.

Now

$$\|(f+g) - (f_1+g_1)\|_\infty \le \|f - f_1\|_\infty + \|g - g_1\|_\infty \le 2\eta \le \epsilon,$$

$$\|cf - cf_1\|_\infty = |c|\|f - f_1\|_\infty \le |c|\eta \le \epsilon,$$

$$\begin{aligned}\|(f \times g) - (f_1 \times g_1)\|_\infty &= \|(f - f_1) \times g + f \times (g - g_1) - (f - f_1) \times (g - g_1)\|_\infty \\ &\le \|(f - f_1) \times g\|_\infty + \|f \times (g - g_1)\|_\infty + \|(f - f_1) \times (g - g_1)\|_\infty \\ &\le \|f - f_1\|_\infty \|g\|_\infty + \|f\|_\infty \|g - g_1)\|_\infty + \|f - f_1\|_\infty \|g - g_1\|_\infty \\ &\le \eta(\|g\|_\infty + \|f\|_\infty + \eta) \le \eta(\|g\|_\infty + \|f\|_\infty + 1) \le \epsilon,\end{aligned}$$

$$\||f| - |f_1|\|_\infty \le \|f - f_1\|_\infty \le \eta \le \epsilon.$$

(i) If A is a linear subspace, then $f_1 + g_1$ and cf_1 belong to A. As ϵ is arbitrary, $f + g$ and cf belong to $\overline{A}$. As f, g and c are arbitrary, $\overline{A}$ is a linear subspace of $\ell^\infty(X)$.

(ii) If A is closed under multiplication, then $f_1 \times g_1 \in A$. As ϵ is arbitrary, $f \times g \in \overline{A}$.

(iii) If the absolute values of functions in A belong to A, then $|f_1| \in A$. As ϵ is arbitrary, $|f| \in \overline{A}$.

281C Lemma There is a sequence $\langle p_n \rangle_{n\in\mathbb{N}}$ of real polynomials such that $\lim_{n\to\infty} p_n(x) = |x|$ uniformly for $x \in [-1, 1]$.

proof (a) By the Binomial Theorem we have

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{4\cdot 2!}x^2 - \frac{1\cdot 3}{2^3\cdot 3!}x^3 - \ldots = -\sum_{n=0}^{\infty} \frac{(2n)!}{(2n-1)(2^n n!)^2}x^n$$

whenever $|x| < 1$, with the convergence being uniform on any interval $[-a, a]$ with $0 \le a < 1$. (For a proof of this, see almost any book on real or complex analysis. If you have no favourite text to hand, you can try to construct a proof from the following facts: (i) the radius of convergence of the series is 1, so on any interval $[-a, a]$, with $0 \le a < 1$, it is uniformly absolutely summable (ii) writing $f(x)$ for the sum of the series for $|x| < 1$, use Lebesgue's Dominated Convergence Theorem to find expressions for the indefinite integrals $\int_0^x f$, $-\int_{-x}^0 f$ and show that these are $\frac{2}{3}(1-(1-x)f(x))$, $\frac{2}{3}(1-(1+x)f(-x))$ for $0 \le x < 1$ (iii) use the Fundamental Theorem of Calculus to show that $f(x) + 2(1-x)f'(x) = 0$ (iv) show that $\frac{d}{dx}\bigl(\frac{f(x)^2}{1-x}\bigr) = 0$ and hence (v) that $f(x)^2 = 1 - x$ whenever $|x| < 1$. Finally, show that because f is continuous and non-zero in $]-1, 1[$, $f(x)$ must be the positive square root of $1 - x$ throughout.)

We have a further fragment of information. If we set

$$q_0(x) = 1, \quad q_1(x) = 1 - \frac{1}{2}x, \quad q_n(x) = -\sum_{k=0}^{n} \frac{(2k)!}{(2k-1)(2^k k!)^2}x^k$$

for $n \ge 2$ and $x \in [0, 1]$, so that q_n is the nth partial sum of the binomial series for $(1-x)^{1/2}$, then we have $\lim_{n\to\infty} q_n(x) = (1-x)^{1/2}$ for every $x \in [0, 1[$. But also every q_n is non-increasing on $[0, 1]$, and $\langle q_n(x)\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence for each $x \in [0, 1]$. So we must have

$$\sqrt{1-x} \le q_n(x)\ \forall\, n \in \mathbb{N},\ x \in [0, 1[,$$

and therefore, because all the q_n are continuous,

$$\sqrt{1-x} \le q_n(x)\ \forall\, n \in \mathbb{N},\ x \in [0, 1].$$

Moreover, given $\epsilon > 0$, set $a = 1 - \frac{1}{4}\epsilon^2$, so that $\sqrt{1-a} = \frac{\epsilon}{2}$. Then there is an $n_0 \in \mathbb{N}$ such that $q_n(x) - \sqrt{1-x} \le \frac{\epsilon}{2}$ for every $x \in [0, a]$ and $n \ge n_0$. In particular, $q_n(a) \le \epsilon$, so $q_n(x) \le \epsilon$ and $q_n(x) - \sqrt{1-x} \le \epsilon$ whenever $x \in [a, 1]$ and $n \ge n_0$. This means that

$$0 \le q_n(x) - \sqrt{1-x} \le \epsilon\ \forall\, n \ge n_0,\ x \in [0, 1];$$

as ϵ is arbitrary, $\langle q_n(x)\rangle_{n\in\mathbb{N}} \to \sqrt{1-x}$ uniformly on $[0, 1]$.

(b) Now set $p_n(x) = q_n(1 - x^2)$ for $x \in \mathbb{R}$. Because each q_n is a real polynomial of degree n, each p_n is a real polynomial of degree $2n$. Next,

$$\begin{aligned}\sup_{|x|\le 1} |p_n(x) - |x|| &= \sup_{|x|\le 1} |q_n(1-x^2) - \sqrt{1-(1-x^2)}| \\ &= \sup_{y\in[0,1]} |q_n(y) - \sqrt{1-y}| \to 0\end{aligned}$$

as $n \to \infty$, so $\lim_{n\to\infty} p_n(x) = |x|$ uniformly for $|x| \le 1$, as required.

281D Corollary Let X be a set, and A a norm-closed linear subspace of $\ell^\infty(X)$ containing χX and such that $f \times g \in A$ whenever f, $g \in A$. Then $|f| \in A$ for every $f \in A$.

proof Set

$$f_1 = \frac{1}{1+\|f\|_\infty}f,$$

so that $f_1 \in A$ and $\|f_1\|_\infty \le 1$. Because A contains χX and is closed under multiplication, $p\circ f_1 \in A$ for every polynomial p with real coefficients. In particular, $g_n = p_n \circ f_1 \in A$ for every n, where $\langle p_n\rangle_{n\in\mathbb{N}}$ is the sequence of 281C. Now, because $|f_1(x)| \le 1$ for every $x \in X$,

$$\|g_n - |f_1|\|_\infty = \sup_{x\in X} |p_n(f_1(x)) - |f_1(x)|| \le \sup_{|y|\le 1} |p_n(y) - |y|| \to 0$$

as $n \to \infty$. Because A is $\|\,\|_\infty$-closed, $|f_1| \in A$; consequently $|f| \in A$, as claimed.

281E Stone-Weierstrass theorem: second form Let X be a topological space and K a compact subset of X. Write $C_b(X)$ for the space of all bounded continuous real-valued functions on X. Let $A \subseteq C_b(X)$ be such that

A is a linear subspace of $C_b(X)$;

$f \times g \in A$ for every f, $g \in A$;

$\chi X \in A$;

whenever x, y are distinct points of K there is an $f \in A$ such that $f(x) \neq f(y)$.

Then for every continuous $h : K \to \mathbb{R}$ and $\epsilon > 0$ there is an $f \in A$ such that

$|f(x) - h(x)| \leq \epsilon$ for every $x \in K$,

if $K \neq \emptyset$, $\inf_{x \in X} f(x) \geq \inf_{x \in K} h(x)$ and $\sup_{x \in X} f(x) \leq \sup_{x \in K} h(x)$.

proof Let $\overline{A}$ be the $\| \|_\infty$-closure of A in $\ell^\infty(X)$. It is helpful to know that $\overline{A} \subseteq C_b(X)$; this is because the uniform limit of continuous functions is continuous. (But if this is new to you, or your memory has faded, don't take time to look it up now; just read '$\overline{A} \cap C_b(X)$' in place of '$\overline{A}$' in the rest of this argument.) By 281B-281D, $\overline{A}$ is a linear subspace of $C_b(X)$ and $|f| \in \overline{A}$ for every $f \in \overline{A}$, so the conditions of 281A apply to $\overline{A}$.

Take a continuous $h : K \to \mathbb{R}$ and an $\epsilon > 0$. The cases in which $K = \emptyset$ or h is constant are trivial, because all constant functions belong to A; so I suppose that $M_0 = \inf_{x \in K} h(x)$ and $M_1 = \sup_{x \in K} h(x)$ are defined and distinct. As observed at the end of the proof of 281A, M_0 and M_1 are finite. Set

$$\eta = \min(\tfrac{1}{3}\epsilon, \tfrac{1}{2}(M_1 - M_2)) > 0, \quad \tilde{h}(x) = \operatorname{med}(M_0 + \eta, h(x), M_1 - \eta) \text{ for } x \in K$$

(definition: 2A1Ac), so that $\tilde{h} : K \to \mathbb{R}$ is continuous and $M_0 + \eta \leq \tilde{h}(x) \leq M_1 - \eta$ for every $x \in K$. By 281A, there is an $f_0 \in \overline{A}$ such that $|f_0(x) - \tilde{h}(x)| \leq \eta$ for every $x \in K$ and $M_0 + \eta \leq f_0(x) \leq M_1 - \eta$ for every $x \in X$. Now there is an $f \in A$ such that $\|f - f_0\|_\infty \leq \eta$, so that

$$|f(x) - h(x)| \leq |f(x) - f_0(x)| + |f_0(x) - \tilde{h}(x)| + |\tilde{h}(x) - h(x)| \leq 3\eta \leq \epsilon$$

for every $x \in K$, while

$$M_0 \leq f_0(x) - \eta \leq f(x) \leq f_0(x) + \eta \leq M_1$$

for every $x \in X$.

281F Corollary: Weierstrass' theorem Let K be any closed bounded subset of $\mathbb{R}$. Then every continuous $h : K \to \mathbb{R}$ can be uniformly approximated on K by polynomials.

proof Apply 281E with $X = K$ (noting that K, being closed and bounded, is compact), and A the set of polynomials with real coefficients, regarded as functions from K to $\mathbb{R}$.

281G Stone-Weierstrass theorem: third form Let X be a topological space and K a compact subset of X. Write $C_b(X;\mathbb{C})$ for the space of all bounded continuous complex-valued functions on X, so that $C_b(X;\mathbb{C})$ is a linear space over $\mathbb{C}$. Let $A \subseteq C_b(X;\mathbb{C})$ be such that

A is a linear subspace of $C_b(X;\mathbb{C})$;

$f \times g \in A$ for every f, $g \in A$;

$\chi X \in A$;

the complex conjugate $\bar{f}$ of f belongs to A for every $f \in A$;

whenever x, y are distinct points of K there is an $f \in A$ such that $f(x) \neq f(y)$.

Then for every continuous $h : K \to \mathbb{C}$ and $\epsilon > 0$ there is an $f \in A$ such that

$|f(x) - h(x)| \leq \epsilon$ for every $x \in K$,

if $K \neq \emptyset$, $\sup_{x \in X} |f(x)| \leq \sup_{x \in K} |h(x)|$.

proof If $K = \emptyset$, or h is identically zero, we can take $f = \mathbf{0}$. So let us suppose that $M = \sup_{x \in K} |h(x)| > 0$.

(a) Set

$$A_{\mathbb{R}} = \{f : f \in A, \ f(x) \text{ is real for every } x \in X\}.$$

Then $A_{\mathbb{R}}$ satisfies the conditions of 281E. **P** (i) Evidently $A_{\mathbb{R}}$ is a subset of $C_b(X) = C_b(X;\mathbb{R})$, is closed under addition, multiplication by real scalars and pointwise multiplication of functions, and contains χX. If x, y are distinct points of K, there is an $f \in A$ such that $f(x) \neq f(y)$. Now

$$\mathcal{R}\mathrm{e}\, f = \frac{1}{2}(f + \bar{f}), \quad \mathcal{I}\mathrm{m}\, f = \frac{1}{2i}(f - \bar{f})$$

both belong to A and are real-valued, so belong to $A_{\mathbb{R}}$, and at least one of them takes different values at x and y. **Q**

(b) Consequently, given a continuous function $h : K \to \mathbb{C}$ and $\epsilon > 0$, we may apply 281E twice to find f_1, $f_2 \in A_{\mathbb{R}}$ such that

$$|f_1(x) - \mathcal{R}\mathrm{e}(h(x))| \le \eta, \quad |f_2(x) - \mathcal{I}\mathrm{m}(h(x))| \le \eta$$

for every $x \in K$, where $\eta = \min(\frac{1}{2}, M, \frac{1}{6}\epsilon) > 0$. Setting $g = f_1 + if_2$, we have $g \in A$ and $|g(x) - h(x)| \le 2\eta$ for every $x \in K$.

(c) Set $M_1 = \|g\|_\infty$. If $M_1 \le M$ we can take $f = g$ and stop. Otherwise, consider the function

$$\phi(t) = \frac{M-\eta}{\max(M,\sqrt{t})}$$

for $t \in [0, M_1^2]$. By Weierstrass' theorem (281F), there is a real polynomial p such that $|\phi(t) - p(t)| \le \frac{\eta}{M_1}$ whenever $0 \le t \le M_1^2$. Note that $|g|^2 = g \times \bar{g} \in A$, so that

$$f = g \times p(|g|^2) \in A.$$

Now

$$|p(t)| \le \phi(t) + \frac{\eta}{M_1} \le \phi(t) + \frac{\eta}{\max(M,\sqrt{t})} = \frac{M}{\max(M,\sqrt{t})}$$

whenever $0 \le t \le M_1^2$, so

$$|f(x)| \le |g(x)|\frac{M}{\max(M,|g(x)|)} \le M$$

for every $x \in X$. Next, if $0 \le t \le \min(M_1, M + 2\eta)^2$,

$$|1 - p(t)| \le \frac{\eta}{M_1} + 1 - \phi(t) \le \frac{\eta}{M} + 1 - \frac{M-\eta}{M+2\eta} \le \frac{4\eta}{M}.$$

Consequently, if $x \in K$, so that

$$|g(x)| \le \min(M_1, |h(x)| + 2\eta) \le \min(M_1, M + 2\eta),$$

we shall have

$$|1 - p(|g(x)|^2)| \le \frac{4\eta}{M},$$

and

$$\begin{aligned} |f(x) - h(x)| &\le |g(x) - h(x)| + |g(x)||1 - p(|g(x)|^2)| \\ &\le 2\eta + \frac{4\eta}{M}(M + 2\eta) \le 2\eta + \frac{4\eta}{M}(M + 1) \le \epsilon, \end{aligned}$$

as required.

Remark Of course we could have saved ourselves effort by settling for

$$\sup_{x\in X} |f(x)| \le 2 \sup_{x\in K} |h(x)|,$$

which would be quite good enough for the applications below.

281H Corollary Let $[a, b] \subseteq \mathbb{R}$ be a non-empty bounded closed interval and $h : [a, b] \to \mathbb{C}$ a continuous function. Then for any $\epsilon > 0$ there are $y_0, \dots, y_n \in \mathbb{R}$ and $c_0, \dots, c_n \in \mathbb{C}$ such that

$$|h(x) - \sum_{k=0}^{n} c_k e^{iy_k x}| \le \epsilon \text{ for every } x \in [a, b],$$

$$\sup_{x\in\mathbb{R}} |\sum_{k=0}^{n} c_k e^{iy_k x}| \le \sup_{x\in[a,b]} |h(x)|.$$

proof Apply 281G with $X = \mathbb{R}$, $K = [a, b]$ and A the linear span of the functions $x \mapsto e^{iyx}$ as y runs over $\mathbb{R}$.

281I Corollary Let S^1 be the unit circle $\{z : |z| = 1\} \subseteq \mathbb{C}$. Then for any continuous function $h : S^1 \to \mathbb{C}$ and $\epsilon > 0$, there are $n \in \mathbb{N}$ and $c_{-n}, c_{-n+1}, \dots , c_0, \dots , c_n \in \mathbb{C}$ such that $|h(z) - \sum_{k=-n}^{n} c_k z^k| \le \epsilon$ for every $z \in S^1$.

proof Apply 281G with $X = K = S^1$ and A the linear span of the functions $z \mapsto z^k$ for $k \in \mathbb{Z}$.

281J Corollary Let $h : [-\pi, \pi] \to \mathbb{C}$ be a continuous function such that $h(\pi) = h(-\pi)$. Then for any $\epsilon > 0$ there are $n \in \mathbb{N}$, $c_{-n}, \dots , c_n \in \mathbb{C}$ such that $|h(x) - \sum_{k=-n}^{n} c_k e^{ikx}| \le \epsilon$ for every $x \in [-\pi, \pi]$.

proof The point is that $\tilde{h} : S^1 \to \mathbb{C}$ is continuous on S^1, where $\tilde{h}(z) = h(\arg z)$; this is because arg is continuous everywhere except at -1, and

$$\lim_{x\downarrow-\pi} h(x) = h(-\pi) = h(\pi) = \lim_{x\uparrow\pi} h(x),$$

so

$$\lim_{z\in S^1, z\to-1} \tilde{h}(z) = h(\pi) = \tilde{h}(-1).$$

Now by 281I there are $c_{-n}, \dots , c_n \in \mathbb{C}$ such that $|\tilde{h}(z) - \sum_{k=-n}^{n} c_k z^k| \le \epsilon$ for every $z \in S^1$, and these coefficients serve equally for h.

281K Corollary Suppose that $r \ge 1$ and that $K \subseteq \mathbb{R}^r$ is a non-empty closed bounded set. Let $h : K \to \mathbb{C}$ be a continuous function, and $\epsilon > 0$. Then there are $y_0, \dots , y_n \in \mathbb{Q}^r$ and $c_0, \dots , c_n \in \mathbb{C}$ such that

$$|h(x) - \textstyle\sum_{k=0}^{n} c_k e^{iy_k \cdot x}| \le \epsilon \text{ for every } x \in K,$$

$$\sup_{x\in\mathbb{R}^r} |\textstyle\sum_{k=0}^{n} c_k e^{iy_k \cdot x}| \le \sup_{x\in K} |h(x)|,$$

writing $y \cdot x = \sum_{j=1}^{r} \eta_j \xi_j$ when $y = (\eta_1, \dots , \eta_r)$ and $x = (\xi_1, \dots , \xi_r)$ belong to $\mathbb{R}^r$.

proof Apply 281G with $X = \mathbb{R}^r$ and A the linear span of the functions $x \mapsto e^{iy\cdot x}$ as y runs over $\mathbb{Q}^r$.

281L Corollary Suppose that $r \ge 1$ and that $K \subseteq \mathbb{R}^r$ is a non-empty closed bounded set. Let $h : K \to \mathbb{R}$ be a continuous function, and $\epsilon > 0$. Then there are $y_0, \dots , y_n \in \mathbb{R}^r$ and $c_0, \dots , c_n \in \mathbb{C}$ such that, writing $g(x) = \sum_{k=0}^{n} c_k e^{iy_k \cdot x}$, g is real-valued and

$$|h(x) - g(x)| \le \epsilon \text{ for every } x \in K,$$

$$\inf_{y\in K} h(y) \le g(x) \le \sup_{y\in K} h(y) \text{ for every } x \in \mathbb{R}^r.$$

proof Apply 281E with $X = \mathbb{R}^r$ and A the set of *real*-valued functions on $\mathbb{R}^r$ which are *complex* linear combinations of the functions $x \mapsto e^{iy\cdot x}$; as remarked in part (a) of the proof of 281G, A satisfies the conditions of 281E.

281M Weyl's Equidistribution Theorem We are now ready for one of the basic results of number theory. I shall actually apply it to provide an example in §285 below, but (at least in the one-variable case) it is surely on the (rather long) list of things which every pure mathematician should know. For the sake of the application I have in mind, I give the full r-dimensional version, but you may wish to take it in the first place with $r = 1$.

It will be helpful to have a notation for 'fractional part'. For any real number x, write $<x>$ for that number in $[0, 1[$ such that $x - <x>$ is an integer. Now for the theorem.

281N Theorem Let $\eta_1, \dots , \eta_r$ be real numbers such that $1, \eta_1, \dots , \eta_r$ are linearly independent over $\mathbb{Q}$. Then whenever $0 \le \alpha_j \le \beta_j \le 1$ for each $j \le r$,

$$\lim_{n\to\infty} \frac{1}{n+1} \#(\{m : m \le n, <m\eta_j> \in [\alpha_j, \beta_j] \text{ for every } j \le r\}) = \textstyle\prod_{j=1}^{r} (\beta_j - \alpha_j).$$

Remark Thus the theorem says that the long-term proportion of the r-tuples $(<m\eta_1>,\dots,<m\eta_r>)$ which belong to the interval $[a,b]\subseteq[\mathbf{0},\mathbf{1}]$ is just the Lebesgue measure $\mu[a,b]$ of the interval. Of course the condition '$1,\eta_1,\dots,\eta_r$ are linearly independent over $\mathbb{Q}$' is necessary as well as sufficient (281Xg).

proof (a) Write $y=(\eta_1,\dots,\eta_r)\in\mathbb{R}^r$,

$$<my>=(<m\eta_1>,\dots,<m\eta_r>)\in[\mathbf{0},\mathbf{1}[\,=[0,1[^r$$

for each $m\in\mathbb{N}$. Set $I=[\mathbf{0},\mathbf{1}]=[0,1]^r$, and for any function $f:I\to\mathbb{R}$ write

$$\overline{L}(f)=\limsup_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^n f(<my>),$$

$$\underline{L}(f)=\liminf_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^n f(<my>);$$

and for $f:I\to\mathbb{C}$ write

$$L(f)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^n f(<my>)$$

if the limit exists. It will be worth noting that for non-negative functions f, g, $h:I\to\mathbb{R}$ such that $h\le f+g$,

$$\overline{L}(h)\le\overline{L}(f)+\overline{L}(g),$$

and that $L(cf+g)=cL(f)+L(g)$ for any two functions f, $g:I\to\mathbb{C}$ such that $L(f)$ and $L(g)$ exist, and any $c\in\mathbb{C}$.

(b) I mean to show that $L(f)$ exists and is equal to $\int_I f$ for (many) continuous functions f. The key step is to consider functions of the form

$$f(x)=e^{2\pi ik\cdot x},$$

where $k=(\kappa_1,\dots,\kappa_r)\in\mathbb{Z}^r$. In this case, if $k\ne\mathbf{0}$,

$$k\cdot y=\sum_{j=1}^r\kappa_j\eta_j\notin\mathbb{Z}$$

because $1,\eta_1,\dots,\eta_r$ are linearly independent over $\mathbb{Q}$. So

$$L(f)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^n e^{2\pi ik\cdot<my>}=\lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^n e^{2\pi imk\cdot y}$$

(because $mk\cdot y-k\cdot<my>=\sum_{j=1}^r\kappa_j(m\eta_j-<m\eta_j>)$ is an integer)

$$=\lim_{n\to\infty}\frac{1-e^{2\pi i(n+1)k\cdot y}}{(n+1)(1-e^{2\pi ik\cdot y})}$$

(because $e^{2\pi ik\cdot y}\ne1$)

$$=0,$$

because $|1-e^{2\pi i(n+1)k\cdot y}|\le2$ for every n. Of course we can also calculate the integral of f over I, which is

$$\int_I f(x)dx=\int_I e^{2\pi ik\cdot x}dx=\int_I\prod_{j=1}^r e^{2\pi i\kappa_j\xi_j}dx$$

(writing $x=(\xi_1,\dots,\xi_r)$)

$$=\int_0^1\dots\int_0^1\prod_{j=1}^r e^{2\pi i\kappa_j\xi_j}d\xi_r\dots d\xi_1$$

$$=\int_0^1 e^{2\pi i\kappa_r\xi_r}d\xi_r\dots\int_0^1 e^{2\pi i\kappa_1\xi_1}d\xi_1=0$$

because at least one κ_j is non-zero, and for this j we must have

$$\int_0^1 e^{2\pi i\kappa_j\xi_j}d\xi_j = \frac{1}{2\pi i\kappa_j}(e^{2\pi i\kappa_j}-1) = 0.$$

So we have $L(f) = \int_I f = 0$ when $k \ne \mathbf{0}$. On the other hand, if $k = \mathbf{0}$, then f is constant with value 1, so

$$L(f) = \lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^n f(<my>) = \lim_{n\to\infty} 1 = 1 = \int_I f(x)dx.$$

(c) Now write $\partial I = [\mathbf{0},\mathbf{1}] \setminus]\mathbf{0},\mathbf{1}[$, the boundary of I. If $f : I \to \mathbb{C}$ is continuous and $f(x) = 0$ for $x \in \partial I$, then $L(f) = \int_I f$. **P** As in 281I, let S^1 be the unit circle $\{z : z \in \mathbb{C}, |z| = 1\}$, and set $K = (S^1)^r \subseteq \mathbb{C}^r$. If we think of K as a subset of $\mathbb{R}^{2r}$, it is closed and bounded. Let $\phi : K \to I$ be given by

$$\phi(\zeta_1,\dots,\zeta_r) = (\frac{1}{2}+\frac{\arg\zeta_1}{2\pi},\dots,\frac{1}{2}+\frac{\arg\zeta_r}{2\pi})$$

for $\zeta_1,\dots,\zeta_r \in S^1$. Then $h = f\phi : K \to \mathbb{C}$ is continuous, because ϕ is continuous on $(S^1\setminus\{-1\})^r$ and

$$\lim_{w\to z} f\phi(w) = f\phi(z) = 0$$

for any $z \in K\setminus(S^1\setminus\{-1\})^r$. (Compare 281J.) Now apply 281G with $X = K$ and A the set of polynomials in $\zeta_1,\dots,\zeta_r,\zeta_1^{-1},\dots,\zeta_r^{-1}$ to see that, given $\epsilon > 0$, there is a function of the form

$$g(z) = \sum_{k\in J} c_k\zeta_1^{\kappa_1}\dots\zeta_r^{\kappa_r},$$

for some finite set $J \subseteq \mathbb{Z}^r$ and constants $c_k \in \mathbb{C}$ for $k \in J$, such that

$$|g(z) - h(z)| \le \epsilon \text{ for every } z \in K.$$

Set

$$\tilde g(x) = g(e^{\pi i(2\xi_1-1)},\dots,e^{\pi i(2\xi_r-1)}) = \sum_{k\in J} c_k e^{\pi i k\,.\,(2x-\mathbf{1})} = \sum_{k\in J}(-1)^{k.\mathbf{1}}c_k e^{2\pi i k\,.\,x},$$

so that $\tilde g\phi = g$, and see that

$$\sup_{x\in I}|\tilde g(x) - f(x)| = \sup_{z\in K}|g(z) - h(z)| \le \epsilon.$$

Now $\tilde g$ is of the form dealt with in (a), so we must have $L(\tilde g) = \int_I \tilde g$. Let n_0 be such that

$$\Bigl|\int_I \tilde g - \frac{1}{n+1}\sum_{m=0}^n \tilde g(<my>)\Bigr| \le \epsilon$$

for every $n \ge n_0$. Then

$$|\int_I f - \int_I \tilde g| \le \int_I |f - \tilde g| \le \epsilon$$

and

$$\begin{aligned}|\frac{1}{n+1}\sum_{m=0}^n \tilde g(<my>) - \frac{1}{n+1}\sum_{m=0}^n f(<my>)| &\le \frac{1}{n+1}\sum_{m=0}^n |\tilde g(<my>) - f(<my>)|\\ &\le \frac{1}{n+1}(n+1)\epsilon = \epsilon\end{aligned}$$

for every $n \in \mathbb{N}$. So for $n \ge n_0$ we must have

$$|\frac{1}{n+1}\sum_{m=0}^n f(<my>) - \int_I f| \le 3\epsilon.$$

As ϵ is arbitrary, $L(f) = \int_I f$, as required. **Q**

(d) Observe next that if a, $b \in]\mathbf{0},\mathbf{1}[=]0,1[^r$, and $\epsilon > 0$, there are continuous functions f_1, f_2 such that

$$f_1 \le \chi[a,b] \le f_2 \le \chi\,]\mathbf{0},\mathbf{1}[, \quad \int_I f_2 - \int_I f_1 \le \epsilon.$$

P This is elementary. For $n \in \mathbb{N}$, define $h_n : \mathbb{R} \to [0,1]$ by setting $h_n(\xi) = 0$ if $\xi \le 0$, $2^n\xi$ if $0 \le \xi \le 2^{-n}$ and 1 if $\xi \ge 2^{-n}$. Set

$$f_{1n}(x) = \prod_{j=1}^r h_n(\xi_j - \alpha_j)h_n(\beta_j - \xi_j),$$

$$f_{2n}(x) = \prod_{j=1}^r (1 - h_n(\alpha_j - \xi_j))(1 - h_n(\xi_j - \beta_j))$$

for $x = (\xi_1, \dots, \xi_r) \in \mathbb{R}^r$. (Compare the proof of 242Oa.) Then $f_{1n} \le \chi[a,b] \le f_{2n}$ for each n, $f_{2n} \le \chi\,]\mathbf{0},\mathbf{1}[$ for all n so large that

$$2^{-n} \le \min(\min_{j \le r} \alpha_j, \min_{j\le r}(1-\beta_j)),$$

and $\lim_{n\to\infty} f_{2n}(x) - f_{1n}(x) = 0$ for every x, so

$$\lim_{n\to\infty} \int_I f_{2n} - \int_I f_{1n} = 0.$$

Thus we can take $f_1 = f_{1n}$, $f_2 = f_{2n}$ for any n large enough. **Q**

(e) It follows that if $a, b \in \,]\mathbf{0},\mathbf{1}[$ and $a \le b$, $L(\chi[a,b]) = \mu[a,b]$. **P** Let $\epsilon > 0$. Take f_1, f_2 as in (d). Then, using (c),

$$\overline{L}(\chi[a,b]) \le \overline{L}(f_2) = L(f_2) = \int_I f_2 \le \int_I f_1 + \epsilon \le \mu[a,b] + \epsilon,$$

$$\underline{L}(\chi[a,b]) \ge \underline{L}(f_1) = L(f_1) = \int_I f_1 \ge \int_I f_2 - \epsilon \ge \mu[a,b] - \epsilon,$$

so

$$\mu[a,b] - \epsilon \le \underline{L}(\chi[a,b]) \le \overline{L}(\chi[a,b]) \le \mu[a,b] + \epsilon.$$

As ϵ is arbitrary,

$$\mu[a,b] = \overline{L}(\chi[a,b]) = \underline{L}(\chi[a,b]) = L(\chi[a,b]),$$

as required. **Q**

(f) To complete the proof, take any $a, b \in I$ with $a \le b$. For $0 < \epsilon \le \frac{1}{2}$, set $I_\epsilon = [\epsilon\mathbf{1}, (1-\epsilon)\mathbf{1}]$, so that I_ϵ is a closed interval included in $]\mathbf{0},\mathbf{1}[$ and $\mu I_\epsilon = (1-2\epsilon)^r$. Of course $L(\chi I) = \mu I = 1$, so

$$L(\chi(I \setminus I_\epsilon)) = L(\chi I) - L(\chi I_\epsilon) = 1 - \mu I_\epsilon,$$

and

$$\begin{aligned}
\mu[a,b] - 1 + \mu I_\epsilon &\le \mu[a,b] + \mu I_\epsilon - \mu([a,b] \cup I_\epsilon) = \mu([a,b] \cap I_\epsilon) \\
&= L(\chi([a,b] \cap I_\epsilon)) \le \underline{L}(\chi([a,b])) \\
&\le \overline{L}(\chi([a,b])) \le \overline{L}(\chi([a,b] \cap I_\epsilon)) + \overline{L}(\chi(I \setminus I_\epsilon)) \\
&= L(\chi([a,b] \cap I_\epsilon)) + 1 - \mu I_\epsilon \\
&= \mu([a,b] \cap I_\epsilon) + 1 - \mu I_\epsilon \le \mu[a,b] + 1 - \mu I_\epsilon.
\end{aligned}$$

As ϵ is arbitrary,

$$\mu[a,b] = \overline{L}(\chi[a,b]) = \underline{L}(\chi[a,b]) = L(\chi[a,b]),$$

as stated.

281X Basic exercises (a) Let A be the set of those bounded continuous functions $f : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ which are expressible in the form $f(x,y) = \sum_{k=0}^n g_k(x) g'_k(y)$, where all the g_k, g'_k are continuous functions from $\mathbb{R}^r$ to $\mathbb{R}$. Show that for any bounded continuous function $h : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ and any bounded set $K \subseteq \mathbb{R}^r \times \mathbb{R}^r$ and any $\epsilon > 0$, there is an $f \in A$ such that $|f(x,y) - h(x,y)| \le \epsilon$ for every $(x,y) \in K$ and $\sup_{x,y\in\mathbb{R}^r} |f(x,y)| \le \sup_{x,y\in\mathbb{R}^r} |h(x,y)|$.

(b) Let K be a closed bounded set in $\mathbb{R}^r$, where $r \ge 1$, and $h : K \to \mathbb{R}$ a continuous function. Show that for any $\epsilon > 0$ there is a polynomial p in r variables such that $|h(x) - p(x)| \le \epsilon$ for every $x \in K$.

>(c) Let $[a,b]$ be a non-empty closed interval of $\mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ a continuous function. Show that for any $\epsilon > 0$ there are $y_0, \dots, y_n, a_0, \dots, a_n, b_0, \dots, b_n \in \mathbb{R}$ such that

$$|h(x) - \textstyle\sum_{k=0}^n (a_k \cos y_k x + b_k \sin y_k x)| \le \epsilon \text{ for every } x \in [a,b],$$

$$\sup_{x\in\mathbb{R}} |\textstyle\sum_{k=0}^n (a_k \cos y_k x + b_k \sin y_k x)| \le \sup_{x\in[a,b]} |h(x)|.$$

(d) Let h be a complex-valued function on $]-\pi,\pi]$ such that $|h|^p$ is integrable, where $1 \le p < \infty$. Show that for every $\epsilon > 0$ there is a function of the form $x \mapsto f(x) = \sum_{k=-n}^{n} c_k e^{ikx}$, where $c_{-k}, \dots, c_k \in \mathbb{C}$, such that $\int_{-\pi}^{\pi} |h - f|^p \le \epsilon$. (Compare 244H.)

>(e) Let $h : [-\pi,\pi] \to \mathbb{R}$ be a continuous function such that $h(\pi) = h(-\pi)$, and $\epsilon > 0$. Show that there are $a_0, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ such that

$$|h(x) - \frac{1}{2}a_0 - \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)| \le \epsilon$$

for every $x \in [-\pi,\pi]$.

(f) Let K be a non-empty closed bounded set in $\mathbb{R}^r$, where $r \ge 1$, and $h : K \to \mathbb{R}$ a continuous function. Show that for any $\epsilon > 0$ there are $y_0, \dots, y_n \in \mathbb{R}^r$, $a_0, \dots, a_n, b_0, \dots, b_n \in \mathbb{R}$ such that

$$|h(x) - \sum_{k=0}^{n}(a_k \cos(y_k \centerdot x) + b_k \sin(y_k \centerdot x))| \le \epsilon \text{ for every } x \in K,$$

$$\sup_{x\in\mathbb{R}} |\sum_{k=0}^{n}(a_k \cos(y_k \centerdot x) + b_k \sin(y_k \centerdot x))| \le \sup_{x\in K} |h(x)|,$$

interpreting $y \centerdot x$ as in 281K.

(g) Let $y_1, \dots, y_r$ be real numbers such that $1, y_1, \dots, y_r$ are not linearly independent over $\mathbb{Q}$. Show that there is a non-trivial interval $[a,b] \subseteq [\mathbf{0},\mathbf{1}] \subseteq \mathbb{R}^r$ such that $(<my_1>, \dots, <my_r>) \notin [a,b]$ for every $m \in \mathbb{Z}$.

(h) Let $\eta_1, \dots, \eta_r$ be real numbers such that $1, \eta_1, \dots, \eta_r$ are linearly independent over $\mathbb{Q}$. Suppose that $0 \le \alpha_j \le \beta_j \le 1$ for each $j \le r$. Show that for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$|\prod_{j=1}^{r}(\beta_j - \alpha_j) - \frac{1}{n+1}\#(\{m : k \le m \le k+n,\ <m\eta_j> \in [\alpha_j,\beta_j] \text{ for every } j \le r\})| \le \epsilon$$

whenever $n \ge n_0$ and $k \in \mathbb{N}$. (*Hint*: in the proof of 281N, set

$$\overline{L}(f) = \limsup_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{n+1}\sum_{m=k}^{k+n} f(<my>).)$$

281Y Further exercises (a) Show that under the hypotheses of 281A, there is an $f \in \overline{A}$, the $\| \|_\infty$-closure of A in $C_b(X)$, such that $f\restriction K = h$. (*Hint*: take $f = \lim_{n\to\infty} f_n$ where

$$\|f_{n+1} - f_n\|_\infty \le \sup_{x\in K} |f_n(x) - h(x)| \le 2^{-n}$$

for every $n \in \mathbb{N}$.)

(b) Let X be a topological space and $K \subseteq X$ a compact subset. Suppose that for any distinct points x, y of K there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) \ne f(y)$. Show that for any $r \in \mathbb{N}$ and any continuous $h : K \to \mathbb{R}^r$ there is a continuous $f : X \to \mathbb{R}^r$ extending h. (*Hint*: consider $r = 1$ first.)

(c) Let $\langle X_i \rangle_{i\in I}$ be any family of compact Hausdorff spaces, and X their product as topological spaces. For each i, write $C(X_i)$ for the set of continuous functions from X_i to $\mathbb{R}$, and $\pi_i : X \to X_i$ for the coordinate map. Show that the subalgebra of $C(X)$ generated by $\{f\pi_i : i \in I,\ f \in C(X_i)\}$ is $\| \|_\infty$-dense in $C(X)$. (*Note*: you will need to know that X is compact, and that if Z is any compact Hausdorff space then for any distinct z, $w \in Z$ there is an $f \in C(Z)$ such that $f(z) \ne f(w)$. For references see 3A3J and 3A3Bf in the next volume.)

(d) Let X be a topological space and K a compact subset of X. Let A be a linear subspace of the space $C_b(X)$ of bounded real-valued continuous functions on X such that $|f| \in A$ for every $f \in A$. Let $h : K \to \mathbb{R}$ be a continuous function such that whenever x, $y \in K$ there is an $f \in A$ such that $f(x) = h(x)$ and $f(y) = h(y)$. Show that for every $\epsilon > 0$ there is an $f \in A$ such that $|f(x) - h(x)| \le \epsilon$ for every $x \in K$.

(e) Let X be a compact topological space and write $C(X)$ for the set of continuous functions from X to $\mathbb{R}$. Suppose that $h \in C(X)$, and let $A \subseteq C(X)$ be such that

A is a linear subspace of $C(X)$;

either $|f| \in A$ for every $f \in A$ *or* $f \times g \in A$ for every $f, g \in A$ *or* $f \times f \in A$ for every $f \in A$;

whenever $x, y \in X$ and $\delta > 0$ there is an $f \in A$ such that $|f(x)-h(x)| \le \delta$ and $|f(y)-h(y)| \le \delta$.

Show that for every $\epsilon > 0$ there is an $f \in A$ such that $|h(x) - f(x)| \le \epsilon$ for every $x \in X$.

(f) Let X be a compact topological space and A a $\| \|_\infty$-closed linear subspace of the space $C(X)$ of continuous functions from X to $\mathbb{R}$. Show that the following are equiveridical:

(i) $|f| \in A$ for every $f \in A$;

(ii) $f \times f \in A$ for every $f \in A$;

(iii) $f \times g \in A$ for all $f, g \in A$,

and that in this case A is closed in $C(X)$ for the topology defined by the pseudometrics

$$(f, g) \mapsto |f(x) - g(x)| : C(X) \times C(X) \to [0, \infty[$$

as x runs over X (the 'topology of pointwise convergence' on $C(X)$).

(g) Show that under the hypotheses of 281G there is an $f \in \overline{A}$, the $\| \|_\infty$-closure of A in $C_b(X; \mathbb{C})$, such that $f \restriction K = h$ and (if $K \ne \emptyset$) $\|f\|_\infty = \sup_{x \in K} |h(x)|$.

(h) Let $y \in \mathbb{R}$ be irrational. Show that for any Riemann integrable function $f : [0, 1] \to \mathbb{R}$,

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^n f(<my>),$$

writing $<my>$ for the fractional part of my. (*Hint*: recall *Riemann's criterion:* for any $\epsilon > 0$, there are $a_0, \dots, a_n$ with $0 = a_0 \le a_1 \le \dots \le a_n = 1$ and

$$\sum\{a_j - a_{j-1} : j \le n,\ \sup_{x\in[a_{j-1},a_j]} f(x) - \inf_{x\in[a_{j-1},a_j]} f(x) \ge \epsilon\} \le \epsilon.)$$

(i) Let $\langle t_n \rangle_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$. Show that the following are equiveridical: (i) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^n f(t_k) = \int_0^1 f$ for every continuous function $f : [0, 1] \to \mathbb{R}$; (ii) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^n f(t_k) = \int_0^1 f$ for every Riemann integrable function $f : [0, 1] \to \mathbb{R}$; (iii) $\liminf_{n\to\infty} \frac{1}{n+1} \#(\{k : k \le n,\ t_k \in G\}) \ge \mu G$ for every open set $G \subseteq [0, 1]$, where μ is Lebesgue measure on $\mathbb{R}$; (iv) $\lim_{n\to\infty} \frac{1}{n+1} \#(\{k : k \le n,\ t_k \le \alpha\}) = \alpha$ for every $\alpha \in [0, 1]$; (v) $\lim_{n\to\infty} \frac{1}{n+1} \#(\{k : k \le n,\ t_k \in E\}) = \mu E$ for every $E \subseteq [0, 1]$ such that $\mu(\operatorname{int} E) = \mu\overline{E}$ (vi) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^n e^{2\pi i m t_k} = 0$ for every $m \ge 1$. (Cf. 273J. Such sequences $\langle t_n \rangle_{n\in\mathbb{N}}$ are called **equidistributed** or **uniformly distributed**.)

(j) Show that the sequence $\langle < \ln(n + 1)> \rangle_{n\in\mathbb{N}}$ is not equidistributed.

(k) Give $[0, 1]^\mathbb{N}$ its product measure λ. Show that λ-almost every sequence $\langle t_n \rangle_{n\in\mathbb{N}} \in [0, 1]^\mathbb{N}$ is equidistributed in the sense of 281Yi. (*Hint*: 273J.)

(l) Let $f : [0, 1]^2 \to \mathbb{C}$ be a continuous function. Show that if $\gamma \in \mathbb{R}$ is irrational then $\lim_{a\to\infty} \frac{1}{a} \int_0^a f(<t>, <\gamma t>)dt = \int_{[0,1]^2} f$. (*Hint*: first consider functions of the form $x \mapsto e^{2\pi i k \cdot x}$.)

(m) A sequence $\langle t_n \rangle_{n\in\mathbb{N}}$ in $[0, 1]$ is **well-distributed** (with respect to Lebesgue measure μ) if

$$\liminf_{n\to\infty} \inf_{l\in\mathbb{N}} \frac{1}{n+1} \#(\{k : l \le k \le l + n,\ t_k \in G\}) \ge \mu G$$

for every open set $G \subseteq [0, 1]$ (i) Show that $\langle t_n \rangle_{n\in\mathbb{N}}$ is well-distributed iff $\lim_{n\to\infty} \sup_{l\in\mathbb{N}} |\int_0^1 f - \frac{1}{n+1} \sum_{k=l}^{l+n} f(t_k)| = 0$ for every continuous $f : [0, 1] \to \mathbb{R}$. (ii) Show that $\langle <n\alpha> \rangle_{n\in\mathbb{N}}$ is well-distributed for every irrational α.

281 Notes and comments I have given three statements (281A, 281E and 281G) of the Stone-Weierstrass theorem, with an acknowledgment (281F) of Weierstrass' own version, and three further forms (281Ya, 281Yd, 281Yg) in the exercises. Yet another will appear in §4A6 in Volume 4. Faced with such a multiplicity, you may wish to try your own hand at writing out theorems which will cover some or all of these versions. I myself see no way of doing it without setting up a confusing list of alternative hypotheses and conclusions. At which point, I ask 'what is a theorem, anyway?', and answer, it is a stopping-place on our journey; it is a place where we can rest, and congratulate ourselves on our achievement; it is a place which we can learn to recognise, and use as a starting point for new adventures; it is a place we can describe, and share with others. For some theorems, like Fermat's last theorem, there is a canonical statement, an exactly locatable point. For others, like the Stone-Weierstrass theorem here, we reach a mass of closely related results, all depending on some arrangement of the arguments laid out in 281A-281G and 281Ya (which introduces a new idea), and all useful in different ways. I suppose, indeed, that most authors would prefer the versions 281Ya and 281Yg, which eliminate the variable ϵ which appears in 281A, 281E and 281G, at the expense of taking a closed subspace A. But I find that the corollaries which will be useful later (281H-281L) are more naturally expressed in terms of linear subspaces which are not closed.

The applications of the theorem, or the theorems, or the method – choose your own expression – are legion; only a few of them are here. An apparently innocent one is in 281Xa and, in a different variant, in 281Yc; these are enormously important in their own domains. In this volume the principal application will be to 285L below, depending on 281K, and it is perhaps right to note that there is an alternative approach to this particular result, based on ideas in 282G. But I offer Weyl's equidistribution theorem (281M-281N) as evidence that we can expect to find good use for these ideas in almost any branch of mathematics.

Version of 24.9.09

282 Fourier series

Out of the enormous theory of Fourier series, I extract a few results which may at least provide a basis for further study. I give the definitions of Fourier and Fejér sums (282A), with five of the most important results concerning their convergence (282G, 282H, 282J, 282L, 282O). On the way I include the Riemann-Lebesgue lemma (282E). I end by mentioning convolutions (282Q).

282A Definition Let f be an integrable complex-valued function defined almost everywhere in $]-\pi,\pi]$.

(a) The **Fourier coefficients** of f are the complex numbers

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

for $k \in \mathbb{Z}$.

(b) The **Fourier sums** of f are the functions

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

for $x \in\,]-\pi,\pi]$, $n \in \mathbb{N}$.

(c) The **Fourier series** of f is the series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$, or (because we ordinarily consider the symmetric partial sums s_n) the series $c_0 + \sum_{k=1}^{\infty}(c_k e^{ikx} + c_{-k}e^{-ikx})$.

(d) The **Fejér sums** of f are the functions

$$\sigma_m = \frac{1}{m+1}\sum_{n=0}^{m} s_n$$

for $m \in \mathbb{N}$.

(e) It will be convenient to have a further phrase available. If f is any function with $\operatorname{dom} f \subseteq\,]-\pi,\pi]$, its **periodic extension** is the function $\tilde{f}$, with domain $\bigcup_{k\in\mathbb{Z}}(\operatorname{dom} f + 2k\pi)$, such that $\tilde{f}(x) = f(x-2k\pi)$ whenever $k \in \mathbb{Z}$ and $x \in \operatorname{dom} f + 2k\pi$.

282B Remarks I have made two more or less arbitrary choices here.

(a) I have chosen to express Fourier series in their 'complex' form rather than their 'real' form. From the point of view of pure measure theory (and, indeed, from the point of view of the nineteenth-century origins of the subject) there are gains in elegance from directing attention to real functions f and looking at the real coefficients

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \text{ for } k \in \mathbb{N},$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \text{ for } k \ge 1.$$

If we do this we have

$$c_0 = \frac{1}{2}a_0,$$

and for $k \ge 1$ we have

$$c_k = \frac{1}{2}(a_k - ib_k), \quad c_{-k} = \frac{1}{2}(a_k + ib_k), \quad a_k = c_k + c_{-k}, \quad b_k = i(c_k - c_{-k}),$$

so that the Fourier sums become

$$s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

The advantage of this is that real functions f correspond to real coefficients a_k, b_k, so that it is obvious that if f is real-valued so are its Fourier and Fejér sums. The disadvantages are that we have to use a variety of trigonometric equalities which are rather more complicated than the properties of the complex exponential function which they reflect, and that we are farther away from the natural generalizations to locally compact abelian groups. So both electrical engineers and harmonic analysts tend to prefer the coefficients c_k.

(b) I have taken the functions f to be defined on the interval $]-\pi,\pi]$ rather than on the circle $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$. There would be advantages in elegance of language in using S^1, though I do not recall often seeing the formula

$$c_k = \int z^k f(z)dz$$

which is the natural translation of $c_k = \frac{1}{2\pi}\int e^{ikx}f(x)dx$ under the substitution $x = \arg z$, $dx = 2\pi\nu(dz)$. However, applications of the theory tend to deal with periodic functions on the real line, so I work with $]-\pi,\pi]$, and accept the fact that its group operation $+_{2\pi}$, writing $x +_{2\pi} y$ for whichever of $x+y$, $x+y+2\pi$, $x+y-2\pi$ belongs to $]-\pi,\pi]$, is less familiar than multiplication on S^1.

(c) The remarks in (b) are supposed to remind you of §255.

(d) Observe that if $f =_{\text{a.e.}} g$ then f and g have the same Fourier coefficients, Fourier sums and Fejér sums. This means that we could, if we wished, regard the c_k, s_n and σ_m as associated with a member of $L^1_{\mathbb{C}}$, the space of equivalence classes of integrable functions (§242), rather than as associated with a particular function f. Since however the s_n and σ_m appear as actual functions, and since many of the questions we are interested in refer to their values at particular points, it is more natural to express the theory in terms of integrable functions f rather than in terms of members of $L^1_{\mathbb{C}}$.

282C The problems (a) Under what conditions, and in what senses, do the Fourier and Fejér sums s_n and σ_m of a function f converge to f?

(b) How do the properties of the double-ended sequence $\langle c_k\rangle_{k\in\mathbb{Z}}$ reflect the properties of f, and vice versa?

Remark The theory of Fourier series has been one of the leading topics of analysis for nearly two hundred years, and innumerable further problems have contributed greatly to our understanding. (For instance: can one characterize those sequences $\langle c_k\rangle_{k\in\mathbb{Z}}$ which are the Fourier coefficients of some integrable function?) But in this outline I will concentrate on the question (a) above, with one and a half results (282K, 282Rb) addressing (b), which will give us more than enough material to work on.

While most people would feel that the Fourier sums are somehow closer to what we really want to know, it turns out that the Fejér sums are easier to analyse, and there are advantages in dealing with them first. So while you may wish to look ahead to the statements of 282J, 282L and 282O for an idea of where we are going, the first half of this section will be largely about Fejér sums. Note that in any case in which we know that the Fourier sums converge (which is quite common; see, for instance, the examples in 282Xh and 282Xo), then if we know that the Fejér sums converge to f, we can deduce that the Fourier sums also do, by 273Ca.

The first step is a basic lemma showing that both the Fourier and Fejér sums of a function f can be thought of as convolutions of f with kernels describable in terms of familiar functions.

282D Lemma Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-ikx}dx,\quad s_n(x)=\sum_{k=-n}^{n}c_ke^{ikx},\quad \sigma_m(x)=\frac{1}{m+1}\sum_{n=0}^{m}s_n(x)$$

its Fourier coefficients, Fourier sums and Fejér sums. Write $\tilde f$ for the periodic extension of f (282Ae). For $m\in\mathbb{N}$, write

$$\psi_m(t)=\frac{1-\cos(m+1)t}{2\pi(m+1)(1-\cos t)}$$

for $0<|t|\le\pi$. (If you like, you can set $\psi_m(0)=\frac{m+1}{2\pi}$ to make ψ_m continuous on $[-\pi,\pi]$.)

(a) For each $n\in\mathbb{N}$, $x\in\,]-\pi,\pi]$,

$$\begin{aligned}s_n(x)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-_{2\pi}t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt,\end{aligned}$$

writing $x-_{2\pi}t$ for whichever of $x-t$, $x-t-2\pi$, $x-t+2\pi$ belongs to $]-\pi,\pi]$.

(b) For each $m\in\mathbb{N}$, $x\in\,]-\pi,\pi]$,

$$\begin{aligned}\sigma_m(x)&=\int_{-\pi}^{\pi}\tilde f(x+t)\psi_m(t)dt\\&=\int_0^{\pi}(\tilde f(x+t)+\tilde f(x-t))\psi_m(t)dt\\&=\int_{-\pi}^{\pi}f(x-_{2\pi}t)\psi_m(t)dt.\end{aligned}$$

(c) For any $n\in\mathbb{N}$,

$$\frac{1}{2\pi}\int_{-\pi}^{0}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt=\frac{1}{2},\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt=1.$$

(d) For any $m\in\mathbb{N}$,

(i) $0\le\psi_m(t)\le\dfrac{m+1}{2\pi}$ for every t;

(ii) for any $\delta>0$, $\lim_{m\to\infty}\psi_m(t)=0$ uniformly on $\{t:\delta\le|t|\le\pi\}$;

(iii) $\int_{-\pi}^{0}\psi_m=\int_0^{\pi}\psi_m=\dfrac{1}{2}$, $\quad\int_{-\pi}^{\pi}\psi_m=1$.

proof Really all that these amount to is summing geometric series.

(a) For (a), we have

$$\begin{aligned}\sum_{k=-n}^{n} e^{-ikt} &= \frac{e^{int}-e^{-i(n+1)t}}{1-e^{-it}}\\ &= \frac{e^{i(n+\frac{1}{2})t}-e^{-i(n+\frac{1}{2})t}}{e^{\frac{1}{2}it}-e^{-\frac{1}{2}it}} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}.\end{aligned}$$

So

$$\begin{aligned}s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\big(\sum_{k=-n}^{n} e^{ik(x-t)}\big)dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}dt\\ &= \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x}\tilde f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt\end{aligned}$$

because $\tilde f$ and $t\mapsto\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$ are periodic with period 2π, so that the integral from $-\pi-x$ to $-\pi$ must be the same as the integral from $\pi-x$ to π.

For the expression in terms of $f(x-_{2\pi}t)$, we have

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(x-t)\frac{\sin(n+\frac{1}{2})(-t)}{\sin\frac{1}{2}(-t)}dt$$

(substituting $-t$ for t)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-_{2\pi}t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt$$

because (for x, $t\in\,]-\pi,\pi]$) $f(x-_{2\pi}t)=\tilde f(x-t)$ whenever either is defined, and sin is an odd function.

(b) In the same way, we have

$$\begin{aligned}\sum_{n=0}^{m}\sin(n+\frac{1}{2})t &= \mathcal{I}\mathrm{m}\big(\sum_{n=0}^{m} e^{i(n+\frac{1}{2})t}\big) = \mathcal{I}\mathrm{m}\big(e^{\frac{1}{2}it}\sum_{n=0}^{m} e^{int}\big)\\ &= \mathcal{I}\mathrm{m}\big(e^{\frac{1}{2}it}\frac{1-e^{i(m+1)t}}{1-e^{it}}\big) = \mathcal{I}\mathrm{m}\big(\frac{1-e^{i(m+1)t}}{e^{-\frac{1}{2}it}-e^{\frac{1}{2}it}}\big)\\ &= \mathcal{I}\mathrm{m}\big(\frac{1-e^{i(m+1)t}}{-2i\sin\frac{1}{2}t}\big) = \mathcal{I}\mathrm{m}\big(\frac{i(1-e^{i(m+1)t})}{2\sin\frac{1}{2}t}\big)\\ &= \frac{1-\cos(m+1)t}{2\sin\frac{1}{2}t}.\end{aligned}$$

So

$$\textstyle\sum_{n=0}^{m}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} = \frac{1-\cos(m+1)t}{2\sin^2\frac{1}{2}t} = \frac{1-\cos(m+1)t}{1-\cos t} = 2\pi(m+1)\psi_m(t).$$

Accordingly,

$$
\begin{aligned}
\sigma_m(x) &= \frac{1}{m+1}\sum_{n=0}^{m} s_n(x) \\
&= \frac{1}{m+1}\sum_{n=0}^{m}\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(x+t)\Bigl(\frac{1}{m+1}\sum_{n=0}^{m}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\Bigr)dt \\
&= \int_{-\pi}^{\pi}\tilde f(x+t)\psi_m(t)dt = \int_{-\pi}^{\pi} f(x -_{2\pi} t)\psi_m(t)dt
\end{aligned}
$$

as in (a), because cos and ψ_m are even functions. For the same reason,

$$\int_0^{\pi}\tilde f(x-t)\psi_m(t)dt = \int_{-\pi}^{0}\tilde f(x+t)\psi_m(t)dt,$$

so

$$\sigma_m(x) = \int_0^{\pi}(\tilde f(x+t)+\tilde f(x-t))\psi_m(t)dt.$$

(c) We need only look at where the formula $\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$ came from to see that

$$
\begin{aligned}
\frac{1}{2\pi}\int_I\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt &= \frac{1}{2\pi}\int_I\sum_{k=-n}^{n}e^{ikt}dt \\
&= \frac{1}{2\pi}\int_I(1+2\sum_{k=1}^{n}\cos kt)dt = \frac{1}{2}
\end{aligned}
$$

for both $I=[-\pi,0]$ and $I=[0,\pi]$, because $\int_I \cos kt\,dt = 0$ for every $k\ne 0$.

(d)(i) $\psi_m(t)\ge 0$ for every t because $1-\cos(m+1)t$, $1-\cos t$ are always greater than or equal to 0. For the upper bound, we have, using the constructions in (a) and (b),

$$\Bigl|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\Bigr| = \Bigl|\sum_{k=-n}^{n}e^{ikt}\Bigr| \le 2n+1$$

for every n, so

$$
\begin{aligned}
\psi_m(t) &= \frac{1}{2\pi(m+1)}\sum_{n=0}^{m}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} \\
&\le \frac{1}{2\pi(m+1)}\sum_{n=0}^{m}2n+1 = \frac{m+1}{2\pi}.
\end{aligned}
$$

(ii) If $\delta\le|t|\le\pi$,

$$\psi_m(t)\le\frac{1}{\pi(m+1)(1-\cos t)}\le\frac{1}{\pi(m+1)(1-\cos\delta)}\to 0$$

as $m\to\infty$.

(iii) also follows from the construction in (b), because

$$\int_I\psi_m = \frac{1}{2\pi(m+1)}\sum_{n=0}^{m}\int_I\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt = \frac{1}{m+1}\sum_{n=0}^{m}\frac{1}{2} = \frac{1}{2}$$

for both $I = [-\pi, 0]$ and $I = [0, \pi]$, using (c).

Remarks For a discussion of substitution in integrals, if you feel any need to justify the manipulations in part (a) of the proof, see 263J.

The functions

$$t \mapsto \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}, \quad t \mapsto \frac{1-\cos(m+1)t}{(m+1)(1-\cos t)}$$

are called respectively the **Dirichlet kernel** and the **Fejér kernel**.

I give the formulae in terms of $f(x -_{2\pi} t)$ in (a) and (b) in order to provide a link with the work of 255O.

282E The next step is a vital lemma, with a suitably distinguished name which (you will be glad to know) reflects its importance rather than its difficulty.

The Riemann-Lebesgue lemma Let f be a complex-valued function which is integrable over $\mathbb{R}$. Then

$$\lim_{y\to\infty} \int f(x)e^{-iyx}dx = \lim_{y\to-\infty} \int f(x)e^{-iyx}dx = 0.$$

proof (a) Consider first the case in which $f = \chi\,]a, b[$, where $a < b$. Then

$$|\int f(x)e^{-iyx}dx| = |\int_a^b e^{-iyx}dx| = |\frac{1}{-iy}(e^{-iyb} - e^{-iya})| \le \frac{2}{|y|}$$

if $y \ne 0$. So in this case the result is obvious.

(b) It follows at once that the result is true if f is a step-function with bounded support, that is, if there are $a_0 \le a_1 \ldots \le a_n$ such that f is constant on every interval $]a_{j-1}, a_j[$ and zero outside $[a_0, a_n]$.

(c) Now, for a given integrable f and $\epsilon > 0$, there is a step-function g such that $\int |f - g| \le \epsilon$ (242Oa). So

$$|\int f(x)e^{-iyx}dx - \int g(x)e^{-iyx}dx| \le \int |f(x) - g(x)|dx \le \epsilon$$

for every y, and

$$\limsup_{y\to\infty} |\int f(x)e^{-iyx}dx| \le \epsilon,$$

$$\limsup_{y\to-\infty} |\int f(x)e^{-iyx}dx| \le \epsilon.$$

As ϵ is arbitrary, we have the result.

282F Corollary (a) Let f be a complex-valued function which is integrable over $]-\pi, \pi]$, and $\langle c_k\rangle_{k\in\mathbb{Z}}$ its sequence of Fourier coefficients. Then $\lim_{k\to\infty} c_k = \lim_{k\to-\infty} c_k = 0$.

(b) Let f be a complex-valued function which is integrable over $\mathbb{R}$. Then $\lim_{y\to\infty} \int f(x)\sin yx\,dx = 0$.

proof (a) We need only identify

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

with $\int g(x)e^{-ikx}dx$, where $g(x) = f(x)/2\pi$ for $x \in \operatorname{dom} f$ and 0 for $|x| > \pi$.

(b) This is just because

$$\int f(x)\sin yx\,dx = \frac{1}{2i}(\int f(x)e^{iyx}dx - \int f(x)e^{-iyx}dx).$$

282G We are now ready for theorems on the convergence of Fejér sums. I start with an easy one, almost a warming-up exercise.

Theorem Let $f :]-\pi, \pi] \to \mathbb{C}$ be a continuous function such that $\lim_{t\downarrow-\pi} f(t) = f(\pi)$. Then its sequence $\langle\sigma_m\rangle_{m\in\mathbb{N}}$ of Fejér sums converges uniformly to f on $]-\pi, \pi]$.

proof The conditions on f amount just to saying that its periodic extension $\tilde{f}$ is defined and continuous everywhere on $\mathbb{R}$. Consequently it is bounded and uniformly continuous on any bounded interval, in particular, on the interval $[-2\pi, 2\pi]$. Set $K = \sup_{|t|\le 2\pi} |\tilde{f}(t)| = \sup_{t\in]-\pi,\pi]} |f(t)|$. Write

$$\psi_m(t) = \tfrac{1-\cos(m+1)t}{2\pi(m+1)(1-\cos t)}$$

for $m \in \mathbb{N}$, $0 < |t| \le \pi$, as in 282D.

Given $\epsilon > 0$ we can find a $\delta \in]0, \pi]$ such that $|\tilde{f}(x+t) - \tilde{f}(x)| \le \epsilon$ whenever $x \in [-\pi, \pi]$ and $|t| \le \delta$. Next, we can find an $m_0 \in \mathbb{N}$ such that $M_m \le \frac{\epsilon}{4\pi K}$ for every $m \ge m_0$, where $M_m = \sup_{\delta\le|t|\le\pi} \psi_m(t)$ (282D(d-ii)). Now suppose that $m \ge m_0$ and $x \in]-\pi, \pi]$. Set $g(t) = \tilde{f}(x+t) - f(x)$ for $|t| \le \pi$. Then $|g(t)| \le 2K$ for all $t \in [-\pi, \pi]$ and $|g(t)| \le \epsilon$ if $|t| \le \delta$, so

$$\begin{aligned}|\int_{-\pi}^{\pi} g \times \psi_m| &\le \int_{-\pi}^{-\delta} |g| \times \psi_m + \int_{-\delta}^{\delta} |g| \times \psi_m + \int_{\delta}^{\pi} |g| \times \psi_m \\ &\le 2M_m K(\pi - \delta) + \epsilon \int_{-\delta}^{\delta} \psi_m + 2M_m K(\pi - \delta) \\ &\le 4\pi M_m K + \epsilon \le 2\epsilon.\end{aligned}$$

Consequently, using 282Db and 282D(d-iii),

$$|\sigma_m(x) - f(x)| = |\int_{-\pi}^{\pi} (\tilde{f}(x+t) - f(x))\psi_m(t)dt| \le 2\epsilon$$

for every $m \ge m_0$; and this is true for every $x \in]-\pi, \pi]$. As ϵ is arbitrary, $\langle\sigma_m\rangle_{m\in\mathbb{N}}$ converges to f uniformly on $]-\pi, \pi]$.

282H I come now to a theorem describing the behaviour of the Fejér sums of general functions f. The hypothesis of the theorem may take a little bit of digesting; you can get an idea of its intended scope by glancing at Corollary 282I.

Theorem Let f be a complex-valued function which is integrable over $]-\pi, \pi]$, and $\langle\sigma_m\rangle_{m\in\mathbb{N}}$ its sequence of Fejér sums. Suppose that $x \in]-\pi, \pi]$ and $c \in \mathbb{C}$ are such that

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \int_0^{\delta} |\tilde{f}(x+t) + \tilde{f}(x-t) - 2c| dt = 0,$$

writing $\tilde{f}$ for the periodic extension of f, as usual; then $\lim_{m\to\infty} \sigma_m(x) = c$.

proof Set $\phi(t) = |\tilde{f}(x+t) + \tilde{f}(x-t) - 2c|$ when this is defined, which is almost everywhere, and $\Phi(t) = \int_0^t \phi$, which is defined for every $t \ge 0$, because $\tilde{f}$ is integrable over $]-\pi, \pi]$ and therefore over every bounded interval.

As in 282D, set

$$\psi_m(t) = \tfrac{1-\cos(m+1)t}{2\pi(m+1)(1-\cos t)}$$

for $m \in \mathbb{N}$, $0 < |t| \le \pi$. We have

$$|\sigma_m(x) - c| = |\int_0^{\pi} (\tilde{f}(x+t) + \tilde{f}(x-t) - 2c)\psi_m(t)dt| \le \int_0^{\pi} \phi(t)\psi_m$$

by (b) and (d) of 282D.

Let $\epsilon > 0$. By hypothesis, $\lim_{t\downarrow 0} \Phi(t)/t = 0$; let $\delta \in]0, \pi]$ be such that $\Phi(t) \le \epsilon t$ for every $t \in [0, \delta]$. Take any $m \ge \pi/\delta$. I break the integral $\int_0^{\pi} \phi \times \psi_m$ up into three parts.

(i) For the integral from 0 to $1/m$, we have

$$\int_0^{1/m} \phi \times \psi_m \le \int_0^{1/m} \frac{m+1}{2\pi}\phi = \frac{m+1}{2\pi}\Phi(\frac{1}{m}) \le \frac{\epsilon(m+1)}{2\pi m} \le \epsilon,$$

because $\psi_m(t) \le \frac{m+1}{2\pi}$ for every t (282D(d-i)).

(ii) For the integral from $1/m$ to δ, we have

$$\int_{1/m}^{\delta} \phi \times \psi_m \le \frac{1}{2\pi(m+1)} \int_{1/m}^{\delta} \phi(t) \frac{1}{1-\cos t} dt \le \frac{\pi}{4(m+1)} \int_{1/m}^{\delta} \frac{\phi(t)}{t^2} dt$$

(because $1 - \cos t \ge \frac{2t^2}{\pi^2}$ for $|t| \le \pi$)

$$= \frac{\pi}{4(m+1)} \Bigl(\frac{\Phi(\delta)}{\delta^2} - \frac{\Phi(\frac{1}{m})}{(\frac{1}{m})^2} + \int_{1/m}^{\delta} \frac{2\Phi(t)}{t^3} dt \Bigr)$$

(integrating by parts – see 225F)

$$\le \frac{\pi}{4(m+1)} \Bigl(\frac{\epsilon}{\delta} + \int_{1/m}^{\delta} \frac{2\epsilon}{t^2} dt \Bigr)$$

(because $\Phi(t) \le \epsilon t$ for $0 \le t \le \delta$)

$$\le \frac{\pi}{4(m+1)} \Bigl(\frac{\epsilon}{\delta} + 2\epsilon m \Bigr) \le \frac{\pi\epsilon}{4(m+1)\delta} + \frac{\pi\epsilon}{2} \le \frac{\epsilon}{4} + \frac{\pi\epsilon}{2} \le 2\epsilon.$$

(iii) For the integral from δ to π, we have

$$\int_{\delta}^{\pi} \phi \times \psi_m \le \int_{\delta}^{\pi} \frac{1}{\pi(m+1)(1-\cos\delta)} \phi \to 0 \text{ as } m \to \infty$$

because ϕ is integrable over $[-\pi, \pi]$. There must therefore be an $m_0 \in \mathbb{N}$ such that

$$\int_{\delta}^{\pi} \phi \times \psi_m \le \epsilon$$

for every $m \ge m_0$.

Putting these together, we see that

$$\int_0^{\pi} \phi \times \psi_m \le \epsilon + 2\epsilon + \epsilon = 4\epsilon$$

for every $m \ge \max(m_0, \frac{\pi}{\delta})$. As ϵ is arbitrary, $\lim_{m\to\infty} \sigma_m(x) = c$, as claimed.

282I Corollary Let f be a complex-valued function which is integrable over $]-\pi, \pi]$, and $\langle \sigma_m \rangle_{m\in\mathbb{N}}$ its sequence of Fejér sums.

(a) $f(x) = \lim_{m\to\infty} \sigma_m(x)$ for almost every $x \in \,]-\pi, \pi]$.

(b) $\lim_{m\to\infty} \int_{-\pi}^{\pi} |f - \sigma_m| = 0$.

(c) If g is another integrable function with the same Fourier coefficients, then $f =_{\text{a.e.}} g$.

(d) If $x \in \,]-\pi, \pi[$ is such that $a = \lim_{t\in\operatorname{dom} f, t\uparrow x} f(t)$ and $b = \lim_{t\in\operatorname{dom} f, t\downarrow x} f(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{m\to\infty} \sigma_m(x) = \frac{1}{2}(a+b).$$

(e) If $a = \lim_{t\in\operatorname{dom} f, t\uparrow\pi} f(t)$ and $b = \lim_{t\in\operatorname{dom} f, t\downarrow-\pi} f(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{m\to\infty} \sigma_m(\pi) = \frac{1}{2}(a+b).$$

(f) If f is defined and continuous at $x \in \,]-\pi, \pi[$, then

$$\lim_{m\to\infty} \sigma_m(x) = f(x).$$

(g) If $\tilde{f}$, the periodic extension of f, is defined and continuous at π, then

$$\lim_{m\to\infty} \sigma_m(\pi) = f(\pi).$$

proof (a) We have only to recall that by 223D

$$\begin{aligned}\limsup_{\delta\downarrow 0}\frac{1}{\delta}\int_0^\delta &|f(x+t)+f(x-t)-2f(x)|dt\\ &\le\limsup_{\delta\downarrow 0}\frac{1}{\delta}\Bigl(\int_0^\delta|f(x+t)-f(x)|dt+\int_0^\delta|f(x-t)-f(x)|dt\Bigr)\\ &=\limsup_{\delta\downarrow 0}\frac{1}{\delta}\int_{-\delta}^\delta|f(x+t)-f(x)|dt=0\end{aligned}$$

for almost every $x\in\,]-\pi,\pi[$.

(b) Next observe that, in the language of 255O,

$$\sigma_m=f*\psi_m,$$

by the last formula in 282Db. Consequently, by 255Od,

$$\|\sigma_m\|_1\le\|f\|_1\|\psi_m\|_1,$$

writing $\|\sigma_m\|_1=\int_{-\pi}^{\pi}|\sigma_m|$. But this means that we have

$$f(x)=\lim_{m\to\infty}\sigma_m(x)\text{ for almost every }x,\quad \limsup_{m\to\infty}\|\sigma_m\|_1\le\|f\|_1;$$

and it follows from 245H that $\lim_{m\to\infty}\|f-\sigma_m\|_1=0$.

(c) If g has the same Fourier coefficients as f, then it has the same Fourier and Fejér sums, so we have

$$g(x)=\lim_{m\to\infty}\sigma_m(x)=f(x)$$

almost everywhere.

(d)-(e) Both of these amount to considering $x\in\,]-\pi,\pi]$ such that

$$\lim_{t\in\mathrm{dom}\,\tilde f,t\uparrow x}\tilde f(t)=a,\quad \lim_{t\in\mathrm{dom}\,\tilde f,t\downarrow x}\tilde f(t)=b.$$

Setting $c=\frac{1}{2}(a+b)$, $\phi(t)=|\tilde f(x+t)+\tilde f(x-t)-2c|$ whenever this is defined, we have $\lim_{t\in\mathrm{dom}\,\phi,t\downarrow 0}\phi(t)=0$, so surely $\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^\delta\phi=0$, and the theorem applies.

(f)-(g) are special cases of (d) and (e).

282J I now turn to conditions for the convergence of Fourier sums. Probably the easiest result – one which is both striking and satisfying – is the following.

Theorem Let f be a complex-valued function which is square-integrable over $]-\pi,\pi]$. Let $\langle c_k\rangle_{k\in\mathbb{Z}}$ be its Fourier coefficients and $\langle s_n\rangle_{n\in\mathbb{N}}$ its Fourier sums (282A). Then

(i) $\sum_{k=-\infty}^{\infty}|c_k|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2$,

(ii) $\lim_{n\to\infty}\int_{-\pi}^{\pi}|f-s_n|^2=0$.

proof (a) I recall some notation from 244N/244P. Let $\mathcal{L}^2_{\mathbb{C}}$ be the space of square-integrable complex-valued functions on $]-\pi,\pi]$. For g, $h\in\mathcal{L}^2_{\mathbb{C}}$, write

$$(g|h)=\int_{-\pi}^{\pi}g\times\bar h,\quad \|g\|_2=\sqrt{(g|g)}.$$

Recall that $\|g+h\|_2\le\|g\|_2+\|h\|_2$ for all g, $h\in\mathcal{L}^2_{\mathbb{C}}$ (244Fb/244Pb). For $k\in\mathbb{Z}$, $x\in\,]-\pi,\pi]$ set $e_k(x)=e^{ikx}$, so that

$$(f|e_k)=\int_{-\pi}^{\pi}f(x)e^{-ikx}dx=2\pi c_k.$$

Moreover, if $|k|\le n$,

$$(s_n|e_k) = \sum_{j=-n}^{n} c_j \int_{-\pi}^{\pi} e^{ijx}e^{-ikx}dx = 2\pi c_k,$$

because

$$\int_{-\pi}^{\pi} e^{ijx}e^{-ikx}dx = 2\pi \text{ if } j = k,$$
$$= 0 \text{ if } j \neq k.$$

So

$$(f - s_n|e_k) = 0 \text{ whenever } |k| \le n;$$

in particular,

$$(f - s_n|s_n) = \sum_{k=-n}^{n} \bar{c}_k(f - s_n|e_k) = 0$$

for every $n \in \mathbb{N}$.

(b) Fix $\epsilon > 0$. The next element of the proof is the fact that there are $m \in \mathbb{N}$, $a_{-m}, \dots , a_m \in \mathbb{C}$ such that $\|f - h\|_2 \le \epsilon$, where $h = \sum_{k=-m}^{m} a_k e_k$. **P** By 244Hb/244Pb we know that there is a continuous function $g : [-\pi, \pi] \to \mathbb{C}$ such that $\|f - g\|_2 \le \frac{\epsilon}{3}$. Next, modifying g on a suitably short interval $]\pi - \delta, \pi]$, we can find a continuous function $g_1 : [-\pi, \pi] \to \mathbb{C}$ such that $\|g - g_1\|_2 \le \frac{\epsilon}{3}$ and $g_1(-\pi) = g_1(\pi)$. (Set $M = \sup_{x\in[-\pi,\pi]} |g(x)|$, take $\delta \in]0, 2\pi]$ such that $(2M)^2\delta \le (\epsilon/3)^2$, and set $g_1(\pi - t\delta) = tg(\pi - \delta) + (1 - t)g(-\pi)$ for $t \in [0, 1]$.) Either by the Stone-Weierstrass theorem (281J), or by 282G above, there are $a_{-m}, \dots , a_m$ such that $|g_1(x) - \sum_{k=-m}^{m} a_k e^{ikx}| \le \frac{\epsilon}{3\sqrt{2\pi}}$ for every $x \in [-\pi, \pi]$; setting $h = \sum_{k=-m}^{m} a_k e_k$, we have $\|g_1 - h\|_2 \le \frac{1}{3}\epsilon$, so that

$$\|f - h\|_2 \le \|f - g\|_2 + \|g - g_1\|_2 + \|g_1 - h\|_2 \le \epsilon. \textbf{ Q}$$

(c) Now take any $n \ge m$. Then $s_n - h$ is a linear combination of $e_{-n}, \dots , e_n$, so $(f - s_n|s_n - h) = 0$. Consequently

$$\begin{aligned}\epsilon^2 &\ge (f - h|f - h)\\ &= (f - s_n|f - s_n) + (f - s_n|s_n - h) + (s_n - h|f - s_n) + (s_n - h|s_n - h)\\ &= \|f - s_n\|_2^2 + \|s_n - h\|_2^2 \ge \|f - s_n\|_2^2.\end{aligned}$$

Thus $\|f - s_n\|_2 \le \epsilon$ for every $n \ge m$. As ϵ is arbitrary, $\lim_{n\to\infty} \|f - s_n\|_2^2 = 0$, which proves (ii).

(d) As for (i), we have

$$\sum_{k=-n}^{n} |c_k|^2 = \frac{1}{2\pi} \sum_{k=-n}^{n} \bar{c}_k(s_n|e_k) = \frac{1}{2\pi}(s_n|s_n) = \frac{1}{2\pi}\|s_n\|_2^2.$$

But of course

$$\big|\|s_n\|_2 - \|f\|_2\big| \le \|s_n - f\|_2 \to 0$$

as $n \to \infty$, so

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi} \lim_{n\to\infty} \|s_n\|_2^2 = \frac{1}{2\pi}\|f\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2,$$

as required.

282K Corollary Let $L^2_{\mathbb{C}}$ be the Hilbert space of equivalence classes of square-integrable complex-valued functions on $]-\pi, \pi]$, with the inner product

$$(f^\bullet|g^\bullet) = \int_{-\pi}^{\pi} f \times \bar{g}$$

and norm

$$\|f^\bullet\|_2 = \bigl(\int_{-\pi}^{\pi} |f|^2\bigr)^{1/2},$$

writing $f^\bullet \in L^2_{\mathbb{C}}$ for the equivalence class of a square-integrable function f. Let $\ell^2_{\mathbb{C}}(\mathbb{Z})$ be the Hilbert space of square-summable double-ended complex sequences, with the inner product

$$(\boldsymbol{c}|\boldsymbol{d}) = \sum_{k=-\infty}^{\infty} c_k \bar{d}_k$$

and norm

$$\|\boldsymbol{c}\|_2 = \bigl(\sum_{k=-\infty}^{\infty} |c_k|^2\bigr)^{1/2}$$

for $\boldsymbol{c} = \langle c_k\rangle_{k\in\mathbb{Z}}$, $\boldsymbol{d} = \langle d_k\rangle_{k\in\mathbb{Z}}$ in $\ell^2_{\mathbb{C}}(\mathbb{Z})$. Then we have an inner-product-space isomorphism $S : L^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}(\mathbb{Z})$ defined by saying that

$$S(f^\bullet)(k) = \tfrac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

for every square-integrable function f and every $k \in \mathbb{Z}$.

proof (a) As in 282J, write $\mathcal{L}^2_{\mathbb{C}}$ for the space of square-integrable functions. If f, $g \in \mathcal{L}^2_{\mathbb{C}}$ and $f^\bullet = g^\bullet$, then $f =_{\text{a.e.}} g$, so

$$\tfrac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \tfrac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(x)e^{-ikx}dx$$

for every $k \in \mathbb{N}$. Thus S is well-defined.

(b) S is linear. **P** This is elementary. If f, $g \in \mathcal{L}^2_{\mathbb{C}}$ and $c \in \mathbb{C}$,

$$\begin{aligned} S(f^\bullet + g^\bullet)(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} (f(x) + g(x))e^{-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx + \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(x)e^{-ikx}dx \\ &= S(f^\bullet)(k) + S(g^\bullet)(k) \end{aligned}$$

for every $k \in \mathbb{Z}$, so that $S(f^\bullet + g^\bullet) = S(f^\bullet) + S(g^\bullet)$. Similarly,

$$S(cf^\bullet)(k) = \tfrac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} cf(x)e^{-ikx}dx = \tfrac{c}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = cS(f^\bullet)(k)$$

for every $k \in \mathbb{Z}$, so that $S(cf^\bullet) = cS(f^\bullet)$. **Q**

(c) If $f \in \mathcal{L}^2_{\mathbb{C}}$ has Fourier coefficients c_k, then $S(f^\bullet) = \langle c_k\sqrt{2\pi}\rangle_{k\in\mathbb{Z}}$, so by 282J(i)

$$\|S(f^\bullet)\|_2^2 = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 = \int_{-\pi}^{\pi} |f|^2 = \|f^\bullet\|_2^2.$$

Thus $Su \in \ell^2_{\mathbb{C}}(\mathbb{Z})$ and $\|Su\|_2 = \|u\|_2$ for every $u \in L^2_{\mathbb{C}}$. Because S is linear and norm-preserving, it is surely injective.

(d) It now follows that $(Sv|Su) = (v|u)$ for every u, $v \in L^2_{\mathbb{C}}$. **P** (This is of course a standard fact about Hilbert spaces.) We know that for any $t \in \mathbb{R}$

$$\begin{aligned}\|u\|_2^2 + 2\,\mathcal{R}\mathrm{e}(e^{it}(v|u)) + \|v\|_2^2 &= (u|u) + e^{it}(v|u) + e^{-it}(u|v) + (v|v)\\ &= (u + e^{it}v|u + e^{it}v)\\ &= \|u + e^{it}v\|_2^2 = \|S(u + e^{it}v)\|_2^2\\ &= \|Su\|_2^2 + 2\,\mathcal{R}\mathrm{e}(e^{it}(Sv|Su)) + \|Sv\|_2^2\\ &= \|u\|_2^2 + 2\,\mathcal{R}\mathrm{e}(e^{it}(Sv|Su)) + \|v\|_2^2,\end{aligned}$$

so that $\mathcal{R}\mathrm{e}(e^{it}(Sv|Su)) = \mathcal{R}\mathrm{e}(e^{it}(v|u))$. As t is arbitrary, $(Sv|Su) = (v|u)$. **Q**

(e) Finally, S is surjective. **P** Let $\boldsymbol{c} = \langle c_k\rangle_{k\in\mathbb{Z}}$ be any member of $\ell^2_{\mathbb{C}}(\mathbb{Z})$. Set $c_k^{(n)} = c_k$ if $|k| \le n$, 0 otherwise, and $\boldsymbol{c}^{(n)} = \langle c_k^{(n)}\rangle_{k\in\mathbb{N}}$. Consider

$$s_n = \sum_{k=-n}^{n} c_k e_k, \quad u_n = s_n^\bullet$$

where I write $e_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}$ for $x \in]-\pi,\pi]$. Then $Su_n = \boldsymbol{c}^{(n)}$, by the same calculations as in part (a) of the proof of 282J. Now

$$\|\boldsymbol{c}^{(n)} - \boldsymbol{c}\|_2 = \sqrt{\textstyle\sum_{|k|>n} |c_k|^2} \to 0$$

as $n \to \infty$, so

$$\|u_m - u_n\|_2 = \|\boldsymbol{c}^{(m)} - \boldsymbol{c}^{(n)}\|_2 \to 0$$

as m, $n \to \infty$, and $\langle u_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2_{\mathbb{C}}$. Because $L^2_{\mathbb{C}}$ is complete (244G/244Pb), $\langle u_n\rangle_{n\in\mathbb{N}}$ has a limit $u \in L^2_{\mathbb{C}}$, and now

$$Su = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} \boldsymbol{c}^{(n)} = \boldsymbol{c}. \quad \mathbf{Q}$$

Thus $S : L^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}(\mathbb{Z})$ is an inner-product-space isomorphism.

Remark In the language of Hilbert spaces, all that is happening here is that $\langle e_k^\bullet\rangle_{k\in\mathbb{Z}}$ is a 'Hilbert space basis' or 'complete orthonormal sequence' in $L^2_{\mathbb{C}}$, which is matched by S with the standard basis of $\ell^2_{\mathbb{C}}(\mathbb{Z})$. The only step which calls on non-trivial real analysis, as opposed to the general theory of Hilbert spaces, is the check that the linear subspace generated by $\{e_k^\bullet : k \in \mathbb{Z}\}$ is dense; this is part (b) of the proof of 282J.

Observe that while $S : L^2 \to \ell^2$ is readily described, its inverse is more of a problem. If $\boldsymbol{c} \in \ell^2$, we should like to say that $S^{-1}\boldsymbol{c}$ is the equivalence class of f, where $f(x) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ for every x. This works very well if $\{k : c_k \ne 0\}$ is finite, but for the general case it is less clear how to interpret the sum. It is in fact the case that if $\boldsymbol{c} \in \ell^2$ then

$$g(x) = \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty} \textstyle\sum_{k=-n}^{n} c_k e^{ikx}$$

is defined for almost every $x \in]-\pi,\pi]$, and that $S^{-1}\boldsymbol{c} = g^\bullet$ in L^2; this is, in effect, Carleson's theorem (286V). A proof of Carleson's theorem is out of our reach for the moment. What is covered by the results of this section is that

$$h(x) = \frac{1}{\sqrt{2\pi}}\lim_{m\to\infty} \frac{1}{m+1}\textstyle\sum_{n=0}^{m}\sum_{k=-n}^{n} c_k e^{ikx}$$

is defined for almost every $x \in]-\pi,\pi]$, and that $h^\bullet = S^{-1}\boldsymbol{c}$. (The point is that we know from the result just proved that there is *some* square-integrable f such that $\boldsymbol{c}$ is the sequence of Fourier coefficients of f; now 282Ia declares that the Fejér sums of f converge to f almost everywhere, that is, that $h =_{\text{a.e.}} \frac{1}{\sqrt{2\pi}}f$.)

282L The next result is the easiest, and one of the most useful, theorems concerning pointwise convergence of Fourier sums.

Theorem Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and $\langle s_n\rangle_{n\in\mathbb{N}}$ its sequence of Fourier sums.

(i) If f is differentiable at $x \in]-\pi,\pi[$, then $f(x) = \lim_{n\to\infty} s_n(x)$.

(ii) If the periodic extension $\tilde f$ of f is differentiable at π, then $f(\pi) = \lim_{n\to\infty} s_n(\pi)$.

proof (a) Take $x \in \left]-\pi,\pi\right]$ such that $\tilde f$ is differentiable at x; of course this covers both parts. We have

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\tilde f(x+t)}{\sin\frac{1}{2}t}\sin(n+\frac{1}{2})t\,dt$$

for each n, by 282Da.

(b) Next,

$$\int_{-\pi}^{\pi}\frac{\tilde f(x+t)-\tilde f(x)}{t}dt$$

exists in $\mathbb{C}$, because there is surely some $\delta \in \left]0,\pi\right]$ such that $(\tilde f(x+t)-\tilde f(x))/t$ is bounded on $\{t : 0 < |t| \le \delta\}$, while

$$\int_{-\pi}^{-\delta}\frac{\tilde f(x+t)-\tilde f(x)}{t}dt, \quad \int_{\delta}^{\pi}\frac{\tilde f(x+t)-\tilde f(x)}{t}dt$$

exist because $1/t$ is bounded on those intervals. It follows that

$$\int_{-\pi}^{\pi}\frac{\tilde f(x+t)-\tilde f(x)}{\sin\frac{1}{2}t}dt$$

exists, because $|t| \le \pi|\sin\frac{1}{2}t|$ if $|t| \le \pi$. So by the Riemann-Lebesgue lemma (282Fb),

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\frac{\tilde f(x+t)-\tilde f(x)}{\sin\frac{1}{2}t}\sin(n+\frac{1}{2})t\,dt = 0.$$

(c) Because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde f(x)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt = \tilde f(x)$$

for every n (282Dc),

$$s_n(x) = \tilde f(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\tilde f(x+t)-\tilde f(x)}{\sin\frac{1}{2}t}\sin(n+\frac{1}{2})t\,dt \to \tilde f(x)$$

as $n\to\infty$, as required.

282M Lemma Suppose that f is a complex-valued function, defined almost everywhere and of bounded variation on $\left]-\pi,\pi\right]$. Then $\sup_{k\in\mathbb{Z}}|kc_k| < \infty$, where c_k is the kth Fourier coefficient of f, as in 282A.

proof Set

$$M = \lim_{x\in\operatorname{dom} f, x\uparrow\pi}|f(x)| + \operatorname{Var}_{]-\pi,\pi[}(f).$$

By 224J,

$$\begin{aligned}|kc_k| &= \frac{1}{2\pi}\Bigl|\int_{-\pi}^{\pi}kf(t)e^{-ikt}dt\Bigr| \le \frac{1}{2\pi}M\sup_{c\in[-\pi,\pi]}\Bigl|\int_{-\pi}^{c}ke^{-ikt}dt\Bigr| \\ &= \frac{M}{2\pi}\sup_{c\in[-\pi,\pi]}|e^{-ikc}-e^{ik\pi}| \le \frac{M}{\pi}\end{aligned}$$

for every k.

282N I give another lemma, extracting the technical part of the proof of the next theorem. (Its most natural application is in 282Xn.)

Lemma Let $\langle d_k\rangle_{k\in\mathbb{N}}$ be a complex sequence, and set $t_n = \sum_{k=0}^{n} d_k$, $\tau_m = \frac{1}{m+1}\sum_{n=0}^{m} t_n$ for n, $m \in \mathbb{N}$. Suppose that $\sup_{k\in\mathbb{N}}|kd_k| = M < \infty$. Then for any $j \ge 1$ and any $c \in \mathbb{C}$,

$$|t_n - c| \le \frac{M}{j} + (2j+3)\sup_{m\ge n-n/j}|\tau_m - c|$$

for every $n \ge j^2$.

proof (a) The first point to note is that for any n, $n' \in \mathbb{N}$,

$$|t_n - t_{n'}| \le \tfrac{M|n-n'|}{1+\min(n,n')}.$$

P If $n = n'$ this is trivial. Suppose that $n' < n$. Then

$$|t_n - t_{n'}| = |\sum_{k=n'+1}^{n} d_k| \le \sum_{k=n'+1}^{n} \frac{M}{k} \le \tfrac{M(n-n')}{n'+1} = \tfrac{M|n-n'|}{1+\min(n',n)}.$$

Of course the case $n < n'$ is identical. **Q**

(b) Now take any $n \ge j^2$. Set $\eta = \sup_{m\ge n-n/j}|\tau_m - c|$. Let $m \ge j$ be such that $jm \le n < j(m+1)$; then $n < jm + m$; also

$$n(1-\tfrac{1}{j}) \le m(j+1)(1-\tfrac{1}{j}) \le mj.$$

Set

$$\tau^* = \frac{1}{m}\sum_{n'=jm+1}^{jm+m} t_{n'} = \frac{jm+m+1}{m}\tau_{jm+m} - \frac{jm+1}{m}\tau_{jm}.$$

Then

$$\begin{aligned}|\tau^* - c| &= |\frac{jm+m+1}{m}\tau_{jm+m} - \frac{jm+1}{m}\tau_{jm} - c|\\ &= |\frac{jm+m+1}{m}(\tau_{jm+m} - c) - \frac{jm+1}{m}(\tau_{jm} - c)|\\ &\le \frac{jm+m+1}{m}\eta + \frac{jm+1}{m}\eta \le (2j+3)\eta.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\tau^* - t_n| &= \Bigl|\frac{1}{m}\sum_{n'=jm+1}^{jm+m}(t_{n'} - t_n)\Bigr| \le \frac{1}{m}\sum_{n'=jm+1}^{jm+m}\frac{M|n-n'|}{1+\min(n,n')}\\ &\le \frac{1}{m}\sum_{n'=jm+1}^{jm+m}\frac{Mm}{1+jm} = \frac{Mm}{1+jm} \le \frac{M}{j}.\end{aligned}$$

Putting these together, we have

$$|t_n - c| \le |t_n - \tau^*| + |\tau^* - c| \le \frac{M}{j} + (2j+3)\eta = \frac{M}{j} + (2j+3)\sup_{m\ge n-n/j}|\tau_m - c|,$$

as required.

282O Theorem Let f be a complex-valued function of bounded variation, defined almost everywhere in $]-\pi,\pi]$, and let $\langle s_n\rangle_{n\in\mathbb{N}}$ be its sequence of Fourier sums.

(i) If $x \in]-\pi,\pi[$, then

$$\lim_{n\to\infty} s_n(x) = \frac{1}{2}(\lim_{t\in\mathrm{dom}\,f,t\uparrow x} f(t) + \lim_{t\in\mathrm{dom}\,f,t\downarrow x} f(t)).$$

(ii) $\lim_{n\to\infty} s_n(\pi) = \dfrac{1}{2}(\lim_{t\in\mathrm{dom}\,f,t\uparrow\pi} f(t) + \lim_{t\in\mathrm{dom}\,f,t\downarrow-\pi} f(t))$.

(iii) If f is defined throughout $]-\pi,\pi]$, is continuous, and $\lim_{t\downarrow-\pi} f(t) = f(\pi)$, then $s_n(x) \to f(x)$ uniformly on $]-\pi,\pi]$.

proof (a) Note first that 224F shows that the limits $\lim_{t\in\operatorname{dom} f,t\downarrow x} f(t)$, $\lim_{t\in\operatorname{dom} f,t\uparrow x} f(t)$ required in the formulae above always exist. We know also from 282M that $M=\sup_{k\in\mathbb{Z}}|kc_k|<\infty$, where c_k is the kth Fourier coefficient of f.

Take any $x\in\,]-\pi,\pi]$, and set

$$c=\tfrac{1}{2}(\lim_{t\in\operatorname{dom} f,t\uparrow x}\tilde f(t)+\lim_{t\in\operatorname{dom}\tilde f,t\downarrow x}\tilde f(t)),$$

writing $\tilde f$ for the periodic extension of f, as usual. We know from 282Id-282Ie that $c=\lim_{m\to\infty}\sigma_m(x)$, writing σ_m for the Fejér sums of f. Let $\epsilon>0$. Take any $j\ge\max(2,2M/\epsilon)$, and $m_0\ge 1$ such that $|\sigma_m(x)-c|\le\epsilon/(2j+3)$ for every $m\ge m_0$.

Now if $n\ge\max(j^2,2m_0)$, apply Lemma 282N with

$$d_0=c_0,\quad d_k=c_ke^{ikx}+c_{-k}e^{-ikx}\text{ for }k\ge 1,$$

so that $t_n=s_n(x)$, $\tau_m=\sigma_m(x)$ and $|kd_k|\le 2M$ for every k, n, $m\in\mathbb{N}$. We have $n-n/j\ge\frac{1}{2}n\ge m_0$, so

$$\eta=\sup_{m\ge n-n/j}|\tau_m-c|\le\sup_{m\ge m_0}|\tau_m-c|\le\tfrac{\epsilon}{2j+3}.$$

So 282N tells us that

$$|s_n(x)-c|=|t_n-c|\le\frac{2M}{j}+(2j+3)\sup_{m\ge n-n/j}|\tau_m-c|\le\epsilon+(2j+3)\eta\le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty}s_n(x)=c$, as required.

(b) This proves (i) and (ii) of this theorem. Finally, for (iii), observe that under these conditions $\sigma_m(x)\to f(x)$ uniformly as $m\to\infty$, by 282G. So given $\epsilon>0$ we choose $j\ge\max(2,2M/\epsilon)$ and $m_0\in\mathbb{N}$ such that $|\sigma_m(x)-f(x)|\le\epsilon/(2j+3)$ whenever $m\ge m_0$ and $x\in\,]-\pi,\pi]$. By the same calculation as before,

$$|s_n(x)-f(x)|\le 2\epsilon$$

for every $n\ge\max(j^2,2m_0)$ and every $x\in\,]-\pi,\pi]$. As ϵ is arbitrary, $\lim_{n\to\infty}s_n(x)=f(x)$ uniformly for $x\in\,]-\pi,\pi]$.

282P Corollary Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and $\langle s_n\rangle_{n\in\mathbb{N}}$ its sequence of Fourier sums.

(i) Suppose that $x\in\,]-\pi,\pi[$ is such that f is of bounded variation on some neighbourhood of x. Then

$$\lim_{n\to\infty}s_n(x)=\frac{1}{2}(\lim_{t\in\operatorname{dom} f,t\uparrow x}f(t)+\lim_{t\in\operatorname{dom} f,t\downarrow x}f(t)).$$

(ii) If there is a $\delta>0$ such that f is of bounded variation on both $]-\pi,-\pi+\delta]$ and $[\pi-\delta,\pi]$, then

$$\lim_{n\to\infty}s_n(\pi)=\frac{1}{2}(\lim_{t\in\operatorname{dom} f,t\uparrow\pi}f(t)+\lim_{t\in\operatorname{dom} f,t\downarrow-\pi}f(t)).$$

proof In case (i), take $\delta>0$ such that f is of bounded variation on $[x-\delta,x+\delta]$ and set $f_1(t)=f(t)$ if $x\in\operatorname{dom} f\cap[x-\delta,x+\delta]$, 0 for other $t\in\,]-\pi,\pi]$; in case (ii), set $f_1(t)=f(t)$ if $t\in\operatorname{dom} f$ and $|t|\ge\pi-\delta$, 0 for other $t\in\,]-\pi,\pi]$, and say that $x=\pi$. In either case, f_1 is of bounded variation, so by 282O the Fourier sums $\langle s'_n\rangle_{n\in\mathbb{N}}$ of f_1 converge at x to the value given by the formulae above. But now observe that, writing $\tilde f$ and $\tilde f_1$ for the periodic extensions of f and f_1, $\tilde f-\tilde f_1=0$ on a neighbourhood of x, so

$$\int_{-\pi}^{\pi}\tfrac{\tilde f(x+t)-\tilde f_1(x+t)}{\sin\frac{1}{2}t}dt$$

exists in $\mathbb{C}$, and by 282Fb

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\frac{\tilde f(x+t)-\tilde f_1(x+t)}{\sin\frac{1}{2}t}\sin(n+\frac{1}{2})t\,dt=0,$$

that is, $\lim_{n\to\infty}s_n(x)-s'_n(x)=0$. So $\langle s_n\rangle_{n\in\mathbb{N}}$ also converges to the right limit.

282Q I cannot leave this section without mentioning one of the most important facts about Fourier series, even though I have no space here to discuss its consequences.

Theorem Let f and g be complex-valued functions which are integrable over $]-\pi,\pi]$, and $\langle c_k\rangle_{k\in\mathbb{N}}$, $\langle d_k\rangle_{k\in\mathbb{N}}$ their Fourier coefficients. Let $f*g$ be their convolution, defined by the formula

$$(f*g)(x)=\int_{-\pi}^{\pi}f(x-_{2\pi}t)g(t)dt=\int_{-\pi}^{\pi}\tilde f(x-t)g(t)dt,$$

as in 255O, writing $\tilde f$ for the periodic extension of f. Then the Fourier coefficients of $f*g$ are $\langle 2\pi c_kd_k\rangle_{k\in\mathbb{Z}}$.

proof By 255O(c-i),

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)e^{-ikx}dx&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{-ik(t+u)}f(t)g(u)dtdu\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ikt}f(t)dt\int_{-\pi}^{\pi}e^{-iku}g(u)du=2\pi c_kd_k.\end{aligned}$$

***282R** In my hurry to get to the theorems on convergence of Fejér and Fourier sums, I have rather neglected the elementary manipulations which are essential when applying the theory. One basic result is the following.

Proposition (a) Let $f:[-\pi,\pi]\to\mathbb{C}$ be an absolutely continuous function such that $f(-\pi)=f(\pi)$, and $\langle c_k\rangle_{k\in\mathbb{Z}}$ its sequence of Fourier coefficients. Then the Fourier coefficients of f' are $\langle ikc_k\rangle_{k\in\mathbb{Z}}$.

(b) Let $f:\mathbb{R}\to\mathbb{C}$ be a differentiable function such that f' is absolutely continuous on $[-\pi,\pi]$, and $f(\pi)=f(-\pi)$. If $\langle c_k\rangle_{k\in\mathbb{Z}}$ are the Fourier coefficients of $f\restriction]-\pi,\pi]$, then $\sum_{k=-\infty}^{\infty}|c_k|$ is finite.

proof (a) By 225Cb, f' is integrable over $[-\pi,\pi]$; by 225E, f is an indefinite integral of f'. So 225F tells us that

$$\int_{-\pi}^{\pi}f'(x)e^{-ikx}dx=f(\pi)e^{-ik\pi}-f(-\pi)e^{ik\pi}+ik\int_{-\pi}^{\pi}f(x)e^{-ikx}dx=ikc_k$$

for every $k\in\mathbb{Z}$.

(b)(i) Suppose first that $f'(\pi)=f'(-\pi)$. By (a), applied twice, the Fourier coefficients of f'' are $\langle -k^2c_k\rangle_{k\in\mathbb{Z}}$, so $\sup_{k\in\mathbb{Z}}k^2|c_k|$ is finite; because $\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty$, $\sum_{k=-\infty}^{\infty}|c_k|<\infty$.

(ii) Next, suppose that $f(x)=x^2$ for every x. Then, for $k\neq 0$,

$$\begin{aligned}c_k&=\frac{1}{2\pi}\int x^2e^{-ikx}dx=\frac{1}{2\pi}\big(-\frac{1}{ik}(\pi^2e^{-ik\pi}-\pi^2e^{ik\pi})+\int_{-\pi}^{\pi}\frac{2x}{ik}e^{-ikx}dx\big)\\&=\frac{1}{ik\pi}\big(-\frac{1}{ik}(\pi e^{-ik\pi}+\pi e^{ik\pi})+\frac{1}{ik}\int_{-\pi}^{\pi}e^{ikx}dx\big)=\frac{2}{k^2}(-1)^k,\end{aligned}$$

so $\sum_{k\in\mathbb{Z}}|c_k|\le|c_0|+4\sum_{k=1}^{\infty}\frac{1}{k^2}$ is finite.

(iii) In general, we can express f as f_1+cf_2 where $f_2(x)=x^2$ for every x, $c=\frac{1}{4\pi}(f'(\pi)-f'(-\pi))$, and f_1 satisfies the conditions of (i); so that $\langle c_k\rangle_{k\in\mathbb{Z}}$ is the sum of two summable sequences and is itself summable.

282X Basic exercises >(a) Suppose that $\langle c_k\rangle_{k\in\mathbb{N}}$ is an absolutely summable double-ended sequence of complex numbers. Show that $f(x)=\sum_{k=-\infty}^{\infty}c_ke^{ikx}$ exists for every $x\in\mathbb{R}$, that f is continuous and periodic, and that its Fourier coefficients are the c_k.

(c) Set $\phi_n(t)=\frac{2}{t}\sin(n+\frac{1}{2}t)$ for $t\neq 0$. (This is sometimes called the **modified Dirichlet kernel**.) Show that for any integrable function f on $]-\pi,\pi]$, with Fourier sums $\langle s_n\rangle_{n\in\mathbb{N}}$ and periodic extension $\tilde f$,

$$\lim_{n\to\infty}|s_n(x)-\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_n(t)\tilde f(x+t)dt|=0$$

for every $x\in]-\pi,\pi]$. (*Hint*: show that $\frac{2}{t}-\frac{1}{\sin\frac{1}{2}t}$ is bounded, and use 282E.)

(d) Give a proof of 282Ib from 242O, 255O and 282G.

(e) Give another proof of 282Ic, based on 222D, 281J and an idea in the proof of 242O instead of on 282H.

(f) Use the idea of 255Ya to shorten one of the steps in the proof of 282H, taking

$$g_m(t) = \min(\tfrac{m+1}{2\pi}, \tfrac{\pi}{4(m+1)t^2})$$

for $|t| \le \delta$, so that $g_m \ge \psi_m$ on $[-\delta, \delta]$.

>(g)(i) Let f be a real square-integrable function on $]-\pi, \pi]$, and $\langle a_k \rangle_{k\in\mathbb{N}}$, $\langle b_k \rangle_{k\ge 1}$ its real Fourier coefficients (282Ba). Show that $\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}|f|^2$. (ii) Show that $f \mapsto (\sqrt{\frac{\pi}{2}}a_0, \sqrt{\pi}a_1, \sqrt{\pi}b_1, \dots)$ defines an inner-product-space isomorphism between the real Hilbert space $L^2_{\mathbb{R}}$ of equivalence classes of real square-integrable functions on $]-\pi, \pi]$ and the real Hilbert space $\ell^2_{\mathbb{R}}$ of square-summable sequences.

(h) Show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. (*Hint*: find the Fourier series of f where $f(x) = x/|x|$, and compute the sum of the series at $\frac{\pi}{2}$. Of course there are other methods, e.g., examining the Taylor series for $\arctan \frac{\pi}{4}$.)

(i) Let f be an integrable complex-valued function on $]-\pi, \pi]$, and $\langle s_n \rangle_{n\in\mathbb{N}}$ its sequence of Fourier sums. Suppose that $x \in\,]-\pi, \pi[$, $a \in \mathbb{C}$ are such that $\int_{-\pi}^{\pi}\dfrac{f(t)-a}{t-x}dt$ exists and is finite. Show that $\lim_{n\to\infty} s_n(x) = a$. Explain how this generalizes 282L. What modification is appropriate to obtain a limit $\lim_{n\to\infty} s_n(\pi)$?

(j) Suppose that $\alpha > 0$, $K \ge 0$ and $f :]-\pi, \pi[\, \to \mathbb{C}$ are such that $|f(x) - f(y)| \le K|x - y|^\alpha$ for all x, $y \in\,]-\pi, \pi[$. (Such functions are called **Hölder continuous**.) Show that the Fourier sums of f converge to f everywhere in $]-\pi, \pi[$. (*Hint*: use 282Xi.) (Compare 282Yb.)

(k) In 282L, show that it is enough if $\tilde{f}$ is differentiable with respect to its domain at x or π (see 262Fb), rather than differentiable in the strict sense.

(l) Show that $\lim_{a\to\infty}\int_0^a \frac{\sin t}{t}dt$ exists and is finite. (*Hint*: use 224J to estimate $\int_a^b \frac{\sin t}{t}dt$ for $0 < a \le b$.)

(m) Show that $\int_0^\infty \frac{|\sin t|}{t}dt = \infty$. (*Hint*: show that $\sup_{a\ge 0}|\int_1^a \frac{\cos 2t}{t}dt| < \infty$, and therefore that $\sup_{a\ge 0}\int_1^a \frac{\sin^2 t}{t}dt = \infty$.)

>(n) Let $\langle d_k \rangle_{k\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $\sup_{k\in\mathbb{N}}|kd_k| < \infty$ and

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n} d_k = c \in \mathbb{C}.$$

Show that $c = \sum_{k=0}^{\infty} d_k$. (*Hint*: 282N.)

>(o) Show that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. (*Hint*: (b-ii) of the proof of 282R.)

(p) Let f be an integrable complex-valued function on $]-\pi, \pi]$, and $\langle s_n \rangle_{n\in\mathbb{N}}$ its sequence of Fourier sums. Suppose that $x \in\,]-\pi, \pi[$ is such that

(i) there is an $a \in \mathbb{C}$ such that

either $\int_{-\pi}^{x}\dfrac{a-f(t)}{x-t}dt$ exists in $\mathbb{C}$

or there is some $\delta > 0$ such that f is of bounded variation on $[x-\delta, x]$, and $a = \lim_{t\in\mathrm{dom}\, f, t\uparrow x} f(t)$

(ii) there is a $b \in \mathbb{C}$ such that

either $\int_{x}^{\pi}\dfrac{f(t)-b}{t-x}dt$ exists in $\mathbb{C}$

or there is some $\delta > 0$ such that f is of bounded variation on $[x, x+\delta]$, and $b = \lim_{t\in\mathrm{dom}\, f, t\downarrow x} f(t)$.

Show that $\lim_{n\to\infty} s_n(x) = \frac{1}{2}(a + b)$. What modification is appropriate to obtain a limit $\lim_{n\to\infty} s_n(\pi)$?

>(q) Let f, g be integrable complex-valued functions on $]-\pi,\pi]$, and $\boldsymbol{c}=\langle c_k\rangle_{k\in\mathbb{Z}}$, $\boldsymbol{d}=\langle d_k\rangle_{k\in\mathbb{Z}}$ their sequences of Fourier coefficients. Suppose that *either* $\sum_{k=-\infty}^{\infty}|c_k|<\infty$ *or* $\sum_{k=-\infty}^{\infty}|c_k|^2+|d_k|^2<\infty$. Show that the sequence of Fourier coefficients of $f\times g$ is just the convolution $\boldsymbol{c}*\boldsymbol{d}$ of $\boldsymbol{c}$ and $\boldsymbol{d}$ (255Xk).

(r) In 282Ra, what happens if $f(\pi)\ne f(-\pi)$?

(s) Suppose that $\langle c_k\rangle_{k\in\mathbb{N}}$ is a double-ended sequence of complex numbers such that $\sum_{k=-\infty}^{\infty}|kc_k|<\infty$. Show that $f(x)=\sum_{k=-\infty}^{\infty}c_ke^{ikx}$ exists for every $x\in\mathbb{R}$ and that f is differentiable everywhere.

(t) Let $\langle c_k\rangle_{k\in\mathbb{Z}}$ be a double-ended sequence of complex numbers such that $\sup_{k\in\mathbb{Z}}|kc_k|<\infty$. Show that there is a square-integrable function f on $]-\pi,\pi]$ such that the c_k are the Fourier coefficients of f, that f is the limit almost everywhere of its Fourier sums, and that $f*f*f$ is differentiable. (*Hint*: use 282K to show that there is an f, and 282Xn to show that its Fourier sums converge wherever its Fejér sums do; use 282Q and 282Xs to show that $f*f*f$ is differentiable.)

282Y Further exercises (a) Let f be a non-negative integrable function on $]-\pi,\pi]$, with Fourier coefficients $\langle c_k\rangle_{k\in\mathbb{Z}}$. Show that

$$\sum_{j=0}^{n}\sum_{k=0}^{n}a_j\bar{a}_kc_{j-k}\ge 0$$

for all complex numbers $a_0,\dots,a_n$. (See also 285Xu below.)

(b) Let $f:]-\pi,\pi]\to\mathbb{C}$, $K\ge 0$, $\alpha>0$ be such that $|f(x)-f(y)|\le K|x-y|^\alpha$ for all x, $y\in\,]-\pi,\pi]$. Let c_k, s_n be the Fourier coefficients and sums of f. (i) Show that $\sup_{k\in\mathbb{Z}}|k|^\alpha|c_k|<\infty$. (*Hint*: show that $c_k=\frac{1}{4\pi}\int_{-\pi}^{\pi}(f(x)-\tilde{f}(x+\frac{\pi}{k}))e^{-ikx}dx$.) (ii) Show that if $f(\pi)=\lim_{x\downarrow-\pi}f(x)$ then $s_n\to f$ uniformly. (Compare 282Xj.)

(c) Let f be a measurable complex-valued function on $]-\pi,\pi]$, and suppose that $p\in[1,\infty[$ is such that $\int_{-\pi}^{\pi}|f|^p<\infty$. Let $\langle\sigma_m\rangle_{m\in\mathbb{N}}$ be the sequence of Fejér sums of f. Show that $\lim_{m\to\infty}\int_{-\pi}^{\pi}|f-\sigma_m|^p=0$. (*Hint*: use 245Xl, 255Yk and the ideas in 282Ib.)

(d) Construct a continuous function $h:[-\pi,\pi]\to\mathbb{R}$ such that $h(\pi)=h(-\pi)$ but the Fourier sums of h are unbounded at 0, as follows. Set $\alpha(m,n)=\int_0^{\pi}\frac{\sin(m+\frac{1}{2})t\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}dt$. Show that $\lim_{n\to\infty}\alpha(m,n)=0$ for every m, but $\lim_{n\to\infty}\alpha(n,n)=\infty$. Set $h_0(x)=\sum_{k=0}^{\infty}\delta_k\sin(m_k+\frac{1}{2})x$ for $0\le x\le\pi$, 0 for $-\pi\le x\le 0$, where $\delta_k>0$, $m_k\in\mathbb{N}$ are such that (α) $\delta_k\le 2^{-k}$, $\delta_k|\alpha(m_k,m_n)|\le 2^{-k}$ for every $n<k$ (choosing δ_k) (β) $\delta_k\alpha(m_k,m_k)\ge k$, $\delta_n|\alpha(m_k,m_n)|\le 2^{-n}$ for every $n<k$ (choosing m_k). Now modify h_0 on $[-\pi,0[$ by adding a function of bounded variation.

(e)(i) Show that $\lim_{n\to\infty}\int_{-\pi}^{\pi}|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}|dt=\infty$. (*Hint*: 282Xm.) (ii) Show that for any $\delta>0$ there are $n\in\mathbb{N}$, $f\ge 0$ such that $\int_{-\pi}^{\pi}f\le\delta$, $\int_{-\pi}^{\pi}|s_n|\ge 1$, where s_n is the nth Fourier sum of f. (*Hint*: take n such that $\frac{1}{2\pi}\int_{-\pi}^{\pi}|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}|dt>\frac{1}{\delta}$ and set $f(x)=\frac{\delta}{\eta}$ for $0\le x\le\eta$, 0 otherwise, where η is small.) (iii) Show that there is an integrable function $f:]-\pi,\pi]\to\mathbb{R}$ such that $\sup_{n\in\mathbb{N}}\|s_n\|_1$ is infinite, where $\langle s_n\rangle_{n\in\mathbb{N}}$ is the sequence of Fourier sums of f. (*Hint*: it helps to know the 'Uniform Boundedness Theorem' of functional analysis, but f can also be constructed bare-handed by the method of 282Yd.)

(f) Let $u:[-\pi,\pi]\to\mathbb{R}$ be an absolutely continuous function such that $u(\pi)=u(-\pi)$ and $\int_{-\pi}^{\pi}u=0$. Show that $\|u\|_2\le\|u'\|_2$. (This is **Wirtinger's inequality**.)

(g) For $0\le r<1$, $t\in\mathbb{R}$ set $A_r(t)=\frac{1-r^2}{1-2r\cos t+r^2}$. ($A_r$ is the **Poisson kernel**; see 478Xl[1] in Volume 4.) (i) Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi}A_r=1$. (ii) For a real function f which is integrable over $]-\pi,\pi]$, with real Fourier coefficients a_k, b_k (282Ba), set $S_r(x)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}r^k(a_k\cos kx+b_k\sin kx)$ for $x\in\,]-\pi,\pi]$, $r\in[0,1[$. Show

[1]Later editions only.

that $S_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A_r(x-t)f(t)dt$ for every $x \in]-\pi,\pi]$. (*Hint*: $A_r(t) = 1 + 2\sum_{n=1}^{\infty} r^n \cos nt$.) (iii) Show that $\lim_{r\uparrow 1} S_r(x) = f(x)$ for every $x \in]-\pi,\pi[$ which is in the Lebesgue set of f. (*Hint*: 223Yg.) (iv) Show that $\lim_{r\uparrow 1}\int_{-\pi}^{\pi}|S_r - f| = 0$. (v) Show that if f is defined everywhere on $]-\pi,\pi]$, is continuous, and $f(\pi) = \lim_{x\downarrow -\pi} f(x)$, then $\lim_{r\uparrow 1}\sup_{x\in]-\pi,\pi]}|S_r(x) - f(x)| = 0$.

282 Notes and comments This has been a long section with a potentially confusing collection of results, so perhaps I should recapitulate. Associated with any integrable function on $]-\pi,\pi]$ we have the corresponding Fourier sums, being the symmetric partial sums $\sum_{k=-n}^{n} c_k e^{ikx}$ of the complex series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$, or, equally, the partial sums $\frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ of the real series $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$. The Fourier coefficients c_k, a_k, b_k are the only natural ones, because if the series is to converge with any regularity at all then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Bigl(\sum_{k=-\infty}^{\infty} c_k e^{ikx}\Bigr)e^{-ilx}dx$$

ought to be simultaneously

$$\sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} c_k e^{ikx}e^{-ilx}dx = c_l$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ilx}dx.$$

(Compare the calculations in 282J.) The effect of taking Fejér sums $\sigma_m(x)$ rather than the Fourier sums $s_n(x)$ is to smooth the sequence out; recall that if $\lim_{n\to\infty} s_n(x) = c$ then $\lim_{m\to\infty}\sigma_m(x) = c$, by 273Ca in the last chapter.

Most of the work above is concerned with the question of when Fourier or Fejér sums converge, in some sense, to the original function f. As has happened before, in §245 and elsewhere, we have more than one kind of convergence to consider. *Norm* convergence, for $\|\,\|_1$ or $\|\,\|_2$ or $\|\,\|_\infty$, is the simplest; the three theorems 282G, 282Ib and 282J at least are relatively straightforward. (I have given 282Ib as a corollary of 282Ia; but there is an easier proof from 282G. See 282Xd.) Respectively, we have

if f is continuous (and matches at $\pm\pi$, that is, $f(\pi) = \lim_{t\downarrow -\pi} f(t)$) then $\sigma_m \to f$ uniformly, that is, for $\|\,\|_\infty$ (282G);

if f is any integrable function, then $\sigma_m \to f$ for $\|\,\|_1$ (282Ib);

if f is a square-integrable function, then $s_n \to f$ for $\|\,\|_2$ (282J);

if f is continuous and of bounded variation (and matches at $\pm\pi$), then $s_n \to f$ uniformly (282O).

There are some similar results for other $\|\,\|_p$ (282Yc); but note that the Fourier sums need not converge for $\|\,\|_1$ (282Ye).

Pointwise convergence is harder. The results I give are

if f is any integrable function, then $\sigma_m \to f$ almost everywhere (282Ia);

this relies on some careful calculations in 282H, and also on the deep result 223D. Next we have the results which look at the average of the limits of f from the two sides. Suppose I write

$$f^{\pm}(x) = \frac{1}{2}(\lim_{t\uparrow x} f(t) + \lim_{t\downarrow x} f(t))$$

whenever this is defined, taking $f^{\pm}(\pi) = \frac{1}{2}(\lim_{t\uparrow\pi} f(t) + \lim_{t\downarrow -\pi} f(t))$. Then we have

if f is any integrable function, $\sigma_m \to f^{\pm}$ wherever $f^{\pm}$ is defined (282I);

if f is of bounded variation, $s_n \to f^{\pm}$ everywhere (282O).

Of course these apply at any point at which f is continuous, in which case $f(x) = f^{\pm}(x)$. Yet another result of this type is

if f is any integrable function, $s_n \to f$ at any point at which f is differentiable (282L);

in fact, this can be usefully extended for very little extra labour (282Xi, 282Xp).

I cannot leave this list without mentioning the theorem I have *not* given. This is **Carleson's theorem**:

if f is square-integrable, $s_n \to f$ almost everywhere

(Carleson 66). I will come to this in §286. There is an elementary special case in 282Xt. The result is in fact valid for many other f (see the notes to §286).

The next glaring lacuna in the exposition here is the absence of any examples to show how far these results are best possible. There is no suggestion, indeed, that there are any natural necessary and sufficient conditions for

$s_n \to f$ at every point.

Nevertheless, we have to make an effort to find a continuous function for which this is not so, and the construction of an example by du Bois-Reymond (Bois-Reymond 1876) was an important moment in the history of analysis, not least because it forced mathematicians to realise that some comfortable assumptions about the classification of functions – essentially, that functions are either 'good' or so bad that one needn't trouble with them – were false. The example is instructive but I have had to omit it for lack of space; I give an outline of a possible method in 282Yd. (You can find a detailed construction in Körner 88, chapter 18, and a proof that such a function exists in Dudley 89, 7.4.3.) If you allow general integrable functions, then you can do much better, or perhaps I should say much worse; there is an integrable f such that $\sup_{n\in\mathbb{N}} |s_n(x)| = \infty$ for every $x \in \,]-\pi,\pi]$ (Kolmogorov 1926; see Zygmund 59, §§VIII.3-4).

In 282C I mentioned two types of problem. The first – when is a Fourier series summable? – has at least been treated at length, even though I cannot pretend to have given more than a sample of what is known. The second – how do properties of the c_k reflect properties of f? – I have hardly touched on. I do give what seem to me to be the three most important results in this area. The first is

if f and g have the same Fourier coefficients, they are equal almost everywhere (282Ic).

This at least tells us that we ought in principle to be able to learn almost anything about f by looking at its Fourier series. (For instance, 282Ya describes a necessary and sufficient condition for f to be non-negative almost everywhere.) The second is

f is square-integrable iff $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$;

in fact,

$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi}\int_{\pi}^{\pi} |f|^2$ (282J).

Of course this is fundamental, since it shows that Fourier coefficients provide a natural Hilbert space isomorphism between L^2 and ℓ^2 (282K). I should perhaps remark that while the real Hilbert spaces $L^2_{\mathbb{R}}$, $\ell^2_{\mathbb{R}}$ are isomorphic as inner product spaces (282Xg), they are certianly not isomorphic as Banach lattices; for instance, $\ell^2_{\mathbb{R}}$ has 'atomic' elements $\boldsymbol{c}$ such that if $0 \le \boldsymbol{d} \le \boldsymbol{c}$ then $\boldsymbol{d}$ is a multiple of $\boldsymbol{c}$, while $L^2_{\mathbb{R}}$ does not. Perhaps even more important is

the Fourier coefficients of a convolution $f * g$ are just a scalar multiple of the products of the Fourier coefficients of f and g (282Q);

but to use this effectively we need to study the Banach algebra structure of L^1, and I have no choice but to abandon this path immediately. (It will form a conspicuous part of Chapter 44 in Volume 4.) 282Xt gives an elementary consequence, and 282Xq a very partial description of the relationship between a product $f \times g$ of two functions and the convolution product of their sequences of Fourier coefficients.

The Fejér sums considered in this section are one way of working around the convergence difficulties associated with Fourier sums. When we come to look at Fourier transforms in the next two sections we shall need some further manoeuvres. A different type of smoothing is obtained by using the Poisson kernel in place of the Dirichlet or Fejér kernel (282Yg).

I end these notes with a remark on the number 2π. This enters nearly every formula involving Fourier series, but could I think be removed totally from the present section, at least, by re-normalizing the measure of $]-\pi,\pi]$. If instead of Lebesgue measure μ we took the measure $\nu = \frac{1}{2\pi}\mu$ throughout, then every 2π would disappear. (Compare the remark in 282Bb concerning the possibility of doing integrals over S^1.) But I think most of us would prefer to remember the location of a 2π in every formula than to deal with an unfamiliar measure.

Version of 31.3.13

283 Fourier transforms I

I turn now to the theory of Fourier transforms on $\mathbb{R}$. In the first of two sections on the subject, I present those parts of the elementary theory which can be dealt with using the methods of the previous section on Fourier series. I find no way of making sense of the theory, however, without introducing a fragment of L.Schwartz' theory of distributions, which I present in §284. As in §282, of course, this treatment also is nothing but a start in the topic.

The whole theory can also be done in $\mathbb{R}^r$. I leave this extension to the exercises, however, since there are few new ideas, the formulae are significantly more complicated, and I shall not, in this volume at least, have any use for the multidimensional versions of these particular theorems, though some of the same ideas will appear, in multidimensional form, in §285.

283A Definitions Let f be a complex-valued function which is integrable over $\mathbb{R}$.

(a) The **Fourier transform** of f is the function $\hat{f}:\mathbb{R}\to\mathbb{C}$ defined by setting

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyx}f(x)dx$$

for every $y\in\mathbb{R}$. (Of course the integral is always defined because $x\mapsto e^{-iyx}$ is bounded and continuous, therefore measurable.)

(b) The **inverse Fourier transform** of f is the function $\check{f}:\mathbb{R}\to\mathbb{C}$ defined by setting

$$\check{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iyx}f(x)dx$$

for every $y\in\mathbb{R}$.

283B Remarks (a) It is a mildly vexing feature of the theory of Fourier transforms – vexing, that is, for outsiders like myself – that there is in fact no standard definition of 'Fourier transform'. The commonest definitions are, I think,

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\mp iyx}f(x)dx,$$

$$\hat{f}(y)=\int_{-\infty}^{\infty}e^{\mp iyx}f(x)dx,$$

$$\hat{f}(y)=\int_{-\infty}^{\infty}e^{\mp 2\pi iyx}f(x)dx,$$

corresponding to inverse transforms

$$\check{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\pm iyx}f(x)dx,$$

$$\check{f}(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\pm iyx}f(x)dx,$$

$$\check{f}(y)=\int_{-\infty}^{\infty}e^{\pm 2\pi iyx}f(x)dx.$$

I leave it to you to check that the whole theory can be carried through with any of these six pairs, and to investigate other possibilities (see 283Xa-283Xb below).

(b) The phrases 'Fourier transform', 'inverse Fourier transform' make it plain that $(\hat{f})^\vee$ is supposed to be f, at least some of the time. This is indeed the case, but the class of f for which this is true in the literal sense is somewhat constrained, and we shall have to wait a little while before investigating it.

(c) No amount of juggling with constants, in the manner of (a) above, can make $\hat{f}$ and $\check{f}$ quite the same. However, on the definitions I have chosen, we do have $\check{f}(y) = \hat{f}(-y)$ for every y, so that $\check{f}$ and $\hat{f}$ will share essentially all the properties of interest to us here; in particular, everything in the next proposition will be valid with $^\vee$ in place of $^\wedge$, if you change signs at the right points in parts (c), (h) and (i).

283C Proposition Let f and g be complex-valued functions which are integrable over $\mathbb{R}$.

(a) $(f+g)^\wedge = \hat{f} + \hat{g}$.

(b) $(cf)^\wedge = c\hat{f}$ for every $c \in \mathbb{C}$.

(c) If $c \in \mathbb{R}$ and $h(x) = f(x+c)$ whenever this is defined, then $\hat{h}(y) = e^{icy}\hat{f}(y)$ for every $y \in \mathbb{R}$.

(d) If $c \in \mathbb{R}$ and $h(x) = e^{icx}f(x)$ for every $x \in \operatorname{dom} f$, then $\hat{h}(y) = \hat{f}(y-c)$ for every $y \in \mathbb{R}$.

(e) If $c > 0$ and $h(x) = f(cx)$ whenever this is defined, then $\hat{h}(y) = \frac{1}{c}\hat{f}(\frac{y}{c})$ for every $y \in \mathbb{R}$.

(f) $\hat{f} : \mathbb{R} \to \mathbb{C}$ is continuous.

(g) $\lim_{y\to\infty} \hat{f}(y) = \lim_{y\to-\infty} \hat{f}(y) = 0$.

(h) If $\int_{-\infty}^{\infty} |xf(x)|dx < \infty$, then $\hat{f}$ is differentiable, and its derivative is

$$\hat{f}'(y) = -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}xf(x)dx$$

for every $y \in \mathbb{R}$.

(i) If f is absolutely continuous on every bounded interval and f' is integrable, then $(f')^\wedge(y) = iy\hat{f}(y)$ for every $y \in \mathbb{R}$.

proof (a) and **(b)** are trivial, and **(c)**, **(d)** and **(e)** are elementary substitutions.

(f) If $\langle y_n\rangle_{n\in\mathbb{N}}$ is any convergent sequence in $\mathbb{R}$ with limit y, then

$$\begin{aligned}\hat{f}(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \lim_{n\to\infty} e^{-iy_nx}f(x)dx \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iy_nx}f(x)dx = \lim_{n\to\infty} \hat{f}(y_n)\end{aligned}$$

by Lebesgue's Dominated Convergence Theorem, because $|e^{-iy_nx}f(x)| \le |f(x)|$ for every $n \in \mathbb{N}$ and $x \in \operatorname{dom} f$. As $\langle y_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\hat{f}$ is continuous.

(g) This is just the Riemann-Lebesgue lemma (282E).

(h) The point is that $|\frac{\partial}{\partial y}e^{-iyx}f(x)| = |xf(x)|$ whenever $x \in \operatorname{dom} f$ and $y \in \mathbb{R}$. So by 123D

$$\begin{aligned}\hat{f}'(y) &= \frac{1}{\sqrt{2\pi}}\frac{d}{dy}\int_{-\infty}^{\infty} e^{-iyx}f(x)dx = \frac{1}{\sqrt{2\pi}}\frac{d}{dy}\int_{\operatorname{dom} f} e^{-iyx}f(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\operatorname{dom} f} \frac{\partial}{\partial y}e^{-iyx}f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -ixe^{-iyx}f(x)dx \\ &= -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-iyx}f(x)dx.\end{aligned}$$

(i) Because f is absolutely continuous on every bounded interval,

$$f(x) = f(0) + \int_0^x f' \text{ for } x \ge 0, \quad f(x) = f(0) - \int_x^0 f' \text{ for } x \le 0.$$

Because f' is integrable,

$$\lim_{x\to\infty} f(x) = f(0) + \int_0^{\infty} f', \quad \lim_{x\to-\infty} f(x) = f(0) - \int_{-\infty}^0 f'$$

both exist. Because f also is integrable, both limits must be zero. Now we can integrate by parts (225F) to see that

$$(f')^\wedge(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} f'(x)dx = \frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{-iyx} f'(x)dx$$
$$= \frac{1}{\sqrt{2\pi}} \big(\lim_{a\to\infty} e^{-iya} f(a) - \lim_{a\to-\infty} e^{-iya} f(a) \big) + \frac{iy}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} f(x)dx$$
$$= iy\hat{f}(y).$$

283D Lemma (a) $\lim_{a\to\infty} \int_0^a \frac{\sin x}{x} dx = \frac{\pi}{2}$, $\lim_{a\to\infty} \int_{-a}^a \frac{\sin x}{x} dx = \pi$.

(b) There is a $K < \infty$ such that $|\int_a^b \frac{\sin cx}{x} dx| \le K$ whenever $a \le b$ and $c \in \mathbb{R}$.

proof (a)(i) Set

$$F(a) = \int_0^a \frac{\sin x}{x} dx \text{ if } a \ge 0, \quad F(a) = -\int_{-a}^0 \frac{\sin x}{x} dx \text{ if } a \le 0,$$

so that $F(a) = -F(-a)$ and $\int_a^b \frac{\sin x}{x} dx = F(b) - F(a)$ for all $a \le b$.

If $0 < a \le b$, then by 224J

$$|\int_a^b \frac{\sin x}{x} dx| \le (\frac{1}{b} + \frac{1}{a} - \frac{1}{b}) \sup_{c\in[a,b]} |\int_a^c \sin x\, dx| \le \frac{1}{a} \sup_{c\in[a,b]} |\cos a - \cos c| \le \frac{2}{a}.$$

In particular, $|F(n) - F(m)| \le \frac{2}{m}$ if $0 < m \le n$ in $\mathbb{N}$, and $\langle F(n)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence with limit γ say; now

$$|\gamma - F(a)| = \lim_{n\to\infty} |F(n) - F(a)| \le \frac{2}{a}$$

for every $a > 0$, so $\lim_{a\to\infty} F(a) = \gamma$. Of course we also have

$$\lim_{a\to\infty} \int_{-a}^a \frac{\sin x}{x} dx = \lim_{a\to\infty} (F(a) - F(-a)) = \lim_{a\to\infty} 2F(a) = 2\gamma.$$

(ii) So now I have to calculate γ. For this, observe first that

$$2\gamma = \lim_{a\to\infty} \int_{-\pi a}^{\pi a} \frac{\sin x}{x} dx = \lim_{a\to\infty} \int_{-\pi}^{\pi} \frac{\sin at}{t} dt$$

(substituting $x = t/a$). Next,

$$\lim_{t\to 0} \frac{1}{t} - \frac{1}{2\sin\frac{1}{2}t} = \lim_{u\to 0} \frac{\sin u - u}{2u \sin u} = 0,$$

so

$$\int_{-\pi}^{\pi} \Big| \frac{1}{t} - \frac{1}{2\sin\frac{1}{2}t} \Big| dt < \infty,$$

and by the Riemann-Lebesgue lemma (282Fb)

$$\lim_{a\to\infty} \int_{-\pi}^{\pi} \Big(\frac{1}{t} - \frac{1}{2\sin\frac{1}{2}t} \Big) \sin at\, dt = 0.$$

But we know that

$$\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} dt = \pi$$

for every n (using 282Dc), so we must have

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{\sin t}{t} dt = \lim_{a\to\infty} \int_{-\pi}^{\pi} \frac{\sin at}{t} dt = \lim_{a\to\infty} \int_{-\pi}^{\pi} \frac{\sin at}{2\sin\frac{1}{2}t} dt$$
$$= \lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} dt = \pi,$$

and $\gamma = \pi/2$, as claimed.

(b) Because F is continuous and

$$\lim_{a\to\infty} F(a) = \gamma = \frac{\pi}{2}, \quad \lim_{a\to-\infty} F(a) = -\gamma = -\frac{\pi}{2},$$

F is bounded; say $|F(a)| \le K_1$ for all $a \in \mathbb{R}$. Try $K = 2K_1$. Now suppose that $a < b$ and $c \in \mathbb{R}$. If $c > 0$, then

$$|\int_a^b \frac{\sin cx}{x} dx| = |\int_{ac}^{bc} \frac{\sin t}{t} dt| = |F(bc) - F(ac)| \le 2K_1 = K,$$

substituting $x = t/c$. If $c < 0$, then

$$|\int_a^b \frac{\sin cx}{x} dx| = | - \int_a^b \frac{\sin(-c)x}{x} dx| \le K;$$

while if $c = 0$ then

$$|\int_a^b \frac{\sin cx}{x} dx| = 0 \le K.$$

283E The hardest work of this section will lie in the 'pointwise inversion theorems' 283I and 283K below. I begin however with a relatively easy, and at least equally important, result, showing (among other things) that an integrable function f can (essentially) be recovered from its Fourier transform.

Lemma Whenever $c < d$ in $\mathbb{R}$,

$$\begin{aligned}\lim_{a\to\infty} \int_{-a}^{a} e^{-iyx} \frac{e^{idy} - e^{icy}}{y} dy &= 2\pi i \text{ if } c < x < d,\\ &= \pi i \text{ if } x = c \text{ or } x = d,\\ &= 0 \text{ if } x < c \text{ or } x > d.\end{aligned}$$

proof We know that for any $b > 0$

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{\sin bx}{x} dx = \lim_{a\to\infty} \int_{-ab}^{ab} \frac{\sin t}{t} dt = \pi$$

(subsituting $x = t/b$), and therefore that for any $b < 0$

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{\sin bx}{x} dx = -\lim_{a\to\infty} \int_{-a}^{a} \frac{\sin(-b)x}{x} dx = -\pi.$$

Now consider, for $x \in \mathbb{R}$,

$$\lim_{a\to\infty} \int_{-a}^{a} e^{-iyx} \frac{e^{idy} - e^{icy}}{y} dy.$$

First note that all the integrals $\int_{-a}^{a}$ exist, because

$$\lim_{y\to 0} \frac{e^{idy} - e^{icy}}{y} = i(d - c)$$

is finite, and the integrand is certainly continuous except at 0. Now we have

$$\begin{aligned}&\int_{-a}^{a} e^{-iyx} \frac{e^{idy} - e^{icy}}{y} dy\\ &\qquad = \int_{-a}^{a} \frac{e^{i(d-x)y} - e^{i(c-x)y}}{y} dy\\ &\qquad = \int_{-a}^{a} \frac{\cos(d-x)y - \cos(c-x)y}{y} dy + i \int_{-a}^{a} \frac{\sin(d-x)y - \sin(c-x)y}{y} dy\\ &\qquad = i \int_{-a}^{a} \frac{\sin(d-x)y - \sin(c-x)y}{y} dy\end{aligned}$$

because cos is an even function, so

$$\int_{-a}^{a} \frac{\cos(d-x)y-\cos(c-x)y}{y}dy = 0$$

for every $a \geq 0$. (Once again, this integral exists because

$$\lim_{y\to 0} \frac{\cos(d-x)y-\cos(c-x)y}{y} = 0.)$$

Accordingly

$$\begin{aligned}\lim_{a\to\infty}\int_{-a}^{a} e^{-iyx}\frac{e^{idy}-e^{icy}}{y}dy &= i\lim_{a\to\infty}\int_{-a}^{a}\frac{\sin(d-x)y}{y}dy - i\lim_{a\to\infty}\int_{-a}^{a}\frac{\sin(c-x)y}{y}dy\\ &= i\pi - i\pi = 0 \text{ if } x < c,\\ &= i\pi - 0 = \pi i \text{ if } x = c,\\ &= i\pi + i\pi = 2\pi i \text{ if } c < x < d,\\ &= 0 + i\pi = \pi i \text{ if } x = d,\\ &= -i\pi + i\pi = 0 \text{ if } x > d.\end{aligned}$$

283F Theorem Let f be a complex-valued function which is integrable over $\mathbb{R}$, and $\hat{f}$ its Fourier transform. Then whenever $c \leq d$ in $\mathbb{R}$,

$$\int_{c}^{d} f = \frac{i}{\sqrt{2\pi}} \lim_{a\to\infty}\int_{-a}^{a}\frac{e^{icy}-e^{idy}}{y}\hat{f}(y)dy.$$

proof If $c = d$ this is trivial; let us suppose that $c < d$.

(a) Writing

$$\theta_a(x) = \int_{-a}^{a} e^{-iyx}\,\frac{e^{idy}-e^{icy}}{y}dy$$

for $x \in \mathbb{R}$ and $a \geq 0$, 283E tells us that

$$\lim_{a\to\infty}\theta_a(x) = 2\pi i\theta(x)$$

where $\theta = \frac{1}{2}(\chi[c,d] + \chi]c,d[)$ takes the value 1 inside the interval $[c,d]$, 0 outside and the value $\frac{1}{2}$ at the endpoints. At the same time,

$$|\theta_a(x)| = |\int_{-a}^{a}\frac{\sin(d-x)y-\sin(c-x)y}{y}dy|$$

(see the proof of 283E)

$$\leq |\int_{-a}^{a}\frac{\sin(d-x)y}{y}dy| + |\int_{-a}^{a}\frac{\sin(c-x)y}{y}dy| \leq 2K$$

for all $a \geq 0$ and $x \in \mathbb{R}$, where K is the constant of 283Db. Consequently $|f \times \theta_a| \leq 2K|f|$ everywhere on $\operatorname{dom} f$, for every $a \geq 0$, and (applying Lebesgue's Dominated Convergence Theorem to sequences $\langle f \times \theta_{a_n}\rangle_{n\in\mathbb{N}}$, where $a_n \to \infty$)

$$\lim_{a\to\infty}\int f \times \theta_a = 2\pi i\int f \times \theta = 2\pi i\int_{c}^{d} f.$$

(b) Now consider the limit in the statement of the theorem. We have

$$\begin{aligned}\int_{-a}^{a}\frac{e^{icy}-e^{idy}}{y}\hat{f}(y)dy &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\int_{-\infty}^{\infty}\frac{e^{icy}-e^{idy}}{y}e^{-iyx}f(x)dxdy\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-a}^{a}\frac{e^{icy}-e^{idy}}{y}e^{-iyx}f(x)dydx\\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\times\theta_a\end{aligned}$$

by Fubini's and Tonelli's theorems (252H), using the fact that $(e^{icy}-e^{idy})/y$ is bounded to see that

$$\int_{-\infty}^{\infty}\int_{-a}^{a}\Bigl|\tfrac{e^{icy}-e^{idy}}{y}e^{-iyx}f(x)\Bigr|dydx$$

is finite. Accordingly

$$\begin{aligned}\frac{i}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a}\frac{e^{icy}-e^{idy}}{y}\hat{f}(y)dy &= -\frac{i}{2\pi}\lim_{a\to\infty}\int_{-\infty}^{\infty}f\times\theta_a\\ &= -\frac{i}{2\pi}2\pi i\int_c^d f=\int_c^d f,\end{aligned}$$

as required.

283G Corollary If f and g are complex-valued functions which are integrable over $\mathbb{R}$, then $\hat{f}=\hat{g}$ iff $f=_{\text{a.e.}}g$.

proof If $f=_{\text{a.e.}}g$ then of course

$$\hat{f}(y)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyx}f(x)dx=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyx}g(x)dx=\hat{g}(y)$$

for every $y\in\mathbb{R}$. Conversely, if $\hat{f}=\hat{g}$, then by the last theorem

$$\int_c^d f=\int_c^d g$$

for all $c\le d$, so $f=g$ almost everywhere, by 222D.

283H Lemma Let f be a complex-valued function which is integrable over $\mathbb{R}$, and $\hat{f}$ its Fourier transform. Then

$$\tfrac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{ixy}\hat{f}(y)dy=\tfrac{1}{\pi}\int_{-\infty}^{\infty}\tfrac{\sin a(x-t)}{x-t}f(t)dt=\tfrac{1}{\pi}\int_{-\infty}^{\infty}\tfrac{\sin at}{t}f(x-t)dt$$

whenever $a>0$ and $x\in\mathbb{R}$.

proof We have

$$\int_{-a}^{a}\int_{-\infty}^{\infty}|e^{ixy}e^{-iyt}f(t)|dtdy\le 2a\int_{-\infty}^{\infty}|f(t)|dt<\infty,$$

so (because the function $(t,y)\mapsto e^{ixy}e^{-iyt}f(t)$ is surely jointly measurable) we may reverse the order of integration, and get

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{ixy}\hat{f}(y)dy &= \frac{1}{2\pi}\int_{-a}^{a}\int_{-\infty}^{\infty}e^{ixy}e^{-iyt}f(t)dt\,dy\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\int_{-a}^{a}e^{i(x-t)y}dy\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin(x-t)a}{x-t}f(t)dt=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin au}{u}f(x-u)du,\end{aligned}$$

substituting $t=x-u$.

283I Theorem Let f be a complex-valued function which is integrable over $\mathbb{R}$, and suppose that f is differentiable at $x \in \mathbb{R}$. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{ixy} \hat{f}(y)dy = \frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{-ixy} \check{f}(y)dy.$$

proof Set $g(u) = f(x)$ if $|u| \le 1$, 0 otherwise, and observe that $\lim_{u\to 0} \frac{1}{u}(f(x-u) - g(u)) = -f'(x)$ is finite, so that there is a $\delta \in]0,1]$ such that

$$K = \sup_{0<|u|\le\delta} \Bigl|\frac{f(x-u)-g(u)}{u}\Bigr| < \infty.$$

Consequently

$$\begin{aligned}\int_{-\infty}^{\infty} \Bigl|\frac{f(x-u)-g(u)}{u}\Bigr| du &\le \frac{1}{\delta}\int_{-\infty}^{-\delta} |f(x-u)|du + \frac{1}{\delta}\int_{-1}^{1} |g| \\ &\qquad + \int_{-\delta}^{\delta} K + \frac{1}{\delta}\int_{\delta}^{\infty} |f(x-u)|du \\ &\le \frac{1}{\delta}\int_{-\infty}^{\infty} |f| + \frac{2}{\delta}|f(x)| + 2\delta K < \infty.\end{aligned}$$

By the Riemann-Lebesgue lemma (282Fb again),

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} \frac{\sin au}{u}(f(x-u) - g(u))du = 0.$$

If we now examine $\int \frac{\sin au}{u} g(u)du$, we get

$$\int_{-\infty}^{\infty} \frac{\sin au}{u} g(u)du = \int_{-1}^{1} \frac{\sin au}{u} f(x)du = f(x)\int_{-a}^{a} \frac{\sin v}{v} dv,$$

substituting $u = v/a$. So we get

$$\begin{aligned}\lim_{a\to\infty} \int_{-\infty}^{\infty} \frac{\sin au}{u} f(x-u)du &= \lim_{a\to\infty} \int_{-\infty}^{\infty} \frac{\sin au}{u} g(u)du \\ &= \lim_{a\to\infty} f(x)\int_{-a}^{a} \frac{\sin v}{v} dv = \pi f(x),\end{aligned}$$

by 283Da. Accordingly

$$\frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{ixy} \hat{f}(y)dy = \frac{1}{\pi} \lim_{a\to\infty} \int_{-\infty}^{\infty} \frac{\sin au}{u} f(x-u)du = f(x),$$

using 283H. As for the second equality,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{-ixy} \check{f}(y)dy &= \frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{-ixy} \hat{f}(-y)dy \\ &= \frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{ixu} \hat{f}(u)du = f(x)\end{aligned}$$

(substituting $y = -u$).

Remark Compare 282L.

283J Corollary Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function such that f is differentiable and $\hat{f}$ is integrable. Then $f = (\hat{f})^\vee = (\check{f})^\wedge$.

proof Because $\hat{f}$ is integrable,

$$\overset{\wedge\vee}{f}(x) = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ixy}\hat{f}(y)dy = f(x)$$

for every $x \in \mathbb{R}$. Similarly,

$$\overset{\vee\wedge}{f}(x) = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ixy}\check{f}(y)dy = f(x).$$

Remark See also 283Wk below.

283K The next proposition gives a class of functions to which the last corollary can be applied.

Proposition Suppose that f is a twice-differentiable function from $\mathbb{R}$ to $\mathbb{C}$ such that f, f' and f'' are all integrable. Then $\hat{f}$ is integrable.

proof Because f' and f'' are integrable, f and f' are absolutely continuous on any bounded interval (225L). So by 283Ci we have

$$(f'')^\wedge(y) = iy(f')^\wedge(y) = -y^2\hat{f}(y)$$

for every $y \in \mathbb{R}$. At the same time, by 283Cf-283Cg, $(f'')^\wedge$ and $\hat{f}$ must be bounded; say $|\hat{f}(y)|+|(f'')^\wedge(y)| \le K$ for every $y \in \mathbb{R}$. Now

$$|\hat{f}(y)| \le \frac{K}{1+y^2}$$

for every y, so that

$$\int_{-\infty}^{\infty} |\hat{f}| \le K\int_{-\infty}^{-1} \frac{1}{y^2}dy + 2K + K\int_{1}^{\infty} \frac{1}{y^2}dy = 4K < \infty.$$

Remark Compare 282Rb.

283L I turn now to the result corresponding to 282O, using a slightly different approach.

Theorem Let f be a complex-valued function which is integrable over $\mathbb{R}$, with Fourier transform $\hat{f}$ and inverse Fourier transform $\check{f}$, and suppose that f is of bounded variation on some neighbourhood of $x \in \mathbb{R}$. Set $a = \lim_{t\in\mathrm{dom}\, f,t\uparrow x} f(t)$, $b = \lim_{t\in\mathrm{dom}\, f,t\downarrow x} f(t)$. Then

$$\frac{1}{\sqrt{2\pi}} \lim_{\gamma\to\infty} \int_{-\gamma}^{\gamma} e^{ixy}\hat{f}(y)dy = \frac{1}{\sqrt{2\pi}} \lim_{\gamma\to\infty} \int_{-\gamma}^{\gamma} e^{-ixy}\check{f}(y)dy = \frac{1}{2}(a+b).$$

proof (a) The limits $\lim_{t\in\mathrm{dom}\, f,t\uparrow x} f(t)$ and $\lim_{t\in\mathrm{dom}\, f,t\downarrow x} f(t)$ exist because f is of bounded variation near x (224F). Recall from 283Db that there is a constant $K < \infty$ such that

$$|\int_{\gamma}^{\delta} \frac{\sin cx}{x}dx| \le K$$

whenever $\gamma \le \delta$ and $c \in \mathbb{R}$.

(b) Let $\epsilon > 0$. The hypothesis is that there is some $\delta > 0$ such that $\mathrm{Var}_{[x-\delta,x+\delta]}(f) < \infty$. Consequently

$$\lim_{\eta\downarrow 0} \mathrm{Var}_{]x,x+\eta]}(f) = \lim_{\eta\downarrow 0} \mathrm{Var}_{[x-\eta,x[}(f) = 0$$

(224E). There is therefore an $\eta > 0$ such that

$$\max(\mathrm{Var}_{[x-\eta,x[}(f), \mathrm{Var}_{]x,x+\eta]}(f)) \le \epsilon.$$

Of course

$$|f(t) - f(u)| \le \mathrm{Var}_{[x-\eta,x[}(f) \le \epsilon$$

whenever t, $u \in \mathrm{dom}\, f$ and $x - \eta \le t \le u < x$, so we shall have

$$|f(t) - a| \le \epsilon \text{ for every } t \in \mathrm{dom}\, f \cap [x-\eta, x[,$$

and similarly

$$|f(t)-b|\le\epsilon \text{ whenever } t\in\operatorname{dom}f\cap\,]x,x+\eta].$$

(c) Now set

$$g_1(t)=f(t) \text{ when } t\in\operatorname{dom}f \text{ and } |x-t|>\eta,\ 0 \text{ otherwise},$$

$$g_2(t)=a \text{ when } x-\eta\le t<x,\ b \text{ when } x<t\le x+\eta,\ 0 \text{ otherwise},$$

$$g_3=f-g_1-g_2.$$

Then $f=g_1+g_2+g_3$; each g_j is integrable; g_1 is zero on a neighbourhood of x;

$$\sup_{t\in\operatorname{dom}g_3,t\ne x}|g_3(t)|\le\epsilon,$$

$$\operatorname{Var}_{[x-\eta,x[}(g_3)\le\epsilon,\quad \operatorname{Var}_{]x,x+\eta]}(g_3)\le\epsilon.$$

(d) Consider the three parts g_1, g_2, g_3 separately.

(i) For the first, we have

$$\lim_{\gamma\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\gamma}^{\gamma}e^{ixy}\hat g_1(y)dy=0$$

by 283I.

(ii) Next,

$$\frac{1}{\sqrt{2\pi}}\int_{-\gamma}^{\gamma}e^{ixy}\hat g_2(y)dy=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(x-t)\gamma}{x-t}g_2(t)dt$$

(by 283H)

$$=\frac{a}{\pi}\int_{x-\eta}^{x}\frac{\sin(x-t)\gamma}{x-t}dt+\frac{b}{\pi}\int_{x}^{x+\eta}\frac{\sin(x-t)\gamma}{x-t}dt$$

$$=\frac{a}{\pi}\int_0^{\gamma\eta}\frac{\sin u}{u}du+\frac{b}{\pi}\int_0^{\gamma\eta}\frac{\sin u}{u}du$$

(substituting $t=x-\frac{1}{\gamma}u$ in the first integral, $t=-x+\frac{1}{\gamma}u$ in the second)

$$\to\frac{a+b}{2}\text{ as }\gamma\to\infty$$

by 283Da.

(iii) As for the third, we have, for any $\gamma>0$,

$$\begin{aligned}\Bigl|\frac{1}{\sqrt{2\pi}}\int_{-\gamma}^{\gamma}e^{ixy}\hat g_3(y)dy\Bigr|&=\frac{1}{\pi}\Bigl|\int_{-\infty}^{\infty}\frac{\sin(x-t)\gamma}{x-t}g_3(t)dt\Bigr|=\frac{1}{\pi}\Bigl|\int_{-\infty}^{\infty}\frac{\sin t\gamma}{t}g_3(x-t)dt\Bigr|\\&\le\frac{1}{\pi}\Bigl|\int_{-\eta}^{0}\frac{\sin t\gamma}{t}g_3(x-t)dt\Bigr|+\frac{1}{\pi}\Bigl|\int_0^{\eta}\frac{\sin t\gamma}{t}g_3(x-t)dt\Bigr|\\&\le\frac{K}{\pi}\Bigl(\sup_{t\in\operatorname{dom}g_3\cap]x-\eta,x[}|g_3(t)|+\operatorname*{Var}_{]x-\eta,x[}(g_3)\\&\qquad+\sup_{t\in\operatorname{dom}g_3\cap]x,x+\eta[}|g_3(t)|+\operatorname*{Var}_{]x,x+\eta[}(g_3)\Bigr)\\&\le4\epsilon\frac{K}{\pi},\end{aligned}$$

using 224J again to bound the integrals in terms of the variation and supremum of g_3 and integrals of $\frac{\sin\gamma t}{t}$ over subintervals.

(e) We therefore have

$$\begin{aligned}\limsup_{\gamma\to\infty}\Big|\tfrac{1}{\sqrt{2\pi}}\int_{-\gamma}^{\gamma}e^{ixy}\hat{f}(y)dy-\tfrac{a+b}{2}\Big|\\ &\le\limsup_{\gamma\to\infty}\frac{1}{\sqrt{2\pi}}\Big|\int_{-\gamma}^{\gamma}e^{ixy}\hat{g}_1(y)dy\Big|\\ &\qquad+\limsup_{\gamma\to\infty}\Big|\frac{1}{\sqrt{2\pi}}\int_{-\gamma}^{\gamma}e^{ixy}\hat{g}_2(y)dy-\tfrac{a+b}{2}\Big|\\ &\qquad+\limsup_{\gamma\to\infty}\frac{1}{\sqrt{2\pi}}\Big|\int_{-\gamma}^{\gamma}e^{ixy}\hat{g}_3ydy\Big|\\ &\le 0+0+\frac{4K}{\pi}\epsilon\end{aligned}$$

by the calculations in (d). As ϵ is arbitrary,

$$\lim_{\gamma\to\infty}\tfrac{1}{\sqrt{2\pi}}\textstyle\int_{-\gamma}^{\gamma}e^{ixy}\hat{f}(y)dy-\tfrac{a+b}{2}=0.$$

(f) This is the first half of the theorem. But of course the second half follows at once, because

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\lim_{\gamma\to\infty}\int_{-\gamma}^{\gamma}e^{-ixy}\check{f}(y)dy&=\frac{1}{\sqrt{2\pi}}\lim_{\gamma\to\infty}\int_{-\gamma}^{\gamma}e^{-ixy}\hat{f}(-y)dy\\ &=\frac{1}{\sqrt{2\pi}}\lim_{\gamma\to\infty}\int_{-\gamma}^{\gamma}e^{ixy}\hat{f}(y)dy=\frac{a+b}{2}.\end{aligned}$$

Remark You will see that this argument uses some of the same ideas as those in 282O-282P. It is more direct because (i) I am not using any concept corresponding to Fejér sums (though a very suitable one is available; see 283Xf) (ii) I do not trouble to give the result concerning uniform convergence of the Fejér integrals when f is continuous and of bounded variation (283Xj) (iii) I do not give any pointer to the significance of the fact that if f is of bounded variation then $\sup_{y\in\mathbb{R}}|y\hat{f}(y)|<\infty$ (283Xk).

283M Corresponding to 282Q, we have the following.

Theorem Let f and g be complex-valued functions which are integrable over $\mathbb{R}$, and $f*g$ their convolution product, defined by setting

$$(f*g)(x)=\textstyle\int_{-\infty}^{\infty}f(t)g(x-t)dt$$

whenever this is defined (255E). Then

$$(f*g)^\wedge(y)=\sqrt{2\pi}\hat{f}(y)\hat{g}(y),\quad (f*g)^\vee(y)=\sqrt{2\pi}\check{f}(y)\check{g}(y)$$

for every $y\in\mathbb{R}$.

proof For any y,

$$\begin{aligned}(f*g)^\wedge(y)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyx}(f*g)(x)dx\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-iy(t+u)}f(t)g(u)dtdu\end{aligned}$$

(using 255G)

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyt}f(t)dt\int_{-\infty}^{\infty}e^{-iyu}g(u)du=\sqrt{2\pi}\hat{f}(y)\hat{g}(y).$$

Now, of course,

$$(f*g)^\vee(y)=(f*g)^\wedge(-y)=\sqrt{2\pi}\hat{f}(-y)\hat{g}(-y)=\sqrt{2\pi}\check{f}(y)\check{g}(y).$$

283N I show how to compute a special Fourier transform, which will be used repeatedly in the next section.

Lemma For $\sigma > 0$, set $\psi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$ for $x \in \mathbb{R}$. Then its Fourier transform and inverse Fourier transform are

$$\hat{\psi}_\sigma = \check{\psi}_\sigma = \frac{1}{\sigma}\psi_{1/\sigma}.$$

In particular, $\hat{\psi}_1 = \psi_1$.

proof (a) I begin with the special case $\sigma = 1$, using the Maclaurin series

$$e^{-iyx} = \sum_{k=0}^\infty \frac{(-iyx)^k}{k!}$$

and the expressions for $\int_{-\infty}^\infty x^k e^{-x^2/2}dx$ from §263.

Fix $y \in \mathbb{R}$. Writing

$$g_k(x) = \frac{(-iyx)^k}{k!}e^{-x^2/2}, \quad h_n(x) = \sum_{k=0}^n g_k(x), \quad h(x) = e^{|yx|-x^2/2},$$

we see that

$$|g_k(x)| \le \frac{|yx|^k}{k!}e^{-x^2/2},$$

so that

$$|h_n(x)| \le \sum_{k=0}^\infty |g_k(x)| \le e^{|yx|}e^{-x^2/2} = h(x)$$

for every n; moreover, h is integrable, because $|h(x)| \le e^{-|x|}$ whenever $|x| \ge 2(1+|y|)$. Consequently, using Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}\hat{\psi}_1(y) &= \frac{1}{2\pi}\int_{-\infty}^\infty \lim_{n\to\infty} h_n = \frac{1}{2\pi}\lim_{n\to\infty}\int_{-\infty}^\infty h_n \\ &= \frac{1}{2\pi}\sum_{k=0}^\infty \int_{-\infty}^\infty g_k = \frac{1}{2\pi}\sum_{k=0}^\infty \frac{(-iy)^k}{k!}\int_{-\infty}^\infty x^k e^{-x^2/2}dx \\ &= \frac{1}{2\pi}\sum_{j=0}^\infty \frac{(-iy)^{2j}}{(2j)!}\frac{(2j)!}{2^j j!}\sqrt{2\pi}\end{aligned}$$

(by 263H)

$$= \frac{1}{\sqrt{2\pi}}\sum_{j=0}^\infty \frac{(-y^2)^j}{2^j j!} = \frac{1}{\sqrt{2\pi}}e^{-y^2/2} = \psi_1(y),$$

as claimed.

(b) For the general case, $\psi_\sigma(x) = \frac{1}{\sigma}\psi_1(\frac{x}{\sigma})$, so that

$$\hat{\psi}_\sigma(y) = \frac{1}{\sigma}\cdot\sigma\hat{\psi}_1(\sigma y) = \frac{1}{\sigma}\psi_{1/\sigma}(y)$$

by 283Ce. Of course we now have

$$\check{\psi}_\sigma(y) = \hat{\psi}_\sigma(-y) = \frac{1}{\sigma}\psi_{1/\sigma}(y)$$

because $\psi_{1/\sigma}$ is an even function.

283O To lead into the ideas of the next section, I give the following very simple fact.

Proposition Let f and g be two complex-valued functions which are integrable over $\mathbb{R}$. Then $\int_{-\infty}^\infty f \times \hat{g} = \int_{-\infty}^\infty \hat{f} \times g$ and $\int_{-\infty}^\infty f \times \check{g} = \int_{-\infty}^\infty \check{f} \times g$.

proof Of course

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|e^{-ixy}f(x)g(y)|dxdy=\int_{-\infty}^{\infty}|f|\int_{-\infty}^{\infty}|g|<\infty,$$

so

$$\begin{aligned}\int_{\infty}^{\infty} f\times\hat{g}&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-iyx}g(x)dxdy\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-ixy}g(x)dydx=\int_{-\infty}^{\infty}\hat{f}\times g.\end{aligned}$$

For the other half of the proposition, replace every e^{-ixy} in the argument by e^{ixy}.

283W Higher dimensions I offer a series of exercises designed to provide hints on how the work of this section may be done in the r-dimensional case, where $r\ge 1$.

(a) Let f be an integrable complex-valued function defined almost everywhere in $\mathbb{R}^r$. Its **Fourier transform** is the function $\hat{f}:\mathbb{R}^r\to\mathbb{C}$ defined by the formula

$$\hat{f}(y)=\frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,\boldsymbol{.}\,x}f(x)dx,$$

writing $y\,\boldsymbol{.}\,x=\eta_1\xi_1+\ldots+\eta_r\xi_r$ for $x=(\xi_1,\ldots,\xi_r)$ and $y=(\eta_1,\ldots,\eta_r)\in\mathbb{R}^r$, and $\int\ldots dx$ for integration with respect to Lebesgue measure on $\mathbb{R}^r$. Similarly, the **inverse Fourier transform** of f is the function $\check{f}$ given by

$$\check{f}(y)=\frac{1}{(\sqrt{2\pi})^r}\int e^{iy\,\boldsymbol{.}\,x}f(x)dx=\hat{f}(-y).$$

Show that, for any integrable complex-valued function f on $\mathbb{R}^r$,

(i) $\hat{f}:\mathbb{R}^r\to\mathbb{C}$ is continuous;

(ii) $\lim_{\|y\|\to\infty}\hat{f}(y)=0$, writing $\|y\|=\sqrt{y\,\boldsymbol{.}\,y}$ as usual;

(iii) if $\int\|x\||f(x)|dx<\infty$, then $\hat{f}$ is differentiable, and

$$\frac{\partial}{\partial\eta_j}\hat{f}(y)=-\frac{i}{(\sqrt{2\pi})^r}\int e^{-iy\,\boldsymbol{.}\,x}\xi_j f(x)dx$$

for $j\le r$, $y\in\mathbb{R}^r$, always taking ξ_j to be the jth coordinate of $x\in\mathbb{R}^r$;

(iv) if $j\le r$ and $\frac{\partial f}{\partial\xi_j}$ is defined everywhere and is integrable, then $(\frac{\partial f}{\partial\xi_j})\hat{}\,(y)=i\eta_j\hat{f}(y)$ for every $y\in\mathbb{R}^r$. (Use 225L to show that if $e\in\mathbb{R}^r$ is a unit vector, then $\gamma\mapsto f(x+\gamma e)$ is absolutely continuous on every bounded interval for almost every x.)

(b) Show that if $f_1,\ldots,f_r$ are integrable complex-valued functions on $\mathbb{R}$ with Fourier transforms $g_1,\ldots,g_r$, and we write $f(x)=f_1(\xi_1)\ldots f_r(\xi_r)$ for $x=(\xi_1,\ldots,\xi_r)\in\mathbb{R}^r$, then the Fourier transform of f is $y\mapsto g_1(\eta_1)\ldots g_r(\eta_r)$.

(c) Let f be an integrable complex-valued function on $\mathbb{R}^r$, and $\hat{f}$ its Fourier transform. If $c\le d$ in $\mathbb{R}^r$, show that

$$\int_{[c,d]}f=(\frac{i}{\sqrt{2\pi}})^r\lim_{\alpha_1,\ldots,\alpha_r\to\infty}\int_{[-a,a]}\prod_{j=1}^r\frac{e^{i\gamma_j\eta_j}-e^{i\delta_j\eta_j}}{\eta_j}\hat{f}(y)dy,$$

setting $a=(\alpha_1,\ldots)$, $c=(\gamma_1,\ldots)$, $d=(\delta_1,\ldots)$.

(d) Let f be an integrable complex-valued function on $\mathbb{R}^r$, and $\hat{f}$ its Fourier transform. Show that if we write

$$B_\infty(\mathbf{0},a)=\{y:|\eta_j|\le a\text{ for every }j\le r\},$$

then

$$\frac{1}{(\sqrt{2\pi})^r}\int_{B_\infty(\mathbf{0},a)} e^{ix\cdot y}\hat{f}(y)dy = \int \phi_a(t)f(x-t)dt$$

for every $a \ge 0$, where

$$\phi_a(t) = \frac{1}{\pi^r}\prod_{j=1}^r \frac{\sin a\tau_j}{\tau_j}$$

for $t = (\tau_1, \dots, \tau_r) \in \mathbb{R}^r$.

(e) Show that $\int_{\mathbb{R}^r} \frac{1}{1+\|x\|^{r+1}}dx < \infty$.

(f) Let $f : \mathbb{R}^r \to \mathbb{C}$ be an integrable function. Show that if all the partial derivatives $\frac{\partial^k f}{\partial \xi_j^k}$, for $k \le r+1$ and $j \le r$, are defined almost everywhere and integrable, then $\hat{f}$ is integrable.

(g) Show that if f and g are integrable complex-valued functions on $\mathbb{R}^r$, then (defining convolution as in 255L) $(f * g)^\wedge = (\sqrt{2\pi})^r\hat{f} \times \hat{g}$.

(h) Let f and g be integrable complex-valued functions on $\mathbb{R}^r$. Show that $f * \check{g} = (\sqrt{2\pi})^r(\hat{f} \times g)^\vee$.

(i) For $\sigma > 0$, define $\psi_\sigma : \mathbb{R}^r \to \mathbb{C}$ by setting

$$\psi_\sigma(x) = \frac{1}{(\sigma\sqrt{2\pi})^r}e^{-x\,.\,x/2\sigma^2}, \quad (\hat{\psi}_\sigma)^\vee = \psi_\sigma.$$

for every $x \in \mathbb{R}^r$. Show that

$$\hat{\psi}_\sigma = \check{\psi}_\sigma = \frac{1}{\sigma^r}\psi_{1/\sigma}.$$

(j) Defining ψ_σ as in (e), show that $\lim_{\sigma\to 0}(f * \psi_\sigma)(x) = f(x)$ whenever $x \in \mathbb{R}^r$ and $f : \mathbb{R}^r \to \mathbb{C}$ is continuous and either integrable or bounded. (Cf. 261Ye, 262Yi.)

(k) Show that if $f : \mathbb{R}^r \to \mathbb{C}$ is continuous and integrable, and $\hat{f}$ also is integrable, then $f = \check{\hat{f}}$. (*Hint*: Show that both are equal at every point to $\lim_{\sigma\to 0}(\sqrt{2\pi})^r(\hat{f} \times \hat{\psi}_\sigma)^\vee$.)

(l) Show that if f and g are integrable complex-valued functions on $\mathbb{R}^r$, then $\int f \times \hat{g} = \int \hat{f} \times g$.

(m)(i) Show that $\int_{2k\pi}^{2(k+1)\pi} \frac{\sin t}{t\sqrt{t}}dt > 0$ for every $k \in \mathbb{N}$, and hence that $\int_0^\infty \frac{\sin t}{t\sqrt{t}}dt > 0$.

(ii) Set $f_1(\xi) = 1/\sqrt{|\xi|}$ for $0 < |\xi| \le 1$, 0 for other ξ. Show that $\lim_{a\to\infty}\frac{1}{\sqrt{a}}\int_{-a}^a \hat{f}_1(\eta)d\eta$ exists in $\mathbb{R}$ and is greater than 0.

(iii) Construct an integrable function f_2, zero on some neighbourhood of 0, such that there are infinitely many $m \in \mathbb{N}$ for which $|\int_{-m}^m \hat{f}_2(\eta)d\eta| \ge \frac{1}{\sqrt{m}}$. (*Hint*: take $f_2(\xi) = 2^{-k}\sin m_k\xi$ for $k+1 \le \xi < k+2$, for a sufficiently rapidly increasing sequence $\langle m_k\rangle_{k\in\mathbb{N}}$.)

(iv) Set $f(x) = f_1(\xi_1)f_2(\xi_2)$ for $x \in \mathbb{R}^2$. Show that f is integrable, that f is zero in a neighbourhood of $\mathbf{0}$, but that

$$\limsup_{a\to\infty}\frac{1}{2\pi}\left|\int_{B_\infty(\mathbf{0},a)}\hat{f}(y)dy\right| > 0,$$

defining B_∞ as in 283Wd.

283X Basic exercises (a) Confirm that the six alternative definitions of the transforms $\hat{f}$, $\check{f}$ offered in 283B all lead to the same theory; find the constants involved in the new versions of 283Ch, 283Ci, 283L, 283M and 283N.

(b) If we redefined $\hat{f}(y)$ to be $\alpha\int_{-\infty}^{\infty}e^{i\beta xy}f(x)dx$, what would $\check{f}(y)$ be?

(c) Show that nearly every 2π would disappear from the theorems of this section if we defined a measure ν on $\mathbb{R}$ by saying that $\nu E=\frac{1}{\sqrt{2\pi}}\mu E$ for every Lebesgue measurable set E, where μ is Lebesgue measure, and wrote

$$\hat{f}(y)=\int_{-\infty}^{\infty}e^{-iyx}f(x)\nu(dx),\quad \check{f}(y)=\int_{-\infty}^{\infty}e^{iyx}f(x)\nu(dx),$$

$$(f*g)(x)=\int_{-\infty}^{\infty}f(t)g(x-t)\nu(dt).$$

What is $\lim_{a\to\infty}\int_{-a}^{a}\frac{\sin t}{t}\nu(dt)$?

>(d) Let f be an integrable complex-valued function on $\mathbb{R}$, with Fourier transform $\hat{f}$. Show that (i) if $g(x)=f(-x)$ whenever this is defined, then $\hat{g}(y)=\hat{f}(-y)$ for every $y\in\mathbb{R}$; (ii) if $g(x)=\overline{f(x)}$ whenever this is defined, then $\hat{g}(y)=\overline{\hat{f}(-y)}$ for every y.

(e) Let f be an integrable complex-valued function on $\mathbb{R}$, with Fourier transform $\hat{f}$. Show that

$$\int_c^d\hat{f}(y)dy=\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-idx}-e^{-icx}}{x}f(x)dx$$

whenever $c\le d$ in $\mathbb{R}$.

>(f) For an integrable complex-valued function f on $\mathbb{R}$, let its **Fejér integrals** be

$$\sigma_c(x)=\frac{1}{c\sqrt{2\pi}}\int_0^c\Bigl(\int_{-a}^{a}e^{ixy}\hat{f}(y)dy\Bigr)da$$

for $c>0$. Show that

$$\sigma_c(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos ct}{ct^2}f(x-t)dt.$$

(g) Show that $\int_{-\infty}^{\infty}\frac{1-\cos at}{at^2}dt=\pi$ for every $a>0$. (*Hint*: integrate by parts and use 283Da.) Show that

$$\lim_{a\to\infty}\int_\delta^\infty\frac{1-\cos at}{at^2}dt=\lim_{a\to\infty}\sup_{t\ge\delta}\frac{1-\cos at}{at^2}=0$$

for every $\delta>0$.

(h) Let f be an integrable complex-valued function on $\mathbb{R}$, and define its Fejér integrals σ_a as in 283Xf above. Show that if $x\in\mathbb{R}$, $c\in\mathbb{C}$ are such that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^\delta|f(x+t)+f(x-t)-2c|dt=0,$$

then $\lim_{a\to\infty}\sigma_a(x)=c$. (*Hint*: adapt the argument of 282H.)

>(i) Let f be an integrable complex-valued function on $\mathbb{R}$, and define its Fejér integrals σ_a as in 283Xf above. Show that $f(x)=\lim_{a\to\infty}\sigma_a(x)$ for almost every $x\in\mathbb{R}$.

(j) Let $f:\mathbb{R}\to\mathbb{C}$ be a continuous integrable complex-valued function of bounded variation, and define its Fejér integrals σ_a as in 283Xf above. Show that $f(x)=\lim_{a\to\infty}\sigma_a(x)$ uniformly for $x\in\mathbb{R}$.

>(k) Let f be an integrable complex-valued function of bounded variation on $\mathbb{R}$, and $\hat{f}$ its Fourier transform. Show that $\sup_{y\in\mathbb{R}}|y\hat{f}(y)|<\infty$.

(l) Let f and g be integrable complex-valued functions on $\mathbb{R}$. Show that $f*\check{g}=\sqrt{2\pi}(\hat{f}\times g)^\vee$.

(m) Let f be an integrable complex-valued function on $\mathbb{R}$, and fix $x \in \mathbb{R}$. Set

$$\hat{f}_x(y) = \int_{-\infty}^{\infty} f(t) \cos y(x-t) dt$$

for $y \in \mathbb{R}$. Show that

(i) if f is differentiable at x,

$$f(x) = \frac{1}{\pi} \lim_{a\to\infty} \int_0^a \hat{f}_x(y) dy;$$

(ii) if there is a neighbourhood of x in which f has bounded variation, then

$$\frac{1}{\pi} \lim_{a\to\infty} \int_0^a \hat{f}_x(y) dy = \frac{1}{2}(\lim_{t\in \operatorname{dom} f, t\uparrow 0} f(t) + \lim_{t\in \operatorname{dom} f, t\downarrow 0} f(t));$$

(iii) if f is twice differentiable and f', f'' are integrable then $\hat{f}_x$ is integrable and $f(x) = \frac{1}{\pi}\int_0^\infty \hat{f}_x$. (The formula

$$f(x) = \frac{1}{\pi}\int_0^\infty \Bigl(\int_{-\infty}^{\infty} f(t) \cos y(x-t) dt\Bigr) dy,$$

valid for such functions f, is called **Fourier's integral formula**.)

(n) Show that if f is a complex-valued function of bounded variation, defined almost everywhere in $\mathbb{R}$, and converging to 0 (along its domain) at $\pm\infty$, then

$$g(y) = \frac{1}{\sqrt{2\pi}} \lim_{a\to\infty} \int_{-a}^{a} e^{-iyx} f(x) dx$$

is defined in $\mathbb{C}$ for every $y \neq 0$, and that the limit is uniform in any region bounded away from 0.

(o) Let f be an integrable complex-valued function on $\mathbb{R}$. Set

$$\hat{f}_c(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos yx\, f(x) dx, \quad \hat{f}_s(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin yx\, f(x) dx$$

for $y \in \mathbb{R}$. Show that

$$\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ixy} \hat{f}(y) dy = \sqrt{\frac{2}{\pi}}\int_0^a \cos xy\, \hat{f}_c(y) dy + \sqrt{\frac{2}{\pi}}\int_0^a \sin xy\, \hat{f}_s(y) dy$$

for every $x \in \mathbb{R}$ and $a \geq 0$.

(p) Use the fact that $\int_0^a \int_0^\infty e^{-xy} \sin y\, dxdy = \int_0^\infty \int_0^a e^{-xy} \sin y\, dydx$ whenever $a \geq 0$ to show that $\int_0^\infty \frac{1}{1+x^2} dx = \lim_{a\to\infty} \int_0^a \frac{\sin y}{y} dy$.

>(q) Show that if $f(x) = e^{-\sigma|x|}$, where $\sigma > 0$, then $\hat{f}(y) = \frac{2\sigma}{\sqrt{2\pi}(\sigma^2+y^2)}$. Hence, or otherwise, find the Fourier transform of $y \mapsto \frac{1}{1+y^2}$.

(r) Find the inverse Fourier transform of the indicator function of a bounded interval in $\mathbb{R}$. Show that in a formal sense 283F can be regarded as a special case of 283O.

(s) Let f be a non-negative integrable function on $\mathbb{R}$, with Fourier transform $\hat{f}$. Show that

$$\sum_{j=0}^{n} \sum_{k=0}^{n} a_j \bar{a}_k \hat{f}(y_j - y_k) \geq 0$$

whenever $y_0, \dots, y_n$ in $\mathbb{R}$ and $a_0, \dots, a_n \in \mathbb{C}$.

(t) Let f be an integrable complex-valued function on $\mathbb{R}$. Show that $\tilde{f}(x) = \sum_{n=-\infty}^{\infty} f(x+2\pi n)$ is defined in $\mathbb{C}$ for almost every x. (*Hint*: $\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |f(x+2\pi n)| dx < \infty$.) Show that $\tilde{f}$ is periodic. Show that the Fourier coefficients of $\tilde{f}\restriction\,]-\pi,\pi]$ are $\langle \frac{1}{\sqrt{2\pi}}\hat{f}(k)\rangle_{k\in\mathbb{Z}}$.

283Y Further exercises (a) Show that if $f:\mathbb{R}\to\mathbb{C}$ is absolutely continuous in every bounded interval, f' is of bounded variation on $\mathbb{R}$, and $\lim_{x\to\infty}f(x)=\lim_{x\to-\infty}f(x)=0$, then

$$g(y)=\frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a}e^{-iyx}f(x)dx=-\frac{i}{y\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a}e^{-iyx}f'(x)dx$$

is defined, with

$$y^2|g(y)|\le\frac{4}{\sqrt{2\pi}}\operatorname{Var}_{\mathbb{R}}(f'),$$

for every $y\ne0$.

(b) Let $f:\mathbb{R}\to\mathbb{C}$ be an integrable function which is absolutely continuous on every bounded interval, and suppose that its derivative f' is of bounded variation on $\mathbb{R}$. Show that $\hat{f}$ is integrable and that $f=\check{\hat{f}}$. (*Hint*: 225Yd, 283Ci, 283Xk.)

(c) Let $f:\mathbb{R}\to[0,\infty[$ be an even function such that f is convex on $[0,\infty[$ and $\lim_{x\to\infty}f(x)=0$.

(i) Show that, for any $y>0$ and $k\in\mathbb{N}$, $\int_{-2k\pi/y}^{2k\pi/y}e^{-iyx}f(x)dx\ge0$.

(ii) Show that $g(y)=\frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a}e^{-iyx}f(x)dx$ exists in $[0,\infty[$ for every $y\ne0$.

(iii) For $n\in\mathbb{N}$, set $f_n(x)=e^{-|x|/(n+1)}f(x)$ for every x. Show that f_n is integrable and convex on $[0,\infty[$.

(iv) Show that $g(y)=\lim_{n\to\infty}\hat{f}_n(y)$ for every $y\ne0$.

(vi) Show that if f is integrable then

$$\int_{-a}^{a}\hat{f}=\frac{4}{\sqrt{2\pi}}\int_0^\infty\frac{\sin at}{t}f(t)dt\le\frac{4a}{\sqrt{2\pi}}\int_0^{\pi/a}f\le2\sqrt{2\pi}f(0)$$

for every $a\ge0$. Hence show that whether f is integrable or not, g is integrable and $f_n=(\hat{f}_n)^\vee$ for every n.

(vii) Show that $\lim_{a\downarrow0}\sup_{n\in\mathbb{N}}\int_{-a}^{a}\hat{f}_n=0$.

(viii) Show that if f' is bounded (on its domain) then $\{\hat{f}_n:n\in\mathbb{N}\}$ is uniformly integrable (*hint*: use (vii) and 283Ya), so that $\lim_{n\to\infty}\|\hat{f}_n-g\|_1=0$ and $f=\check{g}$.

(ix) Show that if f' is unbounded then for every $\epsilon>0$ we can find h_1, $h_2:\mathbb{R}\to[0,\infty[$, both even, convex and converging to 0 at ∞, such that $f=h_1+h_2$, h_1' is bounded, $\int h_2\le\epsilon$ and $h_2(0)\le\epsilon$. Hence show that in this case also $f=\check{g}$.

(d) Suppose that $f:\mathbb{R}\to\mathbb{R}$ is even, twice differentiable and convergent to 0 at ∞, that f'' is continuous and that $\{x:f''(x)=0\}$ is bounded in $\mathbb{R}$. Show that f is the Fourier transform of an integrable function. (*Hint*: use 283Yc and 283Yb.)

(e) Let $g:\mathbb{R}\to\mathbb{R}$ be an odd function of bounded variation such that $\int_1^\infty\frac{1}{x}g(x)dx=\infty$. Show that g cannot be the Fourier transform of any integrable function f. (*Hint*: show that if $g=\hat{f}$ then

$$-i\int_0^1 f=\frac{2}{\sqrt{2\pi}}\lim_{a\to\infty}\int_0^a\frac{1-\cos x}{x}g(x)dx=\infty.)$$

283 Notes and comments I have tried in this section to give the elementary theory of Fourier transforms of integrable functions on $\mathbb{R}$, with an eye to the extension of the concept which will be attempted in the next section. Following §282, I have given prominence to two theorems (283I and 283L) describing conditions for the inversion of the Fourier transform to return to the original function; we find ourselves looking at improper integrals $\lim_{a\to\infty}\int_{-a}^{a}$, just as earlier we needed to look at symmetric sums $\lim_{n\to\infty}\sum_{k=-n}^{n}$. I do not go quite so far as in §282, and in particular I leave the study of square-integrable functions for the moment, since their Fourier transforms may not be describable by the simple formulae used here.

One of the most fundamental obstacles in the subject is the lack of any effective criteria for determining which functions are the Fourier transforms of integrable functions. (Happily, things are better for square-integrable functions; see 284O-284P.) In 283Yc-283Yd I sketch an argument showing that 'ordinary' non-oscillating *even* functions which converge to 0 at $\pm\infty$ are Fourier transforms of integrable functions. Strikingly, this is not true of *odd* functions; thus $y \mapsto \frac{1}{\ln(e+y^2)}$ is the Fourier transform of an integrable function, but $y \mapsto \frac{\arctan y}{\ln(e+y^2)}$ is not (283Ye).

In 283W I sketch the corresponding theory of Fourier transforms in $\mathbb{R}^r$. There are few surprises. One point to note is that where in the one-dimensional case we ask for a well-behaved second derivative, in the r-dimensional case we may need to differentiate $r+1$ times (283Wf). Another is that we lose the 'localization principle'. In the one-dimensional case, if f is integrable and zero on an interval $]c,d[$, then $\lim_{a\to\infty}\int_{-a}^{a} e^{ixy}\hat{f}(y)dy = 0$ for every $x \in\,]c,d[$; this is immediate from either 283I or 283L. But in higher dimensions the most natural formulation of a corresponding result is false (283Wm).

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284 Fourier transforms II

The basic paradox of Fourier transforms is the fact that while for certain functions (see 283J-283K) we have $(\hat{f})^\vee = f$, 'ordinary' integrable functions f (for instance, the indicator functions of non-trivial intervals) give rise to non-integrable Fourier transforms $\hat{f}$ for which there is no direct definition available for $\hat{f}^\vee$, making it a puzzle to decide in what sense the formula $f = \hat{f}^\vee$ might be true. What now seems by far the most natural resolution of the problem lies in declaring the Fourier transform to be an operation on *distributions* rather than on *functions*. I shall not attempt to describe this theory properly (almost any book on 'Distributions' will cover the ground better than I can possibly do here), but will try to convey the fundamental ideas, so far as they are relevant to the questions dealt with here, in language which will make the transition to a fuller treatment straightforward. At the same time, these methods make it easy to prove strong versions of the 'classical' theorems concerning Fourier transforms.

284A Test functions: Definition Throughout this section, a **rapidly decreasing test function** or **Schwartz function** will be a function $h : \mathbb{R} \to \mathbb{C}$ such that h is **smooth**, that is, differentiable everywhere any finite number of times, and moreover

$$\sup_{x\in\mathbb{R}} |x|^k |h^{(m)}(x)| < \infty$$

for all k, $m \in \mathbb{N}$, writing $h^{(m)}$ for the mth derivative of h.

284B The following elementary facts will be useful.

Lemma (a) If g and h are rapidly decreasing test functions, so are $g+h$ and ch, for any $c \in \mathbb{C}$.

(b) If h is a rapidly decreasing test function and $y \in \mathbb{R}$, then $x \mapsto h(y-x)$ is a rapidly decreasing test function.

(c) If h is any rapidly decreasing test function, then h and h^2 are integrable.

(d) If h is a rapidly decreasing test function, so is its derivative h'.

(e) If h is a rapidly decreasing test function, so is the function $x \mapsto xh(x)$.

(f) For any $\epsilon > 0$, the function $x \mapsto e^{-\epsilon x^2}$ is a rapidly decreasing test function.

proof (a) is trivial.

(b) Write $g(x) = h(y-x)$ for $x \in \mathbb{R}$. Then $g^{(m)}(x) = (-1)^m h^{(m)}(y-x)$ for every m, so g is smooth. For any $k \in \mathbb{N}$,

$$|x|^k \le 2^k(|y|^k + |y-x|^k)$$

for every x, so

$$\begin{aligned}\sup_{x\in\mathbb{R}} |x|^k|g^{(m)}(x)| &= \sup_{x\in\mathbb{R}} |x|^k|h^{(m)}(y-x)| \\ &\le 2^k|y|^k \sup_{x\in\mathbb{R}} |h^{(m)}(y-x)| + 2^k \sup_{x\in\mathbb{R}} |y-x|^k|h^{(m)}(y-x)| \\ &= 2^k|y|^k \sup_{x\in\mathbb{R}} |h^{(m)}(x)| + 2^k \sup_{x\in\mathbb{R}} |x|^k|h^{(m)}(x)| < \infty.\end{aligned}$$

(c) Because

$$M = \sup_{x\in\mathbb{R}} |h(x)| + x^2|h(x)|$$

is finite, we have

$$\int |h| \le \int \frac{M}{1+x^2} dx < \infty.$$

Of course we now have $|h^2| \le M|h|$, so h^2 also is integrable.

(d) This is immediate from the definition, as every derivative of h' is a derivative of h.

(e) Setting $g(x) = xh(x)$, $g^{(m)}(x) = xh^{(m)}(x) + mh^{(m-1)}(x)$ for $m \ge 1$, so

$$\sup_{x\in\mathbb{R}} |x^k g^{(m)}(x)| \le \sup_{x\in\mathbb{R}} |x^{k+1} h^{(m)}(x)| + m \sup_{x\in\mathbb{R}} |x^k h^{(m-1)}(x)|$$

is finite, for all $k \in \mathbb{N}$, $m \ge 1$.

(f) If $h(x) = e^{-\epsilon x^2}$, then for each $m \in \mathbb{N}$ we have $h^{(m)}(x) = p_m(x)h(x)$, where $p_0(x) = 1$ and $p_{m+1}(x) = p'_m(x) - 2\epsilon x p_m(x)$, so that p_m is a polynomial. Because $e^{\epsilon x^2} \ge \epsilon^{k+1}x^{2k+2}/(k+1)!$ for all x, $k \ge 0$,

$$\lim_{|x|\to\infty} |x|^k h(x) = \lim_{x\to\infty} x^k/e^{\epsilon x^2} = 0$$

for every k, and $\lim_{|x|\to\infty} p(x)h(x) = 0$ for every polynomial p; consequently

$$\lim_{|x|\to\infty} x^k h^{(m)}(x) = \lim_{|x|\to\infty} x^k p_m(x)h(x) = 0$$

for all k, m, and h is a rapidly decreasing test function.

284C Proposition Let $h : \mathbb{R} \to \mathbb{C}$ be a rapidly decreasing test function. Then $\hat{h} : \mathbb{R} \to \mathbb{C}$ and $\check{h} : \mathbb{R} \to \mathbb{C}$ are rapidly decreasing test functions, and $\check{\hat{h}} = \hat{\check{h}} = h$.

proof (a) Let k, $m \in \mathbb{N}$. Then $\sup_{x\in\mathbb{R}}(|x|^m + |x|^{m+2})|h^{(k)}(x)| < \infty$ and $\int_{-\infty}^{\infty} |x^m h^{(k)}(x)|dx < \infty$. We may therefore use 283Ch-283Ci to see that $y \mapsto i^{k+m}y^k\hat{h}^{(m)}(y)$ is the Fourier transform of $x \mapsto x^m h^{(k)}(x)$, and therefore that $\lim_{|y|\to\infty} y^k\hat{h}^{(m)}(y) = 0$, by 283Cg, so that (because $\hat{h}^{(m)}$ is continuous) $\sup_{y\in\mathbb{R}} |y^k\hat{h}^{(m)}(y)|$ is finite. As k and m are arbitrary, $\hat{h}$ is a rapidly decreasing test function.

(b) Since $\check{h}(y) = \hat{h}(-y)$ for every y, it follows at once that $\check{h}$ is a rapidly decreasing test function.

(c) By 283J, it follows from (a) and (b) that $\check{\hat{h}} = \hat{\check{h}} = h$.

284D Definition I will use the phrase **tempered function** on $\mathbb{R}$ to mean a measurable complex-valued function f, defined almost everywhere in $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} \frac{1}{1+|x|^k}|f(x)|dx < \infty$$

for some $k \in \mathbb{N}$.

284E As in 284B I spell out some elementary facts.

Lemma (a) If f and g are tempered functions, so are $|f|$, $f+g$ and cf, for any $c \in \mathbb{C}$.

(b) If f is a tempered function then it is integrable over any bounded interval.

(c) If f is a tempered function and $x \in \mathbb{R}$, then $t \mapsto f(x+t)$ and $t \mapsto f(x-t)$ are both tempered functions.

proof (a) is elementary; if

$$\int_{-\infty}^{\infty} \frac{1}{1+|x|^j} f(x)dx < \infty, \quad \int_{-\infty}^{\infty} \frac{1}{1+|x|^k} g(x)dx < \infty,$$

then

$$\int_{-\infty}^{\infty} \frac{1}{1+|x|^{j+k}}|(f+g)(x)|dx < \infty$$

because

$$1+|x|^{j+k} \ge \max(1,|x|^{j+k}) \ge \max(1,|x|^j,|x|^k) \ge \frac{1}{2}\max(1+|x|^j,1+|x|^k)$$

for all x.

(b) If

$$\int_{-\infty}^{\infty} \frac{1}{1+|x|^k}|f(x)|dx = M < \infty,$$

then for any $a \le b$

$$\int_a^b |f| \le M(1+|a|^k+|b|^k)(b-a) < \infty.$$

(c) The idea is the same as in 284Bb. If $k \in \mathbb{N}$ is such that

$$\int_{-\infty}^{\infty} \frac{1}{1+|t|^k}|f(t)|dt = M < \infty,$$

then we have

$$1+|x+t|^k \le 2^k(1+|x|^k)(1+|t|^k)$$

so that

$$\frac{1}{1+|t|^k} \le 2^k(1+|x|^k)\frac{1}{1+|x+t|^k}$$

for every t, and

$$\int_{-\infty}^{\infty} \frac{|f(x+t)|}{1+|t|^k}dt \le 2^k(1+|x|^k)\int_{-\infty}^{\infty} \frac{|f(x+t)|}{1+|x+t|^k}dt \le 2^k(1+|x|^k)M < \infty.$$

Similarly,

$$\int_{-\infty}^{\infty} \frac{|f(x-t)|}{1+|t|^k}dt \le 2^k(1+|x|^k)M < \infty.$$

284F Linking the two concepts, we have the following.

Lemma Let f be a tempered function on $\mathbb{R}$ and h a rapidly decreasing test function. Then $f \times h$ is integrable.

proof Of course $f \times h$ is measurable. Let $k \in \mathbb{N}$ be such that $\int_{-\infty}^{\infty} \frac{1}{1+|x|^k}|f(x)|dx < \infty$. There is an M such that $(1+|x|^k)|h(x)| \le M$ for every $x \in \mathbb{R}$, so that

$$\int_{-\infty}^{\infty} |f \times h| \le M\int_{-\infty}^{\infty} \frac{1}{1+|x|^k}|f(x)|dx < \infty.$$

284G Lemma Suppose that f_1 and f_2 are tempered functions and that $\int f_1 \times h = \int f_2 \times h$ for every rapidly decreasing test function h. Then $f_1 =_{\text{a.e.}} f_2$.

proof (a) Set $g = f_1 - f_2$; then $\int g \times h = 0$ for every rapidly decreasing test function h. Of course g is a tempered function, so is integrable over any bounded interval. By 222D, it will be enough if I can show that $\int_a^b g = 0$ whenever $a < b$, since then we shall have $g = 0$ a.e. on every bounded interval and $f_1 =_{\text{a.e.}} f_2$.

(b) Consider the function $\tilde\phi(x) = e^{-1/x}$ for $x > 0$. Then $\tilde\phi$ is differentiable arbitrarily often everywhere in $]0,\infty[$, $0 < \tilde\phi(x) < 1$ for every $x > 0$, and $\lim_{x\to\infty} \tilde\phi(x) = 1$. Moreover, writing $\tilde\phi^{(m)}$ for the mth derivative of $\tilde\phi$,

$$\lim_{x\downarrow 0} \tilde\phi^{(m)}(x) = \lim_{x\downarrow 0} \frac{1}{x}\tilde\phi^{(m)}(x) = 0$$

for every $m \in \mathbb{N}$. **P** (Compare 284Bf.) We have $\tilde\phi^{(m)}(x) = p_m(\frac{1}{x})\tilde\phi(x)$, where $p_0(t) = 1$ and $p_{m+1}(t) = t^2(p_m(t) - p_m'(t))$, so that p_m is a polynomial for each $m \in \mathbb{N}$. Now for any $k \in \mathbb{N}$,

$$0 \le \limsup_{t\to\infty} t^k e^{-t} \le \lim_{t\to\infty} \frac{(k+1)!t^k}{t^{k+1}} = 0,$$

so

$$\lim_{x\downarrow 0} \tilde\phi^{(m)}(x) = \lim_{t\to\infty} p_m(t)e^{-t} = 0,$$

$$\lim_{x\downarrow 0} \frac{1}{x}\tilde\phi^{(m)}(x) = \lim_{t\to\infty} tp_m(t)e^{-t} = 0. \text{ } \mathbf{Q}$$

(c) Consequently, setting $\phi(x) = 0$ for $x \le 0$, $e^{-1/x}$ for $x > 0$, ϕ is smooth, with mth derivative

$$\phi^{(m)}(x) = 0 \text{ for } x \le 0, \quad \phi^{(m)}(x) = \tilde\phi^{(m)}(x) \text{ for } x > 0.$$

(The proof is an easy induction on m.) Also $0 \le \phi(x) \le 1$ for every $x \in \mathbb{R}$, and $\lim_{x\to\infty} \phi(x) = 1$.

(d) Now take any $a < b$, and for $n \in \mathbb{N}$ set

$$\phi_n(x) = \phi(n(x-a))\phi(n(b-x)).$$

Then ϕ_n will be smooth and $\phi_n(x) = 0$ if $x \notin \,]a,b[$, so surely ϕ_n is a rapidly decreasing test function, and

$$\int_{-\infty}^{\infty} g \times \phi_n = 0.$$

Next, $0 \le \phi_n(x) \le 1$ for every x, n, and if $a < x < b$ then $\lim_{n\to\infty} \phi_n(x) = 1$. So

$$\int_a^b g = \int g \times \chi(]a,b[) = \int g \times (\lim_{n\to\infty} \phi_n) = \lim_{n\to\infty} \int g \times \phi_n = 0,$$

using Lebesgue's Dominated Convergence Theorem. As a and b are arbitrary, $g = 0$ a.e., as required.

284H Definition Let f and g be tempered functions in the sense of 284D. Then I will say that g **represents the Fourier transform of** f if

$$\int_{-\infty}^{\infty} g \times h = \int_{-\infty}^{\infty} f \times \hat h$$

for every rapidly decreasing test function h.

284I Remarks (a) As usual, when shifting definitions in this way, we have some checking to do. If f is an integrable complex-valued function on $\mathbb{R}$ and $\hat f$ is its Fourier transform, then surely $\hat f$ is a tempered function, being a bounded continuous function; and if h is any rapidly decreasing test function, then $\int \hat f \times h = \int f \times \hat h$ by 283O. Thus $\hat f$ 'represents the Fourier transform of f' in the sense of 284H above.

(b) Note also that 284G assures us that if g_1, g_2 are two tempered functions both representing the Fourier transform of f, then $g_1 =_{\text{a.e.}} g_2$, since we must have

$$\int g_1 \times h = \int f \times \hat h = \int g_2 \times h$$

for every rapidly decreasing test function h.

(c) It is I suppose obvious that if f_1, f_2, g_1 and g_2 are tempered functions and g_i represents the Fourier transform of f_i for both i, then $cg_1 + g_2$ represents the Fourier transform of $cf_1 + f_2$ for every $c \in \mathbb{C}$.

(d) Of course the value of this indirect approach is that we can assign Fourier transforms, in a sense, to many more functions. But we must note at once that if g 'represents the Fourier transform of f' then so will any function equal almost everywhere to g; we can no longer expect to be able to speak of 'the' Fourier transform of f as a function. We could say that 'the' Fourier transform of f is a functional ϕ on the space of rapidly decreasing test functions, defined by setting $\phi(h) = \int f \times \hat h$; alternatively, we could say that 'the' Fourier transform of f is a member of $L^0_{\mathbb{C}}$, the space of equivalence classes of almost-everywhere-defined measurable functions (241J).

(e) It is now natural to say that g **represents the inverse Fourier transform of** f just when f represents the Fourier transform of g; that is, when $\int f \times h = \int g \times \hat h$ for every rapidly decreasing test function

h. Because $\check{\hat{h}} = \hat{\check{h}} = h$ for every such h (284C), this is the same thing as saying that $\int g \times h = \int f \times \check{h}$ for every rapidly decreasing test function h, which is the other natural expression of what it might mean to say that 'g represents the inverse Fourier transform of f'.

(f) If f, g are tempered functions and we write $\overleftrightarrow{g}(x) = g(-x)$ whenever this is defined, then $\overleftrightarrow{g}$ will also be a tempered function, and we shall always have

$$\int \overleftrightarrow{g} \times \hat{h} = \int g(-x)\hat{h}(x)dx = \int g(x)\hat{h}(-x)dx = \int g \times \check{h},$$

so that

g represents the Fourier transform of f

$\iff \int g \times h = \int f \times \hat{h}$ for every test function h

$\iff \int g \times \check{h} = \int f \times \hat{\check{h}}$ for every h

$\iff \int \overleftrightarrow{g} \times \hat{h} = \int f \times h$ for every h

$\iff \overleftrightarrow{g}$ represents the inverse Fourier transform of f.

Combining this with (d), we get

g represents the Fourier transform of f

$\iff \overleftrightarrow{\overleftrightarrow{f}} = f$ represents the inverse Fourier transform of g

$\iff \overleftrightarrow{f}$ represents the Fourier transform of g.

(g) Yet again, we ought to be conscious that a check is called for: if f is integrable and $\check{f}$ is its inverse Fourier transform as defined in 283Ab, then

$$\int \check{f} \times \hat{h} = \int f \times \check{\hat{h}} = \int f \times h$$

for every rapidly decreasing test function h, so $\check{f}$ 'represents the inverse Fourier transform of f' in the sense given here.

284J Lemma Let f be any tempered function and h a rapidly decreasing test function. Then $f * h$, defined by the formula

$$(f * h)(y) = \int_{-\infty}^{\infty} f(t)h(y-t)dt,$$

is defined everywhere.

proof Take any $y \in \mathbb{R}$. By 284Bb, $t \mapsto h(y-t)$ is a rapidly decreasing test function, so the integral is always defined in $\mathbb{C}$, by 284F.

284K Proposition Let f and g be tempered functions such that g represents the Fourier transform of f, and h a rapidly decreasing test function.

(a) The Fourier transform of the integrable function $f \times h$ is $\frac{1}{\sqrt{2\pi}} g * \hat{h}$.

(b) The Fourier transform of the continuous function $f * h$ is represented by the product $\sqrt{2\pi} g \times \hat{h}$.

proof (a) Of course $f \times h$ is integrable, by 284F, while $g * \hat{h}$ is defined everywhere, by 284C and 284J.

Fix $y \in \mathbb{R}$. Set $h_1(x) = \hat{h}(y-x)$ for $x \in \mathbb{R}$; then h_1 is a rapidly decreasing test function because $\hat{h}$ is (284Bb). Now

$$\begin{aligned}\hat{h}_1(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itx}\hat{h}(y-x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it(y-x)}\hat{h}(x)dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-ity}\int_{-\infty}^{\infty} e^{itx}\hat{h}(x)dx = e^{-ity}\check{\hat{h}}(t) = e^{-ity}h(t),\end{aligned}$$

using 284C. Accordingly

$$\begin{aligned}(f\times h)^\wedge(y)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ity}f(t)h(t)dt\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\hat{h}_1(t)dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)h_1(t)dt\end{aligned}$$

(because g represents the Fourier transform of f)

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)\hat{h}(y-t)dt=\frac{1}{\sqrt{2\pi}}(g*\hat{h})(y).$$

As y is arbitrary, $\frac{1}{\sqrt{2\pi}}g*\hat{h}$ is the Fourier transform of $f\times h$.

(b) Write f_1 for the Fourier transform of $g\times\hat{h}$, $\overset{\leftrightarrow}{f}(x)=f(-x)$ when this is defined, and $\overset{\leftrightarrow}{h}(x)=h(-x)$ for every x, so that $\overset{\leftrightarrow}{f}$ represents the Fourier transform of g, by 284If, and $\overset{\leftrightarrow}{h}$ is the Fourier transform of $\hat{h}$. By (a), we have $f_1=\frac{1}{\sqrt{2\pi}}\overset{\leftrightarrow}{f}*\overset{\leftrightarrow}{h}$. This means that the inverse Fourier transform of $\sqrt{2\pi}g\times\hat{h}$ must be $\sqrt{2\pi}\overset{\leftrightarrow}{f}_1=(\overset{\leftrightarrow}{f}*\overset{\leftrightarrow}{h})^{\leftrightarrow}$; and as

$$\begin{aligned}(\overset{\leftrightarrow}{f}*\overset{\leftrightarrow}{h})^{\leftrightarrow}(y)=(\overset{\leftrightarrow}{f}*\overset{\leftrightarrow}{h})(-y)&=\int_{-\infty}^{\infty}\overset{\leftrightarrow}{f}(t)\overset{\leftrightarrow}{h}(-y-t)dt\\&=\int_{-\infty}^{\infty}f(-t)h(y+t)dt=\int_{-\infty}^{\infty}f(t)h(y-t)dt=(f*h)(y),\end{aligned}$$

the inverse Fourier transform of $\sqrt{2\pi}g\times\hat{h}$ is $f*h$ (which is therefore continuous), and $\sqrt{2\pi}g\times\hat{h}$ must represent the Fourier transform of $f*h$.

Remark Compare 283M. It is typical of the theory of Fourier transforms that we have formulae valid in a wide variety of contexts, each requiring a different interpretation and a different proof.

284L We are now ready for a result corresponding to 282H. I use a different method, or at least a different arrangement of the ideas, through the following fact, which is important in other ways.

Proposition Let f be a tempered function. Writing $\psi_\sigma(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$ for $x\in\mathbb{R}$ and $\sigma>0$, then

$$\lim_{\sigma\downarrow 0}(f*\psi_\sigma)(x)=c$$

whenever $x\in\mathbb{R}$ and $c\in\mathbb{C}$ are such that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^\delta|f(x+t)+f(x-t)-2c|dt=0.$$

proof (a) By 284Bf, every ψ_σ is a rapidly decreasing test function, so that $f*\psi_\sigma$ is defined everywhere, by 284J. We need to know that $\int_{-\infty}^{\infty}\psi_\sigma=1$; this is because (substituting $u=x/\sigma$)

$$\int_{-\infty}^{\infty}\psi_\sigma=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-u^2/2}du=1,$$

by 263G. The argument now follows the lines of 282H. Set

$$\phi(t)=|f(x+t)+f(x-t)-2c|$$

when this is defined, which is almost everywhere, and $\Phi(t)=\int_0^t\phi$, defined for all $t\ge 0$ because f is integrable over every bounded interval (284Eb). We have

$$|(f*\psi_\sigma)(x)-c| = |\int_{-\infty}^{\infty} f(x-t)\psi_\sigma(t)dt - c\int_{-\infty}^{\infty}\psi_\sigma(t)dt|$$
$$= |\int_{-\infty}^{0}(f(x-t)-c)\psi_\sigma(t)dt + \int_0^\infty (f(x-t)-c)\psi_\sigma(t)dt|$$
$$= |\int_0^\infty (f(x+t)-c)\psi_\sigma(t)dt + \int_0^\infty (f(x-t)-c)\psi_\sigma(t)dt|$$

(because ψ_σ is an even function)

$$= |\int_0^\infty (f(x+t)+f(x-t)-2c)\psi_\sigma(t)dt|$$
$$\le \int_0^\infty |f(x+t)+f(x-t)-2c|\psi_\sigma(t)dt = \int_0^\infty \phi\times\psi_\sigma.$$

(b) I should explain why this last integral is finite. Because f is a tempered function, so are the functions $t\mapsto f(x+t)$, $t\mapsto f(x-t)$ (284Ec); of course constant functions are tempered, so $t\mapsto\phi(t) = |f(x+t)+f(x-t)-2c|$ is tempered, and because ψ_σ is a rapidly decreasing test function we may apply 284F to see that the product is integrable.

(c) Let $\epsilon>0$. By hypothesis, $\lim_{t\downarrow 0}\Phi(t)/t=0$; let $\delta>0$ be such that $\Phi(t)\le\epsilon t$ for every $t\in[0,\delta]$. Take any $\sigma\in\,]0,\delta]$. I break the integral $\int_0^\infty\phi\times\psi_\sigma$ up into three parts.

(i) For the integral from 0 to σ, we have

$$\int_0^\sigma \phi\times\psi_\sigma \le \int_0^\sigma \frac{1}{\sigma\sqrt{2\pi}}\phi = \frac{1}{\sigma\sqrt{2\pi}}\Phi(\sigma) \le \frac{\epsilon\sigma}{\sigma\sqrt{2\pi}} \le \epsilon,$$

because $\psi_\sigma(t)\le\frac{1}{\sigma\sqrt{2\pi}}$ for every t.

(ii) For the integral from σ to δ, we have

$$\int_\sigma^\delta \phi\times\psi_\sigma \le \frac{1}{\sigma\sqrt{2\pi}}\int_\sigma^\delta \phi(t)\frac{2\sigma^2}{t^2}dt$$

(because $e^{-t^2/2\sigma^2} = 1/e^{t^2/2\sigma^2} \le 1/(t^2/2\sigma^2) = 2\sigma^2/t^2$ for every $t\ne 0$)

$$= \sigma\sqrt{\frac{2}{\pi}}\int_\sigma^\delta \frac{\phi(t)}{t^2}dt = \sigma\sqrt{\frac{2}{\pi}}\Bigl(\frac{\Phi(\delta)}{\delta^2} - \frac{\Phi(\sigma)}{\sigma^2} + \int_\sigma^\delta \frac{2\Phi(t)}{t^3}dt\Bigr)$$

(integrating by parts – see 225F)

$$\le \sigma\bigl(\frac{\epsilon}{\delta} + \int_\sigma^\delta \frac{2\epsilon}{t^2}dt\bigr)$$

(because $\Phi(t)\le\epsilon t$ for $0\le t\le\delta$ and $\sqrt{2/\pi}\le 1$)

$$\le \sigma\bigl(\frac{\epsilon}{\delta}+\frac{2\epsilon}{\sigma}\bigr) \le 3\epsilon.$$

(iii) For the integral from δ to ∞, we have

$$\int_\delta^\infty \phi\times\psi_\sigma = \frac{1}{\sqrt{2\pi}}\int_\delta^\infty \phi(t)\frac{e^{-t^2/2\sigma^2}}{\sigma}dt.$$

Now for any $t\ge\delta$,

$$\sigma\mapsto\frac{1}{\sigma}e^{-t^2/2\sigma^2} : \,]0,\delta]\to\mathbb{R}$$

is monotonically increasing, because its derivative

$$\frac{d}{d\sigma}\frac{1}{\sigma}e^{-t^2/2\sigma^2} = \frac{1}{\sigma^2}\big(\frac{t^2}{\sigma^2}-1\big)e^{-t^2/2\sigma^2}$$

is positive, and

$$\lim_{\sigma\downarrow 0}\frac{1}{\sigma}e^{-t^2/2\sigma^2} = \lim_{a\to\infty} ae^{-a^2t^2/2} = 0.$$

So we may apply Lebesgue's Dominated Convergence Theorem to see that

$$\lim_{n\to\infty}\int_\delta^\infty \phi(t)\frac{e^{-t^2/2\sigma_n^2}}{\sigma_n}dt = 0$$

whenever $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a sequence in $]0,\delta]$ converging to 0, so that

$$\lim_{\sigma\downarrow 0}\int_\delta^\infty \phi(t)\frac{e^{-t^2/2\sigma^2}}{\sigma}dt = 0.$$

There must therefore be a $\sigma_0\in\,]0,\delta]$ such that

$$\int_\delta^\infty \phi\times\psi_\sigma \le \epsilon$$

for every $\sigma\le\sigma_0$.

(iv) Putting these together, we see that

$$|(f*\psi_\sigma)(x)-c| \le \int_0^\infty \phi\times\psi_\sigma \le \epsilon+3\epsilon+\epsilon = 5\epsilon$$

whenever $0<\sigma\le\sigma_0$. As ϵ is arbitrary, $\lim_{\sigma\downarrow 0}(f*\psi_\sigma)(x) = c$, as claimed.

284M Theorem Let f and g be tempered functions such that g represents the Fourier transform of f. Then

(a)(i) $g(y) = \lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-iyx}e^{-\epsilon x^2}f(x)dx$ for almost every $y\in\mathbb{R}$.

(ii) If $y\in\mathbb{R}$ is such that $a = \lim_{t\in\operatorname{dom}g,t\uparrow y} g(t)$ and $b = \lim_{t\in\operatorname{dom}g,t\downarrow y} g(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-iyx}e^{-\epsilon x^2}f(x)dx = \frac{1}{2}(a+b).$$

(b)(i) $f(x) = \lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ixy}e^{-\epsilon y^2}g(y)dy$ for almost every $x\in\mathbb{R}$.

(ii) If $x\in\mathbb{R}$ is such that $a = \lim_{t\in\operatorname{dom}f,t\uparrow x} f(t)$ and $b = \lim_{t\in\operatorname{dom}f,t\downarrow x} f(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ixy}e^{-\epsilon y^2}g(y)dy = \frac{1}{2}(a+b).$$

proof (a)(i) By 223D,

$$\lim_{\delta\downarrow 0}\frac{1}{2\delta}\int_{-\delta}^\delta |g(y+t)-g(y)|dt = 0$$

for almost every $y\in\mathbb{R}$, because g is integrable over any bounded interval. Fix any such y. Set $\phi(t) = |g(y+t)+g(y-t)-2g(y)|$ whenever this is defined. Then, as in the proof of 282Ia,

$$\int_0^\delta \phi \le \int_{-\delta}^\delta |g(y+t)-g(y)|dt,$$

so $\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^\delta \phi = 0$. Consequently, by 284L,

$$g(y) = \lim_{\sigma\to\infty}(g*\psi_{1/\sigma})(y).$$

We know from 283N that the Fourier transform of ψ_σ is $\frac{1}{\sigma}\psi_{1/\sigma}$ for any $\sigma>0$. Accordingly, by 284K, $g*\psi_{1/\sigma}$ is the Fourier transform of $\sigma\sqrt{2\pi}f\times\psi_\sigma$, that is,

$$(g*\psi_{1/\sigma})(y) = \int_{-\infty}^\infty e^{-iyx}\sigma\psi_\sigma(x)f(x)dx.$$

So

$$\begin{aligned} g(y) &= \lim_{\sigma\to\infty}\int_{-\infty}^{\infty} e^{-iyx}\sigma\psi_\sigma(x)f(x)dx \\ &= \lim_{\sigma\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}e^{-x^2/2\sigma^2}f(x)dx \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}e^{-\epsilon x^2}f(x)dx. \end{aligned}$$

And this is true for almost every y.

(ii) Again, setting $c=\frac{1}{2}(a+b)$, $\phi(t)=|g(y+t)+g(y-t)-2c|$ whenever this is defined, we have $\lim_{t\in\operatorname{dom}\phi,t\downarrow 0}\phi(t)=0$, so of course $\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^\delta\phi=0$, and

$$c=\lim_{\sigma\to\infty}(g*\psi_{1/\sigma})(y)=\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyx}e^{-\epsilon x^2}f(x)dx$$

as before.

(b) This can be shown by similar arguments; or it may be actually deduced from (a), by observing that $x\mapsto\overset{\leftrightarrow}{f}(x)=f(-x)$ represents the Fourier transform of g (see 284Ie), and applying (a) to g and $\overset{\leftrightarrow}{f}$.

284N L^2 spaces We are now ready for results corresponding to 282J-282K.

Lemma Let $\mathcal{L}^2_{\mathbb{C}}$ be the space of square-integrable complex-valued functions on $\mathbb{R}$, and $\mathcal{S}$ the space of rapidly decreasing test functions. Then for every $f\in\mathcal{L}^2_{\mathbb{C}}$ and $\epsilon>0$ there is an $h\in\mathcal{S}$ such that $\|f-h\|_2\le\epsilon$.

proof Set $\phi(x)=e^{-1/x}$ for $x>0$, zero for $x\le 0$; recall from the proof of 284G that ϕ is smooth. For any $a<b$, the functions

$$x\mapsto\phi_n(x)=\phi(n(x-a))\phi(n(b-x))$$

provide a sequence of test functions converging to $\chi\,]a,b[$ from below, so (as in 284G)

$$\inf_{h\in\mathcal{S}}\|\chi\,]a,b[-h\|_2^2\le\lim_{n\to\infty}\int_a^b|1-\phi_n|^2=0.$$

Because $\mathcal{S}$ is a linear space (284Ba), it follows that for every step-function g with bounded support and every $\epsilon>0$ there is an $h\in\mathcal{S}$ such that $\|g-h\|_2\le\frac{1}{2}\epsilon$. But we know from 244H/244Pb that for every $f\in\mathcal{L}^2_{\mathbb{C}}$ and $\epsilon>0$ there is a step-function g with bounded support such that $\|f-g\|_2\le\frac{1}{2}\epsilon$; so there must be an $h\in\mathcal{S}$ such that

$$\|f-h\|_2\le\|f-g\|_2+\|g-h\|_2\le\epsilon.$$

As f and ϵ are arbitrary, we have the result.

284O Theorem (a) Let f be any complex-valued function which is square-integrable over $\mathbb{R}$. Then f is a tempered function and its Fourier transform is represented by another square-integrable function g, and $\|g\|_2=\|f\|_2$.

(b) If f_1 and f_2 are complex-valued functions, square-integrable over $\mathbb{R}$, with Fourier transforms represented by functions g_1, g_2, then

$$\int_{-\infty}^{\infty}f_1\times\bar{f}_2=\int_{-\infty}^{\infty}g_1\times\bar{g}_2.$$

(c) If f_1 and f_2 are complex-valued functions, square-integrable over $\mathbb{R}$, with Fourier transforms represented by functions g_1, g_2, then the integrable function $f_1\times f_2$ has Fourier transform $\frac{1}{\sqrt{2\pi}}g_1*g_2$.

(d) If f_1 and f_2 are complex-valued functions, square-integrable over $\mathbb{R}$, with Fourier transforms represented by functions g_1, g_2, then $\sqrt{2\pi}g_1\times g_2$ represents the Fourier transform of the continuous function f_1*f_2.

proof (a)(i) Consider first the case in which f is a rapidly decreasing test function and g is its Fourier transform; we know that g is also a rapidly decreasing test function, and that f is the inverse Fourier transform of g (284C). Now the complex conjugate $\overline{g}$ of g is given by the formula

$$\overline{g}(y) = \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx} f(x)dx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iyx}\overline{f}(x)dx,$$

so that $\overline{g}$ is the inverse Fourier transform of $\overline{f}$. Accordingly

$$\int f \times \overline{f} = \int \overset{\vee}{g} \times \overline{f} = \int g \times \overset{\vee}{\overline{f}} = \int g \times \overline{g},$$

using 283O for the middle equality.

(ii) Now suppose that $f \in \mathcal{L}^2_{\mathbb{C}}$. I said that f is a tempered function; this is simply because

$$\int_{-\infty}^{\infty} \Bigl(\frac{1}{1+|x|}\Bigr)^2 dx < \infty,$$

so

$$\int_{-\infty}^{\infty} \frac{|f(x)|}{1+|x|} dx < \infty$$

(244Eb). By 284N, there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of rapidly decreasing test functions such that $\lim_{n\to\infty} \|f - f_n\|_2 = 0$. By (i),

$$\lim_{m,n\to\infty} \|\hat{f}_m - \hat{f}_n\|_2 = \lim_{m,n\to\infty} \|f_m - f_n\|_2 = 0,$$

and the sequence $\langle \hat{f}^\bullet_n\rangle_{n\in\mathbb{N}}$ of equivalence classes is a Cauchy sequence in $L^2_{\mathbb{C}}$. Because $L^2_{\mathbb{C}}$ is complete (244G/244Pb), $\langle \hat{f}^\bullet_n\rangle_{n\in\mathbb{N}}$ has a limit in $L^2_{\mathbb{C}}$, which is representable as $g^\bullet$ for some $g \in \mathcal{L}^2_{\mathbb{C}}$. Like f, g must be a tempered function. Of course

$$\|g\|_2 = \lim_{n\to\infty} \|\hat{f}_n\|_2 = \lim_{n\to\infty} \|f_n\|_2 = \|f\|_2.$$

Now if h is any rapidly decreasing test function, h and $\hat{h}$ are square-integrable (284Bc, 284C), so we shall have

$$\int g \times h = \lim_{n\to\infty} \int \hat{f}_n \times h = \lim_{n\to\infty} \int f_n \times \hat{h} = \int f \times \hat{h}.$$

So g represents the Fourier transform of f.

(b) By 284Ib, any functions representing the Fourier transforms of f_1 and f_2 must be equal almost everywhere to square-integrable functions, and therefore square-integrable, with the right norms. It follows as in 282K (part (d) of the proof) that if g_1, g_2 represent the Fourier transforms of f_1, f_2, so that $ag_1 + bg_2$ represents the Fourier transform of $af_1 + bf_2$ and $\|ag_1 + bg_2\|_2 = \|af_1 + bf_2\|_2$ for all a, $b \in \mathbb{C}$, we must have

$$\int f_1 \times \overline{f}_2 = (f_1|f_2) = (g_1|g_2) = \int g_1 \times \overline{g}_2.$$

(c) Of course $f_1 \times f_2$ is integrable because it is the product of two square-integrable functions (244E/244Pb).

(i) Let $y \in \mathbb{R}$ and set $f(x) = \overline{f_2(x)}e^{iyx}$ for $x \in \mathbb{R}$. Then $f \in \mathcal{L}^2_{\mathbb{C}}$. We need to know that the Fourier transform of f is represented by g, where $g(u) = \overline{g_2(y-u)}$. **P** Let h be a rapidly decreasing test function. Then

$$\begin{aligned}\int g \times h &= \int \overline{g_2(y-u)}h(u)du = \int \overline{g_2(u)}h(y-u)du \\ &= \overline{\int g_2 \times h_1} = \overline{\int f_2 \times \hat{h}_1},\end{aligned}$$

where $h_1(u) = \overline{h(y-u)}$. To compute $\hat{h}_1$, we have

$$\begin{aligned}\hat{h}_1(v) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ivu}h_1(u)du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ivu}\overline{h(y-u)}du \\ &= \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ivu}h(y-u)du} = \overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iv(y-u)}h(u)du} = \overline{e^{ivy}\hat{h}(v)}.\end{aligned}$$

So

$$\int g \times h = \overline{\int f_2 \times \hat{h}_1} = \int \overline{f_2(v)\hat{h}_1(v)dv} = \int \overline{f_2(v)}e^{ivy}\hat{h}(v)dv = \int f \times \hat{h};$$

as h is arbitrary, g represents the Fourier transform of f. **Q**

(ii) We now have

$$(f_1 \times f_2)^\wedge(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} f_1(x)f_2(x)dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1 \times \bar{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_1 \times \bar{g}$$

(using part (b))

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_1(u)g_2(y-u)du = \frac{1}{\sqrt{2\pi}}(g_1 * g_2)(y).$$

As y is arbitrary, $(f_1 \times f_2)^\wedge = \frac{1}{\sqrt{2\pi}} g_1 * g_2$, as claimed.

(d) By (c), the Fourier transform of $\sqrt{2\pi} g_1 \times g_2$ is $\overset{\leftrightarrow}{f}_1 * \overset{\leftrightarrow}{f}_2$, writing $\overset{\leftrightarrow}{f}_1(x) = f_1(-x)$, so that $\overset{\leftrightarrow}{f}_1$ represents the Fourier transform of g_1. So the inverse Fourier transform of $\sqrt{2\pi} g_1 \times g_2$ is $(\overset{\leftrightarrow}{f}_1 * \overset{\leftrightarrow}{f}_2)^\leftrightarrow$. But, just as in the proof of 284Kb, $(\overset{\leftrightarrow}{f}_1 * \overset{\leftrightarrow}{f}_2)^\leftrightarrow = f_1 * f_2$, so $f_1 * f_2$ is the inverse Fourier transform of $\sqrt{2\pi} g_1 \times g_2$, and $\sqrt{2\pi} g_1 \times g_2$ represents the Fourier transform of $f_1 * f_2$, as claimed. Also $f_1 * f_2$, being the Fourier transform of an integrable function, is continuous (283Cf; see also 255K).

284P Corollary Writing $L^2_{\mathbb{C}}$ for the Hilbert space of equivalence classes of square-integrable complex-valued functions on $\mathbb{R}$, we have a linear isometry $T : L^2_{\mathbb{C}} \to L^2_{\mathbb{C}}$ given by saying that $T(f^\bullet) = g^\bullet$ whenever f, $g \in \mathcal{L}^2_{\mathbb{C}}$ and g represents the Fourier transform of f.

284Q Remarks (a) 284P corresponds, of course, to 282K, where the similar isometry between $\ell^2_{\mathbb{C}}(\mathbb{Z})$ and $L^2_{\mathbb{C}}(]-\pi,\pi])$ is described. In that case there was a marked asymmetry which is absent from the present situation; because the relevant measure on $\mathbb{Z}$, counting measure, gives non-zero mass to every point, members of $\ell^2_{\mathbb{C}}$ are true functions, and it is not surprising that we have a straightforward formula for $S(f^\bullet) \in \ell^2_{\mathbb{C}}$ for every $f \in \mathcal{L}^2_{\mathbb{C}}(]-\pi,\pi])$. The difficulty of describing $S^{-1} : \ell^2_{\mathbb{C}}(\mathbb{Z}) \to L^2_{\mathbb{C}}(]-\pi,\pi])$ is very similar to the difficulty of describing $T : L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R})$ and its inverse. 284Yg and 286U-286V show just how close this similarity is.

(b) I have spelt out parts (c) and (d) of 284O in detail, perhaps in unnecessary detail, because they give me an opportunity to insist on the difference between '$\sqrt{2\pi} g_1 \times g_2$ *represents* the Fourier transform of $f_1 * f_2$' and '$\frac{1}{\sqrt{2\pi}} g_1 * g_2$ *is* the Fourier transform of $f_1 \times f_2$'. The actual functions g_1 and g_2 are not well-defined by the hypothesis that they represent the Fourier transforms of f_1 and f_2, though their equivalence classes $g_1^\bullet$, $g_2^\bullet \in L^2_{\mathbb{C}}$ are. So the product $g_1 \times g_2$ is also not uniquely defined as a function, though its equivalence class $(g_1 \times g_2)^\bullet = g_1^\bullet \times g_2^\bullet$ is well-defined as a member of $L^1_{\mathbb{C}}$. However the continuous function $g_1 * g_2$ is unaffected by changes to g_1 and g_2 on negligible sets, so is well defined as a function; and since $f_1 \times f_2$ is integrable, and has a true Fourier transform, it is to be expected that $(f_1 \times f_2)^\wedge$ should be exactly equal to $\frac{1}{\sqrt{2\pi}} g_1 * g_2$.

This distinction between 'being' a Fourier transform and 'representing' a Fourier transform echoes a question which arose in 233D concerning conditional expectations. I spoke there of 'a' conditional expectation on T of a function f as being 'a $\mu\restriction$T-integrable function g such that $\int_F g\,d\mu = \int_F f d\mu$ for every $F \in$ T'; the point being that any $\mu\restriction$T-integrable function equal almost everywhere to g would equally be a conditional expectation of f. Here we see that if g represents the Fourier transform of f then any function almost everywhere equal to g will also represent the Fourier transform of f. In 242J I suggested resolving this complication by regarding conditional expectation as a map between L^1 spaces rather than between $\mathcal{L}^1$ spaces. Here, similarly, we could think of the Fourier transform considered in 284H as being a linear operator defined on a certain subspace of $L^0(\mu)$.

In the case of conditional expectations, I think that there are solid reasons for taking the operators on L^1 spaces as the real embodiment of the idea; I will expand on these in Chapter 36 of the next volume. In the case of Fourier transforms, I do not think the arguments have the same force. In 284R below, and in §285, we shall see that there are important cases in which we want to talk about Fourier transforms which cannot be represented by members of L^0, so that this would still be only a half-way house.

(c) Of course 284Oc-284Od also exhibit a characteristic feature of arguments involving Fourier transforms, the extension by continuity of relations valid for test functions.

(d) 284Oa is a version of **Plancherel's theorem**. The formula $\|f\|_2 = \|\hat{f}\|_2$ is **Parseval's identity**.

284R Dirac's delta function Consider the tempered function $\chi\mathbb{R}$ with constant value 1. In what sense, if any, can we assign a Fourier transform to $\chi\mathbb{R}$?

If we examine $\int \chi\mathbb{R} \times \hat{h}$, as suggested in 284H, we get

$$\int_{-\infty}^{\infty} \chi\mathbb{R} \times \hat{h} = \int_{-\infty}^{\infty} \hat{h} = \sqrt{2\pi}\check{\hat{h}}(0) = \sqrt{2\pi}h(0)$$

for every rapidly decreasing test function h. Of course there is no *function* g such that $\int g \times h = \sqrt{2\pi}h(0)$ for every rapidly decreasing test function h, since (using the arguments of 284G) we should have to have $\int_a^b g = \sqrt{2\pi}$ whenever $a < 0 < b$, so that the indefinite integral of g could not be continuous at 0. However there is a *measure* on $\mathbb{R}$ with exactly the right property, the Dirac measure δ_0 concentrated at 0; this is a Radon probability measure (257Xa), and $\int h\,d\delta_0 = h(0)$ for every function h defined at 0. So we shall have

$$\int_{-\infty}^{\infty} \chi\mathbb{R} \times \hat{h} = \sqrt{2\pi}\int h\,d\delta_0$$

for every rapidly decreasing test function h, and we can reasonably say that the measure $\nu = \sqrt{2\pi}\delta_0$ 'represents the Fourier transform of $\chi\mathbb{R}$'.

We note with pleasure at this point that

$$\frac{1}{\sqrt{2\pi}}\int e^{ixy}\nu(dy) = 1$$

for every $x \in \mathbb{R}$, so that $\chi\mathbb{R}$ can be called the inverse Fourier transform of ν.

If we look at the formulae of Theorem 284M, we get ideas consistent with this pairing of $\chi\mathbb{R}$ with ν. We have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}e^{-\epsilon x^2}\chi\mathbb{R}(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}e^{-\epsilon x^2}dx = \frac{1}{\sqrt{2\epsilon}}e^{-y^2/4\epsilon}$$

for every $y \in \mathbb{R}$, using 283N with $\sigma = 1/\sqrt{2\epsilon}$. So

$$\lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}e^{-\epsilon x^2}\chi\mathbb{R}(x)dx = 0$$

for every $y \neq 0$, and the Fourier transform of $\chi\mathbb{R}$ should be zero everywhere except at 0. On the other hand, the functions $y \mapsto \frac{1}{\sqrt{2\epsilon}}e^{-y^2/4\epsilon}$ all have integral $\sqrt{2\pi}$, concentrated more and more closely about 0 as ϵ decreases to 0, so also point us directly to ν, the measure which gives mass $\sqrt{2\pi}$ to 0.

Thus allowing measures, as well as functions, enables us to extend the notion of Fourier transform. Of course we can go very much farther than this. If h is any rapidly decreasing test function, then (because $\hat{\check{h}} = h$)

$$\int_{-\infty}^{\infty} x\hat{h}(x)dx = -i\sqrt{2\pi}h'(0),$$

so that the identity function $x \mapsto x$ can be assigned, as a Fourier transform, the operator $h \mapsto -i\sqrt{2\pi}h'(0)$.

At this point we are entering the true theory of (Schwartzian) distributions or 'generalized functions', and I had better stop. The 'Dirac delta function' is most naturally regarded as the measure δ_0 above; alternatively, as $\frac{1}{\sqrt{2\pi}}\hat{\chi\mathbb{R}}$.

284W The multidimensional case As in §283, I give exercises designed to point the way to the r-dimensional generalization.

(a) A **rapidly decreasing test function** on $\mathbb{R}^r$ is a function $h:\mathbb{R}^r\to\mathbb{C}$ such that (i) h is **smooth**, that is, all repeated partial derivatives

$$\frac{\partial^m h}{\partial\xi_{j_1}\ldots\partial\xi_{j_m}}$$

are defined and continuous everywhere in $\mathbb{R}^r$ (ii)

$$\sup_{x\in\mathbb{R}^r}\|x\|^k|h(x)|<\infty,\quad \sup_{x\in\mathbb{R}^r}\|x\|^k|\frac{\partial^m h}{\partial\xi_{j_1}\ldots\partial\xi_{j_m}}(x)|<\infty$$

for every $k\in\mathbb{N}$, $j_1,\ldots,j_m\le r$. A **tempered function** on $\mathbb{R}^r$ is a measurable complex-valued function f, defined almost everywhere in $\mathbb{R}^r$, such that, for some $k\in\mathbb{N}$,

$$\int_{\mathbb{R}^r}\frac{1}{1+\|x\|^k}|f(x)|dx<\infty.$$

Show that if f is a tempered function on $\mathbb{R}^r$ and h is a rapidly decreasing test function on $\mathbb{R}^r$ then $f\times h$ is integrable.

(b) Show that if h is a rapidly decreasing test function on $\mathbb{R}^r$ so is $\hat{h}$, and that in this case $\check{\hat{h}}=h$.

(c) Show that if f is a tempered function on $\mathbb{R}^r$ and $\int f\times h=0$ for every rapidly decreasing test function h on $\mathbb{R}^r$, then $f=0$ a.e.

(d) If f and g are tempered functions on $\mathbb{R}^r$, I say that g **represents the Fourier transform of** f if $\int g\times h=\int f\times\hat{h}$ for every rapidly decreasing test function h on $\mathbb{R}^r$. Show that if f is integrable then $\hat{f}$ represents the Fourier transform of f in this sense.

(e) Let f be any tempered function on $\mathbb{R}^r$. Writing $\psi_\sigma(x)=\frac{1}{(\sigma\sqrt{2\pi})^r}e^{-x\,.\,x/2\sigma^2}$ for $x\in\mathbb{R}^r$, show that $\lim_{\sigma\downarrow 0}(f*\psi_\sigma)(x)=c$ whenever $x\in\mathbb{R}^r$, $c\in\mathbb{C}$ are such that $\lim_{\delta\downarrow 0}\frac{1}{\delta^r}\int_{B(x,\delta)}|f(t)-c|dt=0$, writing $B(x,\delta)=\{t:\|t-x\|\le\delta\}$.

(f) Let f and g be tempered functions on $\mathbb{R}^r$ such that g represents the Fourier transform of f, and h a rapidly decreasing test function. Show that (i) the Fourier transform of $f\times h$ is $\frac{1}{(\sqrt{2\pi})^r}g*\hat{h}$ (ii) $(\sqrt{2\pi})^r g\times\hat{h}$ represents the Fourier transform of $f*h$.

(g) Let f and g be tempered functions on $\mathbb{R}^r$ such that g represents the Fourier transform of f. Show that

$$g(y)=\lim_{\epsilon\downarrow 0}\frac{1}{(\sqrt{2\pi})^r}\int_{\mathbb{R}^r}e^{-iy\,.\,x}e^{-\epsilon x\,.\,x}f(x)dx$$

for almost every $y\in\mathbb{R}^r$.

(h) Show that for any square-integrable complex-valued function f on $\mathbb{R}^r$ and any $\epsilon>0$ there is a rapidly decreasing test function h such that $\|f-h\|_2\le\epsilon$.

(i) Let $\mathcal{L}^2_{\mathbb{C}}$ be the space of square-integrable complex-valued functions on $\mathbb{R}^r$. Show that

(i) for every $f\in\mathcal{L}^2_{\mathbb{C}}$ there is a $g\in\mathcal{L}^2_{\mathbb{C}}$ which represents the Fourier transform of f, and in this case $\|g\|_2=\|f\|_2$;

(ii) if g_1, $g_2\in\mathcal{L}^2_{\mathbb{C}}$ represent the Fourier transforms of f_1, $f_2\in\mathcal{L}^2_{\mathbb{C}}$, then $\frac{1}{(\sqrt{2\pi})^r}g_1*g_2$ is the Fourier transform of $f_1\times f_2$, and $(\sqrt{2\pi})^r g_1\times g_2$ represents the Fourier transform of f_1*f_2.

(j) Let T be an invertible real $r\times r$ matrix, regarded as a linear operator from $\mathbb{R}^r$ to itself. (i) Show that $\hat{f}=|\det T|(fT)\hat{}\,T^\top$ for every integrable complex-valued function f on $\mathbb{R}^r$. (ii) Show that hT is a rapidly decreasing test function for every rapidly decreasing test function h. (iii) Show that if f, g are a tempered functions and g represents the Fourier transform of f, then $\frac{1}{|\det T|}g(T^\top)^{-1}$ represents the Fourier transform of fT; so that if T is orthogonal, then gT represents the Fourier transform of fT.

284X Basic exercises (a) Show that if g and h are rapidly decreasing test functions, so is $g \times h$.

(b) Show that there are non-zero continuous integrable functions f, $g : \mathbb{R} \to \mathbb{C}$ such that $f * g = 0$ everywhere. (*Hint*: take them to be Fourier transforms of suitable test functions.)

(c) Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a differentiable function such that its derivative f' is a tempered function and, for some $k \in \mathbb{N}$,

$$\lim_{x\to\infty} x^{-k} f(x) = \lim_{x\to-\infty} x^{-k} f(x) = 0.$$

(i) Show that $\int f \times h' = -\int f' \times h$ for every rapidly decreasing test function h. (ii) Show that if g is a tempered function representing the Fourier transform of f, then $y \mapsto iyg(y)$ represents the Fourier transform of f'.

(d) For a tempered function f and $\alpha \in \mathbb{R}$, set

$$(S_\alpha f)(x) = f(x+\alpha), \quad (M_\alpha f)(x) = e^{i\alpha x} f(x), \quad (D_\alpha f)(x) = f(\alpha x)$$

whenever these are defined. (i) Show that $S_\alpha f$, $M_\alpha f$ and (if $\alpha \neq 0$) $D_\alpha f$ are tempered functions. (ii) Show that if g is a tempered function which represents the Fourier transform of f, then $M_{-\alpha} g$ represents the Fourier transform of $S_\alpha f$, $S_{-\alpha} g$ represents the Fourier transform of $M_\alpha f$, $\overset{\leftrightarrow}{\bar{g}} = \bar{\overset{\leftrightarrow}{g}}$ represents the Fourier transform of $\bar{f}$, and if $\alpha \neq 0$ then $\frac{1}{|\alpha|} D_{1/\alpha} g$ represents the Fourier transform of $D_\alpha f$.

(e) Show that if h is a rapidly decreasing test function and f is any measurable complex-valued function, defined almost everywhere in $\mathbb{R}$, such that $\int_{-\infty}^{\infty} |x|^k |f(x)| dx < \infty$ for every $k \in \mathbb{N}$, then the convolution $f * h$ is a rapidly decreasing test function. (*Hint*: show that the Fourier transform of $f * h$ is a test function.)

>(f) Let f be a tempered function such that $\lim_{a\to\infty} \int_{-a}^{a} f$ exists in $\mathbb{C}$. Show that this limit is also equal to $\lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} e^{-\epsilon x^2} f(x) dx$. (*Hint*: set $g(x) = f(x) + f(-x)$. Use 224J to show that if $0 \le a \le b$ then $|\int_a^b g(x) e^{-\epsilon x^2} dx| \le \sup_{c\in[a,b]} |\int_a^c g|$, so that $\lim_{a\to\infty} \int_0^a g(x) e^{-\epsilon x^2} dx$ exists uniformly in ϵ, while $\lim_{\epsilon\downarrow 0} \int_0^a g(x) e^{-\epsilon x^2} dx = \int_0^a g$ for every $a \ge 0$.)

>(g) Let f and g be tempered functions on $\mathbb{R}$ such that g represents the Fourier transform of f. Show that

$$g(y) = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-iyx} f(x) dx$$

at almost all points y for which the limit exists. (*Hint*: 284Xf, 284M.)

>(h) Let f be an integrable complex-valued function on $\mathbb{R}$ such that $\hat{f}$ also is integrable. Show that $\check{\hat{f}} = f$ at any point at which f is continuous.

(i) Show that for every $p \in [1, \infty[$, $f \in \mathcal{L}^p_{\mathbb{C}}$ and $\epsilon > 0$ there is a rapidly decreasing test function h such that $\|f - h\|_p \le \epsilon$.

>(j) Let f and g be square-integrable complex-valued functions on $\mathbb{R}$ such that g represents the Fourier transform of f. Show that

$$\int_c^d f = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{icy} - e^{idy}}{y} g(y) dy$$

whenever $c < d$ in $\mathbb{R}$.

(k) Let f be a measurable complex-valued function, defined almost everywhere in $\mathbb{R}$, such that $\int |f|^p < \infty$, where $1 < p \le 2$. Show that f is a tempered function and that there is a tempered function g representing the Fourier transform of f. (*Hint*: express f as $f_1 + f_2$, where f_1 is integrable and f_2 is square-integrable.) (**Remark** Defining $\|f\|_p$, $\|g\|_q$ as in 244D, where $q = p/(p-1)$, we have $\|g\|_q \le (2\pi)^{(p-2)/2p} \|f\|_p$; see Zygmund 59, XVI.3.2.)

(l) Let f, g be square-integrable complex-valued functions on $\mathbb{R}$ such that g represents the Fourier transform of f.

(i) Show that

$$\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ixy}g(y)dy = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin at}{t}f(x-t)dt$$

whenever $x \in \mathbb{R}$ and $a > 0$. (*Hint*: find the inverse Fourier transform of $y \mapsto e^{-ixy}\chi[-a,a](y)$, and use 284Ob.)

(ii) Show that if $f(x) = 0$ for $x \in \,]c,d[$ then

$$\frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a} e^{ixy}g(y)dy = 0$$

for $x \in \,]c,d[$.

(iii) Show that if f is differentiable at $x \in \mathbb{R}$, then

$$\frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a} e^{ixy}g(y)dy = f(x).$$

(iv) Show that if f has bounded variation over some interval properly containing x, then

$$\frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a} e^{ixy}g(y)dy = \frac{1}{2}(\lim_{t\in\operatorname{dom} f,t\uparrow x} f(t) + \lim_{t\in\operatorname{dom} f,t\downarrow x} f(t)).$$

(m) Let f be an integrable complex function on $\mathbb{R}$. Show that if $\hat{f}$ is square-integrable, so is f.

(n) Let f_1, f_2 be square-integrable complex-valued functions on $\mathbb{R}$ with Fourier transforms represented by g_1, g_2. Show that $\int_{-\infty}^{\infty} f_1(t)f_2(-t)dt = \int_{-\infty}^{\infty} g_1(t)g_2(t)dt$.

(o) Suppose $x \in \mathbb{R}$. Write δ_x for Dirac measure on $\mathbb{R}$ concentrated at x. Describe a sense in which $\sqrt{2\pi}\delta_x$ can be regarded as the Fourier transform of the function $t \mapsto e^{ixt}$.

(p) For any tempered function f and $x \in \mathbb{R}$, let δ_x be the Dirac measure on $\mathbb{R}$ concentrated at x, and set

$$(\delta_x * f)(u) = \int f(u-t)\delta_x(dt) = f(u-x)$$

for every u for which $u - x \in \operatorname{dom} f$ (cf. 257Xe). If g represents the Fourier transform of f, find a corresponding representation of the Fourier transform of $\delta_x * f$, and relate it to the product of g with the Fourier transform of δ_x.

(q)(i) Show that

$$\lim_{\delta\downarrow 0,a\to\infty}\Bigl(\int_{-a}^{-\delta}\frac{1}{x}e^{-iyx}dx + \int_{\delta}^{a}\frac{1}{x}e^{-iyx}dx\Bigr) = -\pi i\operatorname{sgn} y$$

for every $y \in \mathbb{R}$, writing $\operatorname{sgn} y = y/|y|$ if $y \neq 0$ and $\operatorname{sgn} 0 = 0$. (*Hint*: 283Da.)

(ii) Show that

$$\lim_{c\to\infty}\frac{1}{c}\int_0^c\int_{-a}^{a} e^{ixy}\operatorname{sgn} y\,dy\,da = \frac{2i}{x}$$

for every $x \neq 0$.

(iii) Show that for any rapidly decreasing test function h,

$$\begin{aligned}\int_0^\infty \frac{1}{x}(\hat{h}(x) - \hat{h}(-x))dx &= \lim_{\delta\downarrow 0,a\to\infty}\Bigl(\int_{-a}^{-\delta}\frac{1}{x}\hat{h}(x)dx + \int_{\delta}^{a}\frac{1}{x}\hat{h}(x)dx\Bigr)\\ &= -\frac{i\pi}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(y)\operatorname{sgn} y\,dy.\end{aligned}$$

(iv) Show that for any rapidly decreasing test function h,

$$\frac{i\pi}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{h}(x)\operatorname{sgn} x\,dx = \int_0^\infty \frac{1}{y}(h(y) - h(-y))dy.$$

(r) Let $\langle h_n\rangle_{n\in\mathbb{N}}$ be a sequence of rapidly decreasing test functions such that $\phi(f)=\lim_{n\to\infty}\int_{-\infty}^{\infty}h_n\times f$ is defined for every rapidly decreasing test function f. Show that $\lim_{n\to\infty}\int_{-\infty}^{\infty}h_n'\times f$, $\lim_{n\to\infty}\int_{-\infty}^{\infty}\hat{h}_n\times f$ and $\lim_{n\to\infty}\int_{-\infty}^{\infty}(h_n*g)\times f$ are defined for all rapidly decreasing test functions f and g, and are zero if ϕ is identically zero. (*Hint*: 255G will help with the last.)

284Y Further exercises (a) Let f be an integrable complex-valued function on $]-\pi,\pi]$, and $\tilde{f}$ its periodic extension, as in 282Ae. Show that $\tilde{f}$ is a tempered function. Show that for any rapidly decreasing test function h, $\int\tilde{f}\times\hat{h}=\sqrt{2\pi}\sum_{k=-\infty}^{\infty}c_kh(k)$, where $\langle c_k\rangle_{k\in\mathbb{N}}$ is the sequence of Fourier coefficients of f. (*Hint*: begin with the case $f(x)=e^{inx}$. Next show that

$$M=\sum_{k=-\infty}^{\infty}|h(k)|+\sum_{k=-\infty}^{\infty}\sup_{x\in[(2k-1)\pi,(2k+1)\pi]}|\hat{h}(x)|<\infty,$$

and that

$$|\int\tilde{f}\times\hat{h}-\sqrt{2\pi}\sum_{k=-\infty}^{\infty}c_kh(k)|\le M\|f\|_1.$$

Finally apply 282Ib.)

(b) Let f be a complex-valued function, defined almost everywhere in $\mathbb{R}$, such that $f\times h$ is integrable for every rapidly decreasing test function h. Show that f is tempered.

(c) Let f and g be tempered functions on $\mathbb{R}$ such that g represents the Fourier transform of f. Show that

$$\int_c^d f=\frac{i}{\sqrt{2\pi}}\lim_{\sigma\to\infty}\int_{-\infty}^{\infty}\frac{e^{icy}-e^{idy}}{y}e^{-y^2/2\sigma^2}g(y)dy$$

whenever $c\le d$ in $\mathbb{R}$. (*Hint*: set $\theta=\chi[c,d]$. Show that both sides are $\lim_{\sigma\to\infty}\int f\times(\theta*\psi_{1/\sigma})$, defining ψ_σ as in 283N and 284L.)

(d) Show that if $g:\mathbb{R}\to\mathbb{R}$ is an odd function of bounded variation such that $\int_1^{\infty}\frac{1}{x}g(x)dx=\infty$, then g does not represent the Fourier transform of any tempered function. (*Hint*: 283Ye, 284Yc.)

(e) Let $\mathcal{S}$ be the space of rapidly decreasing test functions. For k, $m\in\mathbb{N}$ set $\tau_{km}(h)=\sup_{x\in\mathbb{R}}|x|^k|h^{(m)}(x)|$ for every $h\in\mathcal{S}$, writing $h^{(m)}$ for the mth derivative of h as usual. (i) Show that each τ_{km} is a seminorm and that $\mathcal{S}$ is complete and separable for the metrizable linear space topology $\mathfrak{T}$ they define. (ii) Show that $h\mapsto\hat{h}:\mathcal{S}\to\mathcal{S}$ is continuous for $\mathfrak{T}$. (iii) Show that if f is any tempered function, then $h\mapsto\int f\times h$ is $\mathfrak{T}$-continuous. (iv) Show that if f is an integrable function such that $\int|x^kf(x)|dx<\infty$ for every $k\in\mathbb{N}$, then $h\mapsto f*h:\mathcal{S}\to\mathcal{S}$ is $\mathfrak{T}$-continuous.

(f) Show that if f is a tempered function on $\mathbb{R}$ and

$$\gamma=\lim_{c\to\infty}\frac{1}{c}\int_0^c\int_{-a}^a f(x)dxda$$

is defined in $\mathbb{C}$, then γ is also

$$\lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty}f(x)e^{-\epsilon|x|}dx.$$

(g) Let f, g be square-integrable complex-valued functions on $\mathbb{R}$ such that g represents the Fourier transform of f. Suppose that $m\in\mathbb{Z}$ and that $(2m-1)\pi<x<(2m+1)\pi$. Set $\tilde{f}(t)=f(t+2m\pi)$ for those $t\in]-\pi,\pi]$ such that $t+2m\pi\in\operatorname{dom}f$. Let $\langle c_k\rangle_{k\in\mathbb{Z}}$ be the sequence of Fourier coefficients of $\tilde{f}$. Show that

$$\frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^a e^{ixy}g(y)dy=\lim_{n\to\infty}\sum_{k=-n}^{n}c_ke^{ikx}$$

in the sense that if one limit exists in $\mathbb{C}$ so does the other, and they are then equal. (*Hint*: 284Xl(i), 282Da.)

(h) Show that if f is integrable over $\mathbb{R}$ and there is some $M\ge 0$ such that $f(x)=\hat{f}(x)=0$ for $|x|\ge M$, then $f=0$ a.e. (*Hint*: reduce to the case $M=\pi$. Looking at the Fourier series of $f\restriction]-\pi,\pi]$, show that f is expressible in the form $f(x)=\sum_{k=-m}^{m}c_ke^{ikx}$ for almost every $x\in]-\pi,\pi]$. Now compute $\hat{f}(2n+\frac{1}{2})$ for large n.)

(i) Let ν be a Radon measure on $\mathbb{R}$ which is 'tempered' in the sense that $\int_{-\infty}^{\infty}\frac{1}{1+|x|^k}\nu(dx)$ is finite for some $k\in\mathbb{N}$. (i) Show that every rapidly decreasing test function is ν-integrable. (ii) Show that if ν has bounded support (definition: 256Xf), and h is a rapidly decreasing test function, then $\nu*h$ is a rapidly decreasing test function, where $(\nu*h)(x)=\int_{-\infty}^{\infty}h(x-y)\nu(dy)$ for $x\in\mathbb{R}$. (iii) Show that there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of rapidly decreasing test functions such that $\lim_{n\to\infty}\int_{-\infty}^{\infty}h_n\times f=\int_{-\infty}^{\infty}fd\nu$ for every rapidly decreasing test function f.

(j) Let $\phi:\mathcal{S}\to\mathbb{R}$ be a functional defined by the formula of 284Xr. Show that ϕ is continuous for the topology of 284Ye. (*Note*: it helps to know a little more about metrizable linear topological spaces than is covered in §2A5.)

284 Notes and comments Yet again I must warn you that the material above gives a very restricted view of the subject. I have tried to indicate how the theory of Fourier transforms of 'good' functions – here taken to be the rapidly decreasing test functions – may be extended, through a kind of duality, to a very much wider class of functions, the 'tempered functions'. Evidently, writing $\mathcal{S}$ for the linear space of rapidly decreasing test functions, we can seek to investigate a Fourier transform of any linear functional $\phi:\mathcal{S}\to\mathbb{C}$, writing $\hat{\phi}(h)=\phi(\hat{h})$ for any $h\in\mathcal{S}$. (It is actually commoner at this point to restrict attention to functionals ϕ which are continuous for the standard topology on $\mathcal{S}$, described in 284Ye; these are called **tempered (Schwartzian) distributions**.) By 284F-284G, we can identify some of these functionals with equivalence classes of tempered functions, and then set out to investigate those tempered functions whose Fourier transforms can again be represented by tempered functions.

I suppose the structure of the theory of Fourier transforms is best laid out through the formulae involved. Our aim is to set up pairs $(f,g)=(f,\hat{f})=(\check{g},g)$ in such a way that we have

Inversion: $\check{\hat{h}}=\hat{\check{h}}=h$;

Reversal: $\check{h}(y)=\hat{h}(-y)$;

Linearity: $(h_1+h_2)^\wedge=\hat{h}_1+\hat{h}_2,\quad (ch)^\wedge=c\hat{h}$;

Differentiation: $(h')^\wedge(y)=iy\hat{h}(y)$;

Shift: if $h_1(x)=h(x+c)$ then $\hat{h}_1(y)=e^{iyc}\hat{h}(y)$;

Modulation: if $h_1(x)=e^{icx}h(x)$ then $\hat{h}_1(y)=\hat{h}(y-c)$;

Symmetry: if $h_1(x)=h(-x)$ then $\hat{h}_1(y)=\hat{h}(-y)$;

Complex Conjugate: $(\overline{h})^\wedge(y)=\overline{\hat{h}(-y)}$;

Dilation: if $h_1(x)=h(cx)$, where $c>0$, then $\hat{h}_1(y)=\frac{1}{c}\hat{h}(\frac{y}{c})$;

Convolution: $(h_1*h_2)^\wedge=\sqrt{2\pi}\hat{h}_1\times\hat{h}_2,\quad (h_1\times h_2)^\wedge=\frac{1}{\sqrt{2\pi}}\hat{h}_1*\hat{h}_2$;

Duality: $\int_{-\infty}^{\infty}h_1\times\hat{h}_2=\int_{-\infty}^{\infty}\hat{h}_1\times h_2$;

Parseval: $\int_{-\infty}^{\infty}h_1\times\overline{h_2}=\int_{-\infty}^{\infty}\hat{h}_1\times\overline{\hat{h}_2}$;

and, of course,

$$\hat{h}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyx}h(x)dx,$$

$$\int_c^d\hat{h}(y)dy=\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-icy}-e^{-idy}}{y}h(y)dy.$$

(I have used the letter h in the list above to suggest what is in fact the case, that all the formulae here are valid for rapidly decreasing test functions.) On top of all this, it is often important that the operation $h\mapsto\hat{h}$ should be continuous in some sense.

The challenge of the 'pure' theory of Fourier transforms is to find the widest possible variety of objects h for which the formulae above will be valid, subject to appropriate interpretations of $^\wedge$, $*$ and $\int_{-\infty}^{\infty}$. I must of course remark here that from the very beginnings, the subject has been enriched by its applications in other parts of mathematics, the physical sciences and the social sciences, and that again and again these have suggested further possible pairs $(f, \hat{f})$, making new demands on our power to interpret the rules we seek to follow. Even the theory of distributions does not seem to give a full canonical account of what can be done. First, there are great difficulties in interpreting the 'product' of two arbitrary distributions, making several of the formulae above problematic; and second, it is not obvious that only one kind of distribution need be considered. In this section I have looked at just one space of 'test functions', the space $\mathcal{S}$ of rapidly decreasing test functions; but at least two others are significant, the space $\mathcal{D}$ of smooth functions with bounded support and the space $\mathcal{Z}$ of Fourier transforms of functions in $\mathcal{D}$. The advantage of starting with $\mathcal{S}$ is that it gives a symmetric theory, since $\hat{h} \in \mathcal{S}$ for every $h \in \mathcal{S}$; but it is easy to find objects (e.g., the function $x \mapsto e^{x^2}$, or the function $x \mapsto 1/|x|$) which cannot be interpreted as functionals on $\mathcal{S}$, so that their Fourier transforms must be investigated by other methods, if at all. In 284Xq I sketch some of the arguments which can be used to justify the assertion that the Fourier transform of the function $x \mapsto 1/x$ is, or can be represented by, the function $y \mapsto -i\sqrt{\frac{\pi}{2}}\operatorname{sgn} y$; the general principle in this case being that we approach both 0 and ∞ symmetrically. For a variety of such matching pairs, established by arguments based on the idea in 284Xr, see Lighthill 59, chap. 3.

Accordingly it seems that, after two centuries, we must still proceed by carefully examining particular classes of function, and checking appropriate interpretations of the formulae. In the work above I have repeatedly used the concepts

$$\lim_{a\to\infty} \int_{-a}^{a} f, \quad \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} e^{-\epsilon x^2} f(x) dx$$

as alternative interpretations of $\int_{-\infty}^{\infty} f$. (Of course they are closely related; see 284Xf.) The reasons for using the particular kernel $e^{-\epsilon x^2}$ are that it belongs to $\mathcal{S}$, it is an even function, its Fourier transform is calculable and easy to manipulate, and it is associated with the normal probability density function $\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$, so that any miscellaneous facts we gather have a chance of being valuable elsewhere. But there are applications in which alternative kernels are more manageable – e.g., $e^{-\epsilon|x|}$ (283Xq, 283Yc, 284Yf).

One of the guiding principles here is that purely formal manipulations, along the lines of those in the list above, and (especially) changes in the order of integration, with other exchanges of limit, again and again give rise to formulae which, suitably interpreted, are valid. First courses in analysis are often inhibitory; students are taught to distrust any manipulation which they cannot justify. To my own eye, the delight of this topic lies chiefly in the variety of the arguments demanded by a rigorous approach, the ground constantly shifting with the context; but there is no doubt that cheerful sanguinity is often the best guide to the manipulations which it will be right to try to justify.

This being a book on measure theory, I am of course particularly interested in the possibility of a measure appearing as a Fourier transform. This is what happens if we seek the Fourier transform of the constant function $\chi\mathbb{R}$ (284R). More generally, any periodic tempered function f with period 2π can be assigned a Fourier transform which is a 'signed measure' (for our present purposes, a complex linear combination of measures) concentrated on $\mathbb{Z}$, the mass at each $k \in \mathbb{Z}$ being determined by the corresponding Fourier coefficient of $f{\restriction}\,]-\pi,\pi]$ (284Xo, 284Ya). In the next section I will go farther in this direction, with particular reference to probability distributions on $\mathbb{R}^r$. But the reason why *positive* measures have not forced themselves on our attention so far is that we do not expect to get a positive function as a Fourier transform unless some very special conditions are satisfied, as in 283Yc.

As in §282, I have used the Hilbert space structure of $L^2_{\mathbb{C}}$ as the basis of the discussion of Fourier transforms of functions in $\mathcal{L}^2_{\mathbb{C}}$ (284O-284P). But as with Fourier series, Carleson's theorem (286U) provides a more direct description.

In 284Wj, I offer a calculation based on the change-of-variable formula in 263A to present a multidimensional version of Reversal and Dilation. But what I am really trying to do is to show that Fourier transforms on $\mathbb{R}^r$ are based on the geometry of the Euclidean inner product, not on the Cartesian coordinate system.

Version of 18.9.14

285 Characteristic functions

I come now to one of the most effective applications of Fourier transforms, the use of 'characteristic functions' to analyse probability distributions. It turns out not only that the Fourier transform of a probability distribution determines the distribution (285M) but that many of the things we want to know about a distribution are easily calculated from its transform (285G, 285Xi). Even more strikingly, pointwise convergence of Fourier transforms corresponds (for sequences) to convergence for the vague topology in the space of distributions, so they provide a new and extremely powerful method for proving such results as the Central Limit Theorem and Poisson's theorem (285Q).

As the applications of the ideas here mostly belong to probability theory, I return to probabilists' terminology, as in Chapter 27. There will nevertheless be many points at which it is appropriate to speak of integrals, and there will often be more than one measure in play; so I should say directly that an integral $\int f(x)dx$ will be with respect to Lebesgue measure (usually, but not always, one-dimensional), as in the rest of this chapter, while integrals with respect to other measures will be expressed in the forms $\int f d\nu$ or $\int f(x)\nu(dx)$.

285A Definition (a) Let ν be a Radon probability measure on $\mathbb{R}^r$ (256A). Then the **characteristic function** of ν is the function $\varphi_\nu : \mathbb{R}^r \to \mathbb{C}$ given by the formula

$$\varphi_\nu(y) = \int e^{iy\,.\,x}\nu(dx)$$

for every $y \in \mathbb{R}^r$, writing $y\,.\,x = \eta_1\xi_1 + \ldots + \eta_r\xi_r$ if $y = (\eta_1, \ldots, \eta_r)$ and $x = (\xi_1, \ldots, \xi_r)$.

(b) Let $X_1, \ldots, X_r$ be real-valued random variables on the same probability space. The **characteristic function** of $\boldsymbol{X} = (X_1, \ldots, X_r)$ is the characteristic function $\varphi_{\boldsymbol{X}} = \varphi_{\nu_{\boldsymbol{X}}}$ of their joint probability distribution $\nu_{\boldsymbol{X}}$ as defined in 271C.

285B Remarks (a) By one of the ordinary accidents of history, the definitions of 'characteristic function' and 'Fourier transform' have evolved independently. In 283Ba I remarked that the definition of the Fourier transform remains unfixed, and that the formulae

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{iyx}f(x)dx,$$

$$\check{f}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx}f(x)dx$$

are sometimes used. On the other hand, I think that nearly all authors agree on the definition of the characteristic function as given above. You may feel therefore that I should have followed their lead, and chosen the definition of Fourier transform which best matches the definition of characteristic function. I did not do so largely because I wished to emphasise the symmetry between the Fourier transform and the inverse Fourier transform, and the correspondence between Fourier transforms and Fourier series. The principal advantage of matching the definitions up would be to make the constants in such theorems as 283F, 285Xk the same, and would be balanced by the need to remember different constants for $\hat{f}$ and $\check{f}$ in such results as 283M.

(b) A secondary reason for not trying too hard to make the formulae of this section match directly those of §§283-284 is that the r-dimensional case is at the heart of some of the most important applications of characteristic functions, so that it seems right to introduce it from the beginning; and consequently the formulae of this section will necessarily have new features compared with those in the body of the work so far.

285C Of course there is a direct way to describe the characteristic function of a family $(X_1, \ldots, X_r)$ of random variables, as follows.

Proposition Let $X_1, \ldots, X_r$ be real-valued random variables on the same probability space, and $\nu_{\boldsymbol{X}}$ their joint distribution. Then their characteristic function $\varphi_{\nu_{\boldsymbol{X}}}$ is given by

$$\varphi_{\nu_{\boldsymbol{X}}}(y) = \mathbb{E}(e^{iy\,.\,\boldsymbol{X}}) = \mathbb{E}(e^{i\eta_1X_1}e^{i\eta_2X_2}\ldots e^{i\eta_rX_r})$$

for every $y = (\eta_1, \ldots, \eta_r) \in \mathbb{R}^r$.

proof Apply 271E to the functions h_1, $h_2 : \mathbb{R}^r \to \mathbb{R}$ defined by

$$h_1(x) = \cos(y\,.\,x), \quad h_2(y) = \sin(y\,.\,x),$$

to see that

$$\begin{aligned}\varphi_{\nu_{\boldsymbol{X}}}(y) &= \int h_1(x)\nu_{\boldsymbol{X}}(dx) + i\int h_2(x)\nu_{\boldsymbol{X}}(dx)\\ &= \mathbb{E}(h_1(\boldsymbol{X})) + i\mathbb{E}(h_2(\boldsymbol{X})) = \mathbb{E}(e^{iy\,.\,\boldsymbol{X}}).\end{aligned}$$

285D I ought to spell out the correspondence between Fourier transforms, as defined in 283A, and characteristic functions.

Proposition Let ν be a Radon probability measure on $\mathbb{R}$. Write

$$\hat{\nu}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}\nu(dx)$$

for every $y \in \mathbb{R}$, and φ_ν for the characteristic function of ν.

(a) $\hat{\nu}(y) = \frac{1}{\sqrt{2\pi}}\varphi_\nu(-y)$ for every $y \in \mathbb{R}$.

(b) For any Lebesgue integrable complex-valued function h defined almost everywhere in $\mathbb{R}$,

$$\int_{-\infty}^{\infty} \hat{\nu}(y)h(y)dy = \int_{-\infty}^{\infty} \hat{h}(x)\nu(dx).$$

(c) For any rapidly decreasing test function h on $\mathbb{R}$ (see §284),

$$\int_{-\infty}^{\infty} h(x)\nu(dx) = \int_{-\infty}^{\infty} \check{h}(y)\hat{\nu}(y)dy.$$

(d) If ν is an indefinite-integral measure over Lebesgue measure, with Radon-Nikodým derivative f, then $\hat{\nu}$ is the Fourier transform of f.

proof (a) This is immediate from the definitions of φ_ν and $\hat{\nu}$.

(b) Because

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(y)|\nu(dx)dy = \int_{-\infty}^{\infty} |h(y)|dy < \infty,$$

we may change the order of integration to see that

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{\nu}(y)h(y)dy &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iyx}h(y)\nu(dx)dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iyx}h(y)dy\,\nu(dx) = \int_{-\infty}^{\infty} \hat{h}(x)\nu(dx).\end{aligned}$$

(c) This follows immediately from (b), because $\check{h}$ is integrable and $\hat{\check{h}} = h$ (284C).

(d) The point is just that

$$\int h\,d\nu = \int h(x)f(x)dx$$

for every bounded Borel measurable $h : \mathbb{R} \to \mathbb{R}$ (235K), and therefore for the functions $x \mapsto e^{-iyx} : \mathbb{R} \to \mathbb{C}$. Now

$$\hat{\nu}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}\nu(dx) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}f(x)dx = \hat{f}(y)$$

for every y.

285E Lemma Let X be a normal random variable with expectation a and variance σ^2, where $\sigma > 0$. Then the characteristic function of X is given by the formula

$$\varphi(y) = e^{iya}e^{-\sigma^2y^2/2}.$$

proof This is just 283N with the constants changed. We have

$$\varphi(y) = \mathbb{E}(e^{iyX}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} e^{-(x-a)^2/2\sigma^2} dx$$

(taking the density function for X given in 274Ad, and applying 271Ic)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iy(\sigma t+a)} e^{-t^2/2} dt$$

(substituting $x = \sigma t + a$)

$$= e^{iya}\sqrt{2\pi}\hat{\psi}_1(-y\sigma)$$

(setting $\psi_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, as in 283N)

$$= e^{iya} e^{-\sigma^2 y^2/2}.$$

285F I now give results corresponding to parts of 283C, with an extra refinement concerning independent random variables (285I).

Proposition Let ν be a Radon probability measure on $\mathbb{R}^r$, and φ its characteristic function.

(a) $\varphi(0) = 1$.

(b) $\varphi : \mathbb{R}^r \to \mathbb{C}$ is uniformly continuous.

(c) $\varphi(-y) = \overline{\varphi(y)}$, $|\varphi(y)| \le 1$ for every $y \in \mathbb{R}^r$.

(d) If $r = 1$ and $\int |x| \nu(dx) < \infty$, then $\varphi'(y)$ exists and is equal to $i \int x e^{ixy} \nu(dx)$ for every $y \in \mathbb{R}$.

(e) If $r = 1$ and $\int x^2 \nu(dx) < \infty$, then $\varphi''(y)$ exists and is equal to $-\int x^2 e^{ixy} \nu(dx)$ for every $y \in \mathbb{R}$.

proof (a) $\varphi(0) = \int \chi\mathbb{R}^r \, d\nu = \nu(\mathbb{R}^r) = 1$.

(b) Let $\epsilon > 0$. Let $M > 0$ be such that

$$\nu\{x : \|x\| \ge M\} \le \epsilon,$$

writing $\|x\| = \sqrt{x \centerdot x}$ as usual. Let $\delta > 0$ be such that $|e^{ia} - 1| \le \epsilon$ whenever $|a| \le \delta$. Now suppose that y, $y' \in \mathbb{R}^r$ are such that $\|y - y'\| \le \delta/M$. Then whenever $\|x\| \le M$,

$$|e^{iy \centerdot x} - e^{iy' \centerdot x}| = |e^{iy' \centerdot x}||e^{i(y-y') \centerdot x} - 1| = |e^{i(y-y') \centerdot x} - 1| \le \epsilon$$

because

$$|(y - y') \centerdot x| \le \|y - y'\|\|x\| \le \delta.$$

Consequently, writing B for $\{x : \|x\| \le M\}$,

$$\begin{aligned} |\varphi(y) - \varphi(y')| &\le \int_B |e^{iy \centerdot x} - e^{iy' \centerdot x}| \nu(dx) \\ &\qquad + \int_{\mathbb{R}^r \setminus B} |e^{iy \centerdot x}| \nu(dx) + \int_{\mathbb{R}^r \setminus B} |e^{iy' \centerdot x}| \nu(dx) \\ &\le \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

As ϵ is arbitrary, φ is uniformly continuous.

(c) This is elementary;

$$\varphi(-y) = \int e^{-iy \centerdot x} \nu(dx) = \overline{\int e^{iy \centerdot x} \nu(dx)} = \overline{\varphi(y)},$$

$$|\varphi(y)| = |\int e^{iy \centerdot x} \nu(dx)| \le \int |e^{iy \centerdot x}| \nu(dx) = 1.$$

(d) The point is that $|\frac{\partial}{\partial y} e^{iyx}| = |x|$ for every x, $y \in \mathbb{R}$. So by 123D (applied, strictly speaking, to the real and imaginary parts of the function)

$$\varphi'(y) = \frac{d}{dy} \int e^{iyx} \nu(dx) = \int \frac{\partial}{\partial y} e^{iyx} \nu(dx) = \int ix e^{iyx} \nu(dx).$$

(e) Since we now have $|\frac{\partial}{\partial y}xe^{iyx}| = x^2$ for every x, y, we can repeat the argument to get

$$\varphi''(y) = i\frac{d}{dy}\int xe^{iyx}\nu(dx) = i\int \frac{\partial}{\partial y}xe^{iyx}\nu(dx) = -\int x^2e^{iyx}\nu(dx).$$

285G Corollary (a) Let X be a real-valued random variable with finite expectation, and φ its characteristic function. Then $\varphi'(0) = i\mathbb{E}(X)$.

(b) Let X be a real-valued random variable with finite variance, and φ its characteristic function. Then $\varphi''(0) = -\mathbb{E}(X^2)$.

proof We have only to match X to its distribution ν, and say that

'X has finite expectation'

corresponds to

$$\text{`}\int |x|\nu(dx) = \mathbb{E}(|X|) < \infty\text{'},$$

so that

$$\varphi'(0) = i\int x\,\nu(dx) = i\mathbb{E}(X),$$

and that

'X has finite variance'

corresponds to

$$\text{`}\int x^2\nu(dx) = \mathbb{E}(X^2) < \infty\text{'},$$

so that

$$\varphi''(0) = -\int x^2\,\nu(dx) = -\mathbb{E}(X^2),$$

as in 271E.

285H Remark Observe that there is no result corresponding to 283Cg ('$\lim_{|y|\to\infty}\hat{f}(y) = 0$'). If ν is the Dirac measure on $\mathbb{R}$ concentrated at 0, that is, the distribution of a random variable which is zero almost everywhere, then $\varphi(y) = 1$ for every y.

285I Proposition Let $X_1, \ldots, X_n$ be independent real-valued random variables, with characteristic functions $\varphi_1, \ldots, \varphi_n$. Let φ be the characteristic function of their sum $X = X_1 + \ldots + X_n$. Then

$$\varphi(y) = \prod_{j=1}^n \varphi_j(y)$$

for every $y \in \mathbb{R}$.

proof Let $y \in \mathbb{R}$. By 272E, the variables

$$Y_j = e^{iyX_j}$$

are independent, so by 272R

$$\varphi(y) = \mathbb{E}(e^{iyX}) = \mathbb{E}(e^{iy(X_1+\ldots+X_n)}) = \mathbb{E}(\prod_{j=1}^n Y_j) = \prod_{j=1}^n \mathbb{E}(Y_j) = \prod_{j=1}^n \varphi_j(y),$$

as required.

Remark See also 285R below.

285J There is an inversion theorem for characteristic functions, corresponding to 283F; I give it in 285Xk, with an r-dimensional version in 285Yb. However, this does not seem to be as useful as the following group of results.

Lemma Let ν be a Radon probability measure on $\mathbb{R}^r$, and φ its characteristic function. Then for $1 \le j \le r$ and $a > 0$,

$$\nu\{x : |\xi_j| \ge a\} \le 7a\int_0^{1/a}(1 - \mathcal{R}\mathrm{e}\,\varphi(te_j))dt,$$

where $e_j \in \mathbb{R}^r$ is the jth unit vector.

proof We have

$$\begin{aligned}7a\int_0^{1/a}(1-\mathcal{R}\mathrm{e}\,\varphi(te_j))dt &= 7a\int_0^{1/a}\bigl(1-\mathcal{R}\mathrm{e}\int_{\mathbb{R}^r}e^{it\xi_j}\nu(dx)\bigr)dt\\ &= 7a\int_0^{1/a}\int_{\mathbb{R}^r}1-\cos(t\xi_j)\nu(dx)dt\\ &= 7a\int_{\mathbb{R}^r}\int_0^{1/a}1-\cos(t\xi_j)dt\,\nu(dx)\end{aligned}$$

(because $(x,t)\mapsto 1-\cos(t\xi_j)$ is bounded and $\nu\mathbb{R}^r\cdot\frac{1}{a}$ is finite)

$$\begin{aligned}&= 7a\int_{\mathbb{R}^r}(\frac{1}{a}-\frac{1}{\xi_j}\sin\frac{\xi_j}{a})\nu(dx)\\ &\ge 7a\int_{|\xi_j|\ge a}(\frac{1}{a}-\frac{1}{\xi_j}\sin\frac{\xi_j}{a})\nu(dx)\end{aligned}$$

(because $\frac{1}{\xi}\sin\frac{\xi}{a}\le\frac{1}{a}$ for every $\xi\ne 0$)

$$\ge\nu\{x:|\xi_j|\ge a\},$$

because

$$\frac{\sin\eta}{\eta}\le\frac{\sin 1}{1}\le\frac{6}{7}\text{ if }\eta\ge 1,$$

so

$$a(\frac{1}{a}-\frac{1}{\xi_j}\sin\frac{\xi_j}{a})\ge\frac{1}{7}$$

if $|\xi_j|\ge a$.

285K Characteristic functions and the vague topology The time has come to return to ideas mentioned briefly in 274L. Fix $r\ge 1$ and let P be the set of all Radon probability measures on $\mathbb{R}^r$. For any bounded continuous function $h:\mathbb{R}^r\to\mathbb{R}$, define $\rho_h:P\times P\to\mathbb{R}$ by setting

$$\rho_h(\nu,\nu')=|\int h\,d\nu-\int h\,d\nu'|$$

for ν, $\nu'\in P$. Then the vague topology on P is the topology generated by the pseudometrics ρ_h (274Ld).

285L Theorem Let ν, $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be Radon probability measures on $\mathbb{R}^r$, with characteristic functions φ, $\langle\varphi_n\rangle_{n\in\mathbb{N}}$. Then the following are equiveridical:
(i) $\nu=\lim_{n\to\infty}\nu_n$ for the vague topology;
(ii) $\int h\,d\nu=\lim_{n\to\infty}\int h\,d\nu_n$ for every bounded continuous $h:\mathbb{R}^r\to\mathbb{R}$;
(iii) $\lim_{n\to\infty}\varphi_n(y)=\varphi(y)$ for every $y\in\mathbb{R}^r$.

proof (a) The equivalence of (i) and (ii) is virtually the definition of the vague topology; we have

$$\lim_{n\to\infty}\nu_n=\nu\text{ for the vague topology}$$
$$\iff\lim_{n\to\infty}\rho_h(\nu_n,\nu)=0\text{ for every bounded continuous }h$$

(2A3Mc)

$$\iff\lim_{n\to\infty}|\int h\,d\nu_n-\int h\,d\nu|=0\text{ for every bounded continuous }h.$$

(b) Next, (ii) obviously implies (iii), because

$$\mathcal{R}\mathrm{e}\,\varphi(y) = \int h_y\,d\nu = \lim_{n\to\infty} h_y\,d\nu_n = \lim_{n\to\infty} \mathcal{R}\mathrm{e}\,\varphi_n(y),$$

setting $h_y(x) = \cos(x\centerdot y)$ for each x, and similarly

$$\mathcal{I}\mathrm{m}\,\varphi(y) = \lim_{n\to\infty} \mathcal{I}\mathrm{m}\,\varphi_n(y)$$

for every $y \in \mathbb{R}^r$.

(c) So we are left to prove that (iii)⇒(ii). I start by showing that, given $\epsilon > 0$, there is a closed bounded set K such that

$$\nu_n(\mathbb{R}^r \setminus K) \le \epsilon \text{ for every } n \in \mathbb{N}.$$

P We know that $\varphi(0) = 1$ and that φ is continuous at 0 (285Fb). Let $a > 0$ be so large that whenever $j \le r$ and $|t| \le 1/a$ we have

$$1 - \mathcal{R}\mathrm{e}\,\varphi(te_j) \le \frac{\epsilon}{14r},$$

writing e_j for the jth unit vector, as in 285J. Then

$$7a\int_0^{1/a}(1 - \mathcal{R}\mathrm{e}\,\varphi(te_j))dt \le \frac{\epsilon}{2r}$$

for each $j \le r$. By Lebesgue's Dominated Convergence Theorem (since of course the functions $t \mapsto 1 - \mathcal{R}\mathrm{e}\,\varphi_n(te_j)$ are uniformly bounded on $[0, \frac{1}{a}]$), there is an $n_0 \in \mathbb{N}$ such that

$$7a\int_0^{1/a}(1 - \mathcal{R}\mathrm{e}\,\varphi_n(te_j))dt \le \frac{\epsilon}{r}$$

for every $j \le r$ and $n \ge n_0$. But 285J tells us that now

$$\nu_n\{x : |\xi_j| \ge a\} \le \frac{\epsilon}{r}$$

for every $j \le r$, $n \ge n_0$. On the other hand, there is surely a $b \ge a$ such that

$$\nu_n\{x : |\xi_j| \ge b\} \le \frac{\epsilon}{r}$$

for every $j \le r$, $n < n_0$. So, setting $K = \{x : |\xi_j| \le b \text{ for every } j \le r\}$,

$$\nu_n(\mathbb{R}^r \setminus K) \le \epsilon$$

for every $n \in \mathbb{N}$, as required. **Q**

(d) Now take any bounded continuous $h : \mathbb{R}^r \to \mathbb{R}$ and $\epsilon > 0$. Set $M = 1 + \sup_{x\in\mathbb{R}^r} |h(x)|$, and let K be a bounded closed set such that

$$\nu_n(\mathbb{R}^r \setminus K) \le \frac{\epsilon}{M} \text{ for every } n \in \mathbb{N}, \quad \nu(\mathbb{R}^r \setminus K) \le \frac{\epsilon}{M},$$

using (b) just above. By the Stone-Weierstrass theorem (281K) there are $y_0, \dots, y_m \in \mathbb{Q}^r$ and $c_0, \dots, c_m \in \mathbb{C}$ such that

$$|h(x) - g(x)| \le \epsilon \text{ for every } x \in K,$$

$$|g(x)| \le M \text{ for every } x \in \mathbb{R}^r,$$

writing $g(x) = \sum_{k=0}^m c_k e^{iy_k\centerdot x}$ for $x \in \mathbb{R}^r$. Now

$$\lim_{n\to\infty} \int g\,d\nu_n = \lim_{n\to\infty} \sum_{k=0}^m c_k\varphi_n(y_k) = \sum_{k=0}^m c_k\varphi(y_k) = \int g\,d\nu.$$

On the other hand, for every $n \in \mathbb{N}$,

$$|\int g\,d\nu_n - \int h\,d\nu_n| \le \int_K |g - h|d\nu_n + 2M\nu_n(\mathbb{R} \setminus K) \le 3\epsilon,$$

and similarly $|\int g\,d\nu - \int h\,d\nu| \le 3\epsilon$. Consequently

$$\limsup_{n\to\infty} |\int h\,d\nu_n - \int h\,d\nu| \le 6\epsilon.$$

As ϵ is arbitrary,

$$\lim_{n\to\infty}\int h\,d\nu_n=\int h\,d\nu,$$

and (ii) is true.

285M Corollary (a) Let ν, ν' be two Radon probability measures on $\mathbb{R}^r$ with the same characteristic functions. Then they are equal.

(b) Let $(X_1,\dots,X_r)$ and $(Y_1,\dots,Y_r)$ be two families of real-valued random variables. If

$$\mathbb{E}(e^{i\eta_1X_1+\ldots+i\eta_rX_r})=\mathbb{E}(e^{i\eta_1Y_1+\ldots+i\eta_rY_r})$$

for all $\eta_1,\dots,\eta_r\in\mathbb{R}$, then $(X_1,\dots,X_r)$ has the same joint distribution as $(Y_1,\dots,Y_r)$.

proof (a) Applying 285L with $\nu_n=\nu'$ for every n, we see that $\int h\,d\nu'=\int h\,d\nu$ for every bounded continuous $h:\mathbb{R}^r\to\mathbb{R}$. By 256D(iv), $\nu=\nu'$.

(b) Apply (a) with ν, ν' the two joint distributions.

285N Remarks Probably the most important application of this theorem is to the standard proof of the Central Limit Theorem. I sketch the ideas in 285Xq and 285Yl-285Yo; details may be found in most serious probability texts; two on my shelf are Shiryayev 84, §III.4, and Feller 66, §XV.6. However, to get the full strength of Lindeberg's version of the Central Limit Theorem we have to work quite hard, and I therefore propose to illustrate the method with a version of Poisson's theorem (285Q) instead. I begin with two lemmas which are very frequently used in results of this kind.

285O Lemma Let $c_0,\dots,c_n,d_0,\dots,d_n$ be complex numbers of modulus at most 1. Then

$$|\prod_{k=0}^n c_k-\prod_{k=0}^n d_k|\le\sum_{k=0}^n|c_k-d_k|.$$

proof Induce on n. The case $n=0$ is trivial. For the case $n=1$ we have

$$\begin{aligned}|c_0c_1-d_0d_1|&=|c_0(c_1-d_1)+(c_0-d_0)d_1|\\&\le|c_0||c_1-d_1|+|c_0-d_0||d_1|\le|c_1-d_1|+|c_0-d_0|,\end{aligned}$$

which is what we need. For the inductive step to $n+1$, we have

$$|\prod_{k=0}^{n+1}c_k-\prod_{k=0}^{n+1}d_k|\le|\prod_{k=0}^{n}c_k-\prod_{k=0}^{n}d_k|+|c_{n+1}-d_{n+1}|$$

(by the case just done, because c_{n+1}, d_{n+1}, $\prod_{k=0}^n c_k$ and $\prod_{k=0}^n d_k$ all have modulus at most 1)

$$\le\sum_{k=0}^n|c_k-d_k|+|c_{n+1}-d_{n+1}|$$

(by the inductive hypothesis)

$$=\sum_{k=0}^{n+1}|c_k-d_k|,$$

so the induction continues.

285P Lemma Suppose that $M\ge0$ and $\epsilon>0$. Then there are $\eta>0$ and $y_0,\dots,y_n\in\mathbb{R}$ such that whenever X, Z are two real-valued random variables with $\mathbb{E}(|X|)\le M$, $\mathbb{E}(|Z|)\le M$ and $|\varphi_X(y_j)-\varphi_Z(y_j)|\le\eta$ for every $j\le n$, then $F_X(a)\le F_Z(a+\epsilon)+\epsilon$ for every $a\in\mathbb{R}$, where I write φ_X for the characteristic function of X and F_X for the distribution function of X.

proof The case $M=0$ is trivial, as then both X and Z are zero a.e., so I will suppose henceforth that $M>0$. Set $\delta=\frac{\epsilon}{7}>0$, $b=\dfrac{M}{\delta}$.

(a) Define $h_0 : \mathbb{R} \to [0,1]$ by setting $h_0(x) = \operatorname{med}(0, 1 - \frac{x}{\delta}, 1)$ for $x \in \mathbb{R}$. Then h_0 is continuous. Let $m = \lfloor \frac{b}{\delta} \rfloor$ be the integer part of $\frac{b}{\delta}$, and for $-m \le k \le m+1$ set $h_k(x) = h_0(x - k\delta)$.

By the Stone-Weierstrass theorem (281K again), there are $y_0, \dots, y_n \in \mathbb{R}$ and $c_0, \dots, c_n \in \mathbb{C}$ such that, writing $g_0(x) = \sum_{j=0}^n c_j e^{iy_j x}$,

$$|h_0(x) - g_0(x)| \le \delta \text{ for every } x \in [-b-(m+1)\delta, b+m\delta],$$

$$|g_0(x)| \le 1 \text{ for every } x \in \mathbb{R}.$$

For $-m \le k \le m+1$, set

$$g_k(x) = g_0(x - k\delta) = \sum_{j=0}^n c_j e^{-iy_j k\delta} e^{iy_j x}.$$

Set $\eta = \delta/(1 + \sum_{j=0}^n |c_j|) > 0$.

(b) Now suppose that X, Z are random variables such that $\mathbb{E}(|X|) \le M$, $\mathbb{E}(|Z|) \le M$ and $|\varphi_X(y_j) - \varphi_Z(y_j)| \le \eta$ for every $j \le n$. Then for any k we have

$$\mathbb{E}(g_k(X)) = \mathbb{E}(\sum_{j=0}^n c_j e^{-iy_j k\delta} e^{iy_j X}) = \sum_{j=0}^n c_j e^{-iy_j k\delta} \varphi_X(y_j),$$

and similarly

$$\mathbb{E}(g_k(Z)) = \sum_{j=0}^n c_j e^{-iy_j k\delta} \varphi_Z(y_j),$$

so

$$|\mathbb{E}(g_k(X)) - \mathbb{E}(g_k(Z))| \le \sum_{j=0}^n |c_j||\varphi_X(y_j) - \varphi_Z(y_j)| \le \sum_{j=0}^n |c_j|\eta \le \delta.$$

Next,

$$|h_k(x) - g_k(x)| \le \delta \text{ for every } x \in [-b-(m+1)\delta + k\delta, b+m\delta+k\delta] \supseteq [-b, b],$$

$$|h_k(x) - g_k(x)| \le 2 \text{ for every } x,$$

$$\Pr(|X| \ge b) \le \frac{M}{b} = \delta,$$

so $\mathbb{E}(|h_k(X) - g_k(X)|) \le 3\delta$; and similarly $\mathbb{E}(|h_k(Z) - g_k(Z)|) \le 3\delta$. Putting these together,

$$|\mathbb{E}(h_k(X)) - \mathbb{E}(h_k(Z))| \le 7\delta = \epsilon$$

whenever $-m \le k \le m+1$.

(c) Now suppose that $-b \le a \le b$. Then there is a k such that $-m \le k \le m+1$ and $a \le k\delta \le a + \delta$. Since

$$\chi\,]-\infty, a] \le \chi\,]-\infty, k\delta] \le h_k \le \chi\,]-\infty, (k+1)\delta] \le \chi\,]-\infty, a+2\delta],$$

we must have

$$\Pr(X \le a) \le \mathbb{E}(h_k(X)),$$

$$\mathbb{E}(h_k(Z)) \le \Pr(Z \le a + 2\delta) \le \Pr(Z \le a + \epsilon).$$

But this means that

$$\Pr(X \le a) \le \mathbb{E}(h_k(X)) \le \mathbb{E}(h_k(Z)) + \epsilon \le \Pr(Z \le a + \epsilon) + \epsilon$$

whenever $a \in [-b, b]$.

(d) As for the cases $a \ge b$, $a \le -b$, we surely have

$$b(1 - F_Z(b)) = b\Pr(Z > b) \le \mathbb{E}(|Z|) \le M,$$

so if $a \ge b$ then

$$F_X(a) \le 1 \le F_Z(a) + 1 - F_Z(b) \le F_Z(a) + \frac{M}{b} = F_Z(a) + \delta \le F_Z(a+\epsilon) + \epsilon.$$

Similarly,

$$bF_X(-b) \le \mathbb{E}(|X|) \le M,$$

so

$$F_X(a) \le \delta \le F_Z(a+\epsilon) + \epsilon$$

for every $a \le -b$. This completes the proof.

285Q Law of Rare Events: Theorem For any $M \ge 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that whenever $X_0, \dots, X_n$ are independent $\{0,1\}$-valued random variables with $\Pr(X_k = 1) = p_k \le \delta$ for every $k \le n$ and $\sum_{k=0}^n p_k = \lambda \le M$, and $X = X_0 + \ldots + X_n$, then

$$|\Pr(X = m) - \frac{\lambda^m}{m!}e^{-\lambda}| \le \epsilon$$

for every $m \in \mathbb{N}$.

proof (a) We should begin by calculating some characteristic functions. First, the characteristic function φ_k of X_k will be given by

$$\varphi_k(y) = (1-p_k)e^{iy0} + p_k e^{iy1} = 1 + p_k(e^{iy} - 1).$$

Next, if Z is a Poisson random variable with parameter λ (that is, if $\Pr(Z = m) = \lambda^m e^{-\lambda}/m!$ for every $m \in \mathbb{N}$; all you need to know at this point about the Poisson distribution is that $\sum_{m=0}^\infty \lambda^m e^{-\lambda}/m! = 1$), then its characteristic function φ_Z is given by

$$\varphi_Z(y) = \sum_{m=0}^\infty \frac{\lambda^m}{m!}e^{-\lambda}e^{iym} = e^{-\lambda}\sum_{m=0}^\infty \frac{(\lambda e^{iy})^m}{m!} = e^{-\lambda}e^{\lambda e^{iy}} = e^{\lambda(e^{iy}-1)}.$$

(b) Before getting down to δ's and η's, I show how to estimate $\varphi_X(y) - \varphi_Z(y)$. We know that

$$\varphi_X(y) = \prod_{k=0}^n \varphi_k(y)$$

(using 285I), while

$$\varphi_Z(y) = \prod_{k=0}^n e^{p_k(e^{iy}-1)}.$$

Because $\varphi_k(y)$, $e^{p_k(e^{iy}-1)}$ all have modulus at most 1 (we have

$$|e^{p_k(e^{iy}-1)}| = e^{-p_k(1-\cos y)} \le 1,)$$

285O tells us that

$$|\varphi_X(y) - \varphi_Z(y)| \le \sum_{k=0}^n |\varphi_k(y) - e^{p_k(e^{iy}-1)}| = \sum_{k=0}^n |e^{p_k(e^{iy}-1)} - 1 - p_k(e^{iy}-1)|.$$

(c) So we have a little bit of analysis to do. To estimate $|e^z - 1 - z|$ where $\mathcal{R}\mathrm{e}\, z \le 0$, consider the function

$$g(t) = \mathcal{R}\mathrm{e}(c(e^{tz} - 1 - tz))$$

where $|c| = 1$. We have $g(0) = g'(0) = 0$ and

$$|g''(t)| = |\mathcal{R}\mathrm{e}(c(z^2 e^{tz}))| \le |c||z^2||e^{tz}| \le |z|^2$$

for every $t \ge 0$, so that

$$|g(1)| \le \frac{1}{2}|z|^2$$

by the (real-valued) Taylor theorem with remainder, or otherwise. As c is arbitrary,

$$|e^z - 1 - z| \le \frac{1}{2}|z|^2$$

whenever $\mathcal{R}\mathrm{e}\, z \le 0$. In particular,

$$|e^{p_k(e^{iy}-1)} - 1 - p_k(e^{iy}-1)| \le \frac{1}{2}p_k^2|e^{iy}-1|^2 \le 2p_k^2$$

for each k, and

$$|\varphi_X(y) - \varphi_Z(y)| \le \sum_{k=0}^n |e^{p_k(e^{iy}-1)} - 1 - p_k(e^{iy}-1)| \le 2\sum_{k=0}^n p_k^2$$

for each $y\in\mathbb{R}$.

(d) Now for the detailed estimates. Given $M\ge 0$ and $\epsilon>0$, let $\eta>0$ and $y_0,\dots,y_l\in\mathbb{R}$ be such that

$$\Pr(X\le a)\le\Pr(Z\le a+\frac{1}{2})+\frac{\epsilon}{2}$$

whenever X, Z are real-valued random variables, $\mathbb{E}(|X|)\le M$, $\mathbb{E}(|Z|)\le M$ and $|\varphi_X(y_j)-\varphi_X(y_j)|\le\eta$ for every $j\le l$ (285P). Take $\delta=\frac{\eta}{2M+1}$ and suppose that $X_0,\dots,X_n$ are independent $\{0,1\}$-valued random variables with $\Pr(X_k=1)=p_k\le\delta$ for every $k\le n$, where $\lambda=\sum_{k=0}^n p_k$ is less than or equal to M. Set $X=X_0+\ldots+X_n$ and let Z be a Poisson random variable with parameter λ; then by the arguments of (a)-(c),

$$|\varphi_X(y)-\varphi_Z(y)|\le 2\textstyle\sum_{k=0}^n p_k^2\le 2\delta\sum_{k=0}^n p_k=2\delta\lambda\le\eta$$

for every $y\in\mathbb{R}$. Also

$$\mathbb{E}(|X|)=\mathbb{E}(X)=\textstyle\sum_{k=0}^n p_k=\lambda\le M,$$

$$\mathbb{E}(|Z|)=\mathbb{E}(Z)=\sum_{m=0}^\infty m\frac{\lambda^m}{m!}e^{-\lambda}=e^{-\lambda}\sum_{m=1}^\infty\frac{\lambda^m}{(m-1)!}=e^{-\lambda}\sum_{m=0}^\infty\frac{\lambda^{m+1}}{m!}=\lambda\le M.$$

So

$$\Pr(X\le a)\le\Pr(Z\le a+\frac{1}{2})+\frac{\epsilon}{2},$$

$$\Pr(Z\le a)\le\Pr(X\le a+\frac{1}{2})+\frac{\epsilon}{2}$$

for every a. But as both X and Z take all their values in $\mathbb{N}$,

$$|\Pr(X\le m)-\Pr(Z\le m)|\le\frac{\epsilon}{2}$$

for every $m\in\mathbb{N}$, and

$$|\Pr(X=m)-\frac{\lambda^m}{m!}e^{-\lambda}|=|\Pr(X=m)-\Pr(Z=m)|\le\epsilon$$

for every $m\in\mathbb{N}$, as required.

285R Convolutions Recall from 257A that if ν, $\tilde\nu$ are Radon probability measures on $\mathbb{R}^r$ then they have a convolution $\nu*\tilde\nu$ defined by writing

$$(\nu*\tilde\nu)(E)=(\nu\times\tilde\nu)\{(x,y):x+y\in E\}$$

for every Borel set $E\subseteq\mathbb{R}^r$, which is also a Radon probability measure. We can readily compute the characteristic function $\varphi_{\nu*\tilde\nu}$ from 257B: we have

$$\begin{aligned}\varphi_{\nu*\tilde\nu}(y)&=\int e^{iy\,\cdot\,x}(\nu*\tilde\nu)(dx)=\int e^{iy\,\cdot\,(x+x')}\nu(dx)\tilde\nu(dx')\\&=\int e^{iy\,\cdot\,x}e^{iy\,\cdot\,x'}\nu(dx)\tilde\nu(dx')=\int e^{iy\,\cdot\,x}\nu(dx)\int e^{iy\,\cdot\,x'}\tilde\nu(dx')=\varphi_\nu(y)\varphi_{\tilde\nu}(y)\end{aligned}$$

for every $y\in\mathbb{R}^r$. (Thus convolution of measures corresponds to pointwise multiplication of characteristic functions, just as convolution of functions corresponds to pointwise multiplication of Fourier transforms.) Recalling that the sum of independent random variables corresponds to convolution of their distributions (272T), this gives another way of looking at 285I. Remember also that if ν, $\tilde\nu$ have Radon-Nikodým derivatives f, $\tilde f$ with respect to Lebesgue measure then $f*\tilde f$ is a Radon-Nikodým derivative of $\nu*\tilde\nu$ (257F).

285V Proposition Let ν be a Radon probability measure on $\mathbb{R}^r$ such that $\nu*\nu=\nu$. Then ν is the Dirac measure δ_0 concentrated at 0.

proof By 285R, $\varphi_\nu^2=\varphi_\nu$, so φ_ν is $\{0,1\}$-valued; as $\varphi_\nu(0)=1$ (285Fa) and φ_ν is continuous (285Fb), $\varphi_\nu(y)=1$ for every $y\in\mathbb{R}$, that is, $\varphi_\nu=\varphi_{\delta_0}$ (285H). By 285Ma, $\nu=\delta_0$.

285S The vague topology and pointwise convergence of characteristic functions In 285L we saw that a sequence $\langle\nu_n\rangle_{n\in\mathbb{N}}$ of Radon probability measures on $\mathbb{R}^r$ converges in the vague topology to a Radon probability measure ν if and only if

$$\lim_{n\to\infty}\int e^{iy\cdot x}\nu_n(dx)=\int e^{iy\cdot x}\nu(dx)$$

for every $y\in\mathbb{R}^r$; that is, iff

$$\lim_{n\to\infty}\rho'_y(\nu_n,\nu)=0 \text{ for every } y\in\mathbb{R}^r,$$

writing

$$\rho'_y(\nu,\nu')=|\int e^{iy\cdot x}\nu(dx)-\int e^{iy\cdot x}\nu'(dx)|$$

for Radon probability measures ν, ν' on $\mathbb{R}^r$ and $y\in\mathbb{R}^r$. It is natural to ask whether the pseudometrics ρ'_y actually define the vague topology. Writing $\mathfrak{T}$ for the vague topology and $\mathfrak{S}$ for the topology defined by $\{\rho'_y:y\in\mathbb{R}^r\}$, we surely have $\mathfrak{S}\subseteq\mathfrak{T}$, just because every ρ'_y is one of the pseudometrics used in the definition of $\mathfrak{T}$. Also we know that $\mathfrak{S}$ and $\mathfrak{T}$ give the same convergent sequences, and incidentally that $\mathfrak{T}$ is metrizable (see 285Xt). But all this does not quite amount to saying that the two topologies are the same, and indeed they are not, as the next result shows.

285T Proposition Suppose that $y_0,\dots,y_n\in\mathbb{R}$ and $\eta>0$. Then there are infinitely many $m\in\mathbb{N}$ such that $|1-e^{iy_km}|\le\eta$ for every $k\le n$.

proof Let $\eta_1,\dots,\eta_r\in\mathbb{R}$ be such that $1=\eta_0,\eta_1,\dots,\eta_r$ are linearly independent over $\mathbb{Q}$ and every $y_k/2\pi$ is a linear combination of the η_j over $\mathbb{Q}$; say $y_k=2\pi\sum_{j=0}^r q_{kj}\eta_j$ where every $q_{kj}\in\mathbb{Q}$. Express the q_{kj} as p_{kj}/p where each $p_{kj}\in\mathbb{Z}$ and $p\in\mathbb{N}\setminus\{0\}$. Set $M=\max_{k\le n}\sum_{j=0}^r|p_{kj}|$.

Take any $m_0\in\mathbb{N}$ and let $\delta>0$ be such that $|1-e^{2\pi ix}|\le\eta$ whenever $|x|\le 2\pi M\delta$. By Weyl's Equidistribution Theorem (281N), there are infinitely many m such that $<m\eta_j>\le\delta$ whenever $1\le j\le r$; in particular, there is such an $m\ge m_0$. Set $m_j=\lfloor m\eta_j\rfloor$, so that $|m\eta_j-m_j|\le\delta$ for $0\le j\le r$. Then

$$|mpy_k-2\pi\sum_{j=0}^r p_{kj}m_j|\le 2\pi\sum_{j=0}^r|p_{kj}||m\eta_j-m_j|\le 2\pi M\delta,$$

so that

$$|1-e^{iy_kmp}|=|1-\exp(i(mpy_k-2\pi\sum_{j=0}^r p_{kj}m_j))|\le\eta$$

for every $k\le n$. As $mp\ge m_0$ and m_0 is arbitrary, this proves the result.

285U Corollary The topologies $\mathfrak{S}$ and $\mathfrak{T}$ on the space of Radon probability measures on $\mathbb{R}$, as described in 285S, are different.

proof Let δ_x be the Dirac measure on $\mathbb{R}$ concentrated at x. By 285T, every member of $\mathfrak{S}$ which contains δ_0 also contains δ_m for infinitely many $m\in\mathbb{N}$. On the other hand, the set

$$G=\{\nu:\int e^{-x^2}\nu(dx)>\frac{1}{2}\}$$

is a member of $\mathfrak{T}$, containing δ_0, which does not contain δ_m for any integer $m\ne0$. So $G\in\mathfrak{T}\setminus\mathfrak{S}$ and $\mathfrak{T}\ne\mathfrak{S}$.

285X Basic exercises (a) Let ν be a Radon probability measure on $\mathbb{R}^r$, where $r\ge1$, and suppose that $\int\|x\|\nu(dx)<\infty$. Show that the characteristic function φ of ν is differentiable (in the full sense of 262Fa) and that $\frac{\partial\varphi}{\partial\eta_j}(y)=i\int\xi_je^{iy\cdot x}\nu(dx)$ for every $j\le r$ and $y\in\mathbb{R}^r$, using ξ_j, η_j to represent the coordinates of x and y as usual.

>(b) Let $\boldsymbol{X}=(X_1,\dots,X_r)$ be a family of real-valued random variables, with characteristic function $\varphi_{\boldsymbol{X}}$. Show that the characteristic function φ_{X_j} of X_j is given by

$$\varphi_{X_j}(y)=\varphi_{\boldsymbol{X}}(ye_j) \text{ for every } y\in\mathbb{R},$$

where e_j is the jth unit vector of $\mathbb{R}^r$.

>(c) Let X be a real-valued random variable and φ_X its characteristic function. Show that

$$\varphi_{aX+b}(y) = e^{iyb}\varphi_X(ay)$$

for any a, b, $y \in \mathbb{R}$.

(d) Let X be a real-valued random variable which is not essentially constant, and φ its characteristic function. Show that $|\varphi(y)| < 1$ for all but countably many $y \in \mathbb{R}$. (*Hint*: the support (256Xf) of the distribution of X has distinct points x, x' and if $e^{iyx} \neq e^{iyx'}$ then $|\varphi(y)| < 1$.)

(e) Let X be a real-valued random variable and φ its characteristic function.

(i) Show that for any integrable complex-valued function h on $\mathbb{R}$,

$$\mathbb{E}(\hat{h}(X)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(-y)h(y)dy,$$

writing $\hat{h}$ for the Fourier transform of h.

(ii) Show that for any rapidly decreasing test function h,

$$\mathbb{E}(h(X)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(y)\hat{h}(y)dy.$$

(f) Let ν be a Radon probability measure on $\mathbb{R}$, and suppose that its characteristic function φ is square-integrable. Show that ν is an indefinite-integral measure over Lebesgue measure and that its Radon-Nikodým derivatives are also square-integrable. (*Hint*: use 284O to find a square-integrable f such that $\int f \times h = \frac{1}{\sqrt{2\pi}} \int \varphi \times \hat{h}$ for every rapidly decreasing test function h, and ideas from the proof of 284G to show that $\int_a^b f = \nu\,]a, b[$ whenever $a < b$ in $\mathbb{R}$.)

(g) Let ν be a Radon probability measure on $\mathbb{R}^r$ with bounded support (definition: 256Xf). Show that its characteristic function is smooth.

(h) Let X be a normal random variable with expectation a and variance σ^2. Show that $\mathbb{E}(e^X) = \exp(a + \frac{1}{2}\sigma^2)$.

>(i) Let $\boldsymbol{X} = (X_1, \ldots, X_r)$ be a family of real-valued random variables with characteristic function $\varphi_{\boldsymbol{X}}$. Suppose that $\varphi_{\boldsymbol{X}}$ is expressible in the form

$$\varphi_{\boldsymbol{X}}(y) = \prod_{j=1}^{r} \varphi_j(\eta_j)$$

for some functions $\varphi_1, \ldots, \varphi_r$, writing $y = (\eta_1, \ldots, \eta_r)$ as usual. Show that $X_1, \ldots, X_r$ are independent. (*Hint*: show that the φ_j must be multiples of the characteristic functions of the X_j; now show that the distribution of $\boldsymbol{X}$ has the same characteristic function as the product of the distributions of the X_j.)

(j) Let X_1, X_2 be independent real-valued random variables with the same distribution, and φ the characteristic function of $X_1 - X_2$. Show that $\varphi(t) = \varphi(-t) \geq 0$ for every $t \in \mathbb{R}$.

(k) Let ν be a Radon probability measure on $\mathbb{R}$, with characteristic function φ. Show that

$$\frac{1}{2}(\nu[c, d] + \nu\,]c, d[) = \frac{i}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{-idy} - e^{-icy}}{y}\varphi(y)dy$$

whenever $c < d$ in $\mathbb{R}$. (*Hint*: use part (a) of the proof of 283F.)

(l) Let X be a real-valued random variable and φ_X its characteristic function. Show that

$$\Pr(|X| \geq a) \leq 7a\int_0^{1/a}(1 - \mathcal{R}\mathrm{e}(\varphi_X(y))dy$$

for every $a > 0$.

(m) We say that a set Q of Radon probability measures on $\mathbb{R}$ is **uniformly tight** if for every $\epsilon > 0$ there is an $M \geq 0$ such that $\nu(\mathbb{R} \setminus [-M, M]) \leq \epsilon$ for every $\nu \in Q$. Show that if Q is any uniformly tight family of Radon probability measures on $\mathbb{R}$, and $\epsilon > 0$, then there are $\eta > 0$ and $y_0, \ldots, y_n \in \mathbb{R}$ such that

$$\nu\,]-\infty,a] \le \nu'\,]-\infty,a+\epsilon]+\epsilon$$

whenever ν, $\nu' \in Q$ and $|\varphi_\nu(y_j)-\varphi_{\nu'}(y_j)| \le \eta$ for every $j \le n$, writing φ_ν for the characteristic function of ν.

(n) Let $\langle \nu_n\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability measures on $\mathbb{R}$. Show that it converges for the vague topology to a Radon probability measure ν iff $\{\nu\}\cup\{\nu_n : n\in\mathbb{N}\}$ is uniformly tight in the sense of 285Xm and $\limsup_{n\to\infty}\nu_n\,]-\infty,a] \le \liminf_{n\to\infty}\nu_n\,]-\infty,b]$ whenever $a<b$ in $\mathbb{R}$. (*Hint*: Setting $g(x) = \inf\liminf_{n\to\infty}\nu_n\,]-\infty,b] : x<b\}$ for $x\in\mathbb{R}$, show that the Lebesgue-Stieltjes measure ν associated with g is a probability measure and $\langle\nu_n\rangle_{n\in\mathbb{N}}$ converges to ν for the vague topology.)

>(o) Let ν, ν' be two totally finite Radon measures on $\mathbb{R}^r$ which agree on all closed half-spaces, that is, sets of the form $\{x : x\,\mathbf{.}\,y \ge c\}$ where $y\in\mathbb{R}^r$ is non-zero and $c\in\mathbb{R}$. Show that $\nu=\nu'$. (*Hint*: reduce to the case $\nu\mathbb{R}^r=\nu'\mathbb{R}^r=1$ and use 285M.)

>(p) For $\gamma>0$, the **Cauchy distribution** with centre 0 and scale parameter γ is the Radon probability measure ν_γ defined by the formula

$$\nu_\gamma(E) = \tfrac{\gamma}{\pi}\int_E \tfrac{1}{\gamma^2+t^2}dt.$$

(i) Show that if X is a random variable with distribution ν_γ then $\Pr(X\ge 0)=\Pr(|X|\ge\gamma)=\frac{1}{2}$. (ii) Show that the characteristic function of ν_γ is $y\mapsto e^{-\gamma|y|}$. (*Hint*: 283Xq.) (iii) Show that if X and Y are independent random variables with Cauchy distributions, both centered at 0 and with scale parameters γ, δ respectively, and α, β are not both 0, then $\alpha X+\beta Y$ has a Cauchy distribution centered at 0 and with scale parameter $|\alpha|\gamma+|\beta|\delta$. (iv) Show that if X and Y are independent normally distributed random variables with expectation 0 then X/Y has a Cauchy distribution.

>(q) Let $X_1, X_2, \dots$ be an independent identically distributed sequence of random variables, all with zero expectation and variance 1; let φ be their common characteristic function. For each $n\ge 1$, set $S_n = \frac{1}{\sqrt{n}}(X_1+\ldots+X_n)$.

(i) Show that the characteristic function φ_n of S_n is given by the formula $\varphi_n(y) = (\varphi(\frac{y}{\sqrt{n}}))^n$ for each n.

(ii) Show that $|\varphi_n(y)-e^{-y^2/2}| \le n|\varphi(\frac{y}{\sqrt{n}})-e^{-y^2/2n}|$ for $n\ge 1$ and $y\in\mathbb{R}$.

(iii) Setting $h(y)=\varphi(y)-e^{-y^2/2}$, show that $h(0)=h'(0)=h''(0)=0$ and therefore that $\lim_{n\to\infty} nh(y/\sqrt{n})=0$, so that $\lim_{n\to\infty}\varphi_n(y)=e^{-y^2/2}$ for every $y\in\mathbb{R}$.

(iv) Show that $\lim_{n\to\infty}\Pr(S_n\le a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^a e^{-x^2/2}dx$ for every $a\in\mathbb{R}$.

>(r) A random variable X has a **Poisson distribution** with parameter $\lambda>0$ if $\Pr(X=n)=e^{-\lambda}\lambda^n/n!$ for every $n\in\mathbb{N}$. (i) Show that in this case $\mathbb{E}(X)=\mathrm{Var}(X)=\lambda$. (ii) Show that if X and Y are independent random variables with Poisson distributions then $X+Y$ has a Poisson distribution. (iii) Find a proof of (ii) based on 285Q.

>(s) For $x\in\mathbb{R}^r$, let δ_x be the Dirac measure on $\mathbb{R}^r$ concentrated at x. Show that $\delta_x*\delta_y=\delta_{x+y}$ for all x, $y\in\mathbb{R}^r$.

(t) Let P be the set of Radon probability measures on $\mathbb{R}^r$. For $y\in\mathbb{R}^r$, set $\rho'_y(\nu,\nu')=|\varphi_\nu(y)-\varphi_{\nu'}(y)|$ for all ν, $\nu'\in P$, writing φ_ν for the characteristic function of ν. Set $\psi(x)=\frac{1}{(\sqrt{2\pi})^r}e^{-x\,\mathbf{.}\,x/2}$ for $x\in\mathbb{R}^r$. Show that the vague topology on P is defined by the family $\{\rho_\psi\}\cup\{\rho'_y : y\in\mathbb{Q}^r\}$, defining ρ_ψ as in 285K, and is therefore metrizable. (*Hint*: 281K; cf. 285Xm.)

>(u) Let $\varphi:\mathbb{R}^r\to\mathbb{C}$ be the characteristic function of a Radon probability measure on $\mathbb{R}^r$. Show that $\varphi(0)=1$ and that $\sum_{j=0}^n\sum_{k=0}^n c_j\bar{c}_k\varphi(a_j-a_k)\ge 0$ whenever $a_0,\dots,a_n\in\mathbb{R}^r$ and $c_0,\dots,c_n\in\mathbb{C}$. ('Bochner's theorem' states that these conditions are sufficient, as well as necessary, for φ to be a characteristic function; see 445N in Volume 4.)

(v) Let $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability measures on $\mathbb{R}$ such that $\psi(y) = \lim_{n\to\infty}\varphi_{\nu_n}(y)$ is defined for every $y \in \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{C}$ is continuous at 0. Show that ψ is the characteristic function of a Radon probability measure ν and that $\langle\nu_n\rangle_{n\in\mathbb{N}}$ converges to ν for the vague topology. (*Hint*: as in part (c) of the proof of 285L, show that $\{\nu_n : n \in \mathbb{N}\}$ is uniformly tight. Show that there is a subsequence $\langle\nu_{n_k}\rangle_{k\in\mathbb{N}}$ such that $f(q) = \lim_{k\to\infty}\nu_{n_k}\,]-\infty, q]$ is defined for every $q \in \mathbb{Q}$. Use 285Xn to show that $\langle\nu_{n_k}\rangle_{k\in\mathbb{N}}$ converges for the vague topology.)

285Y Further exercises (a) Let ν be a Radon probability measure on $\mathbb{R}^r$. Write

$$\hat{\nu}(y) = \frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,.\,x}\nu(dx)$$

for every $y \in \mathbb{R}^r$.

(i) Writing φ_ν for the characteristic function of ν, show that $\hat{\nu}(y) = \frac{1}{(\sqrt{2\pi})^r}\varphi_\nu(-y)$ for every $y \in \mathbb{R}^r$.

(ii) Show that $\int h(y)\hat{\nu}(y)dy = \int \hat{h}(x)\nu(dx)$ for any Lebesgue integrable complex-valued function h on $\mathbb{R}^r$, defining the Fourier transform $\hat{h}$ as in 283Wa.

(iii) Show that $\int h(x)\nu(dx) = \int \check{h}(y)\hat{\nu}(y)dy$ for any rapidly decreasing test function h on $\mathbb{R}^r$.

(iv) Show that if ν is an indefinite-integral measure over Lebesgue measure, with Radon-Nikodým derivative f, then $\hat{\nu}$ is the Fourier transform of f.

(b) Let ν be a Radon probability measure on $\mathbb{R}^r$, with characteristic function φ. Show that whenever $c \le d$ in $\mathbb{R}^r$ then

$$\Bigl(\frac{i}{2\pi}\Bigr)^r \lim_{\alpha_1,\dots,\alpha_r\to\infty}\int_{[-a,a]}\Bigl(\prod_{j=1}^r \frac{e^{-i\delta_j\eta_j} - e^{-i\gamma_j\eta_j}}{\eta_j}\Bigr)\varphi(y)dy$$

exists and lies between $\nu\,]c,d[$ and $\nu[c,d]$, writing $a = (\alpha_1,\dots,\alpha_r)$ and $]c,d[\, = \prod_{j\le r}]\gamma_j,\delta_j[$ if $c = (\gamma_1,\dots,\gamma_r)$ and $d = (\delta_1,\dots,\delta_r)$.

(c) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent identically distributed sequence of (not-essentially-constant) random variables. Show that $\lim_{n\to\infty}\Pr(|\sum_{k=0}^n X_k| \le \alpha) = 0$ for every $\alpha \in \mathbb{R}$.

(d) For Radon probability measures ν, ν' on $\mathbb{R}^r$ set

$$\rho(\nu,\nu') = \inf\{\epsilon : \epsilon \ge 0,\ \nu\,]-\infty,a] \le \nu'\,]-\infty,a+\epsilon\mathbf{1}] + \epsilon \le \nu\,]-\infty,a+2\epsilon\mathbf{1}] + 2\epsilon \text{ for every } a \in \mathbb{R}^r\},$$

writing $]-\infty,a] = \{(\xi_1,\dots,\xi_r) : \xi_j \le \alpha_j \text{ for every } j \le r\}$ when $a = (\alpha_1,\dots,\alpha_r)$, and $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^r$. Show that ρ is a metric on the set of Radon probability measures on $\mathbb{R}^r$, and that the topology it defines is the vague topology. (Cf. 274Yc.)

(e) Let $r \ge 1$ and let P be the set of Radon probability measures on $\mathbb{R}^r$. For $m \in \mathbb{N}$ let ρ^*_m be the pseudometric on P defined by setting $\rho^*_m(\nu,\nu') = \sup_{\|y\|\le m}|\varphi_\nu(y) - \varphi_{\nu'}(y)|$ for ν, $\nu' \in P$, writing φ_ν for the characteristic function of ν. Show that $\{\rho^*_m : m \in \mathbb{N}\}$ defines the vague topology on P.

(f) Let $r \ge 1$. We say that a set Q of Radon probability measures on $\mathbb{R}^r$ is **uniformly tight** if for every $\epsilon > 0$ there is a compact set $K \subseteq \mathbb{R}^r$ such that $\nu(\mathbb{R}^r \setminus K) \le \epsilon$ for every $\nu \in Q$. Show that if Q is any uniformly tight family of Radon probability measures on $\mathbb{R}^r$, and $\epsilon > 0$, then there are $\eta > 0$, $y_0,\dots,y_n \in \mathbb{R}^r$ such that $\nu\,]-\infty,a] \le \nu'\,]-\infty,a+\epsilon\mathbf{1}] + \epsilon$ whenever ν, $\nu' \in Q$ and $a \in \mathbb{R}^r$ and $|\varphi_\nu(y_j) - \varphi_{\nu'}(y_j)| \le \eta$ for every $j \le n$, writing φ_ν for the characteristic function of ν.

(g) Show that for any $M \ge 0$ the set of Radon probability measures ν on $\mathbb{R}^r$ such that $\int \|x\|\nu(dx) \le M$ is uniformly tight in the sense of 285Yf.

(h) Let $C_b(\mathbb{R}^r)$ be the Banach space of bounded continuous real-valued functions on $\mathbb{R}^r$.

(i) Show that any Radon probability measure ν on $\mathbb{R}^r$ corresponds to a continuous linear functional $h_\nu : C_b(\mathbb{R}^r) \to \mathbb{R}$, writing $h_\nu(f) = \int f d\nu$ for $f \in C_b(\mathbb{R}^r)$.

(ii) Show that if $h_\nu = h_{\nu'}$ then $\nu = \nu'$.

(iii) Show that the vague topology on the set of Radon probability measures corresponds to the weak* topology on the dual $(C_b(\mathbb{R}^r))^*$ of $C_b(\mathbb{R}^r)$.

(i) Let $r \ge 1$ and let P be the set of Radon probability measures on $\mathbb{R}^r$. For $m \in \mathbb{N}$ let $\tilde{\rho}^*_m$ be the pseudometric on P defined by setting

$$\tilde{\rho}^*_m(\nu,\nu') = \int_{\{y:\|y\|\le m\}} |\varphi_\nu(y) - \varphi_{\nu'}(y)|dy$$

for ν, $\nu' \in P$, writing φ_ν for the characteristic function of ν. Show that $\{\tilde{\rho}^*_m : m \in \mathbb{N}\}$ defines the vague topology on P.

(j) Let (Ω,Σ,μ) be a probability space. Suppose that $\langle X_n\rangle_{n\in\mathbb{N}}$ is a sequence of real-valued random variables on Ω, and X another real-valued random variable on Ω; let φ_{X_n}, φ_X be the corresponding characteristic functions. Show that the following are equiveridical: (i) $\lim_{n\to\infty} \mathbb{E}(h(X_n)) = \mathbb{E}(h(X))$ for every bounded continuous function $h : \mathbb{R} \to \mathbb{R}$; (ii) $\lim_{n\to\infty} \varphi_{X_n}(y) = \varphi_X(y)$ for every $y \in \mathbb{R}$.

(k) Let (Ω,Σ,μ) be a probability space, and P the set of Radon probability measures on $\mathbb{R}$. (i) Show that we have a function $\psi : L^0(\mu) \to P$ defined by saying that $\psi(X^\bullet)$ is the distribution of X whenever X is a real-valued random variable on Ω. (ii) Show that ψ is continuous for the topology of convergence in measure on $L^0(\mu)$ and the vague topology on P. (Compare 271Yd.)

(l) Let X be a real-valued random variable with finite variance. Show that for any $\eta \ge 0$,

$$|\varphi(y) - 1 - iy\mathbb{E}(X) + \frac{1}{2}y^2\mathbb{E}(X^2)| \le \frac{1}{6}\eta|y|^3\mathbb{E}(X^2) + y^2\mathbb{E}(\psi_\eta(X)),$$

writing φ for the characteristic function of X and $\psi_\eta(x) = 0$ for $|x| \le \eta$, x^2 for $|x| > \eta$.

(m) Suppose that $\epsilon \ge \delta > 0$ and that $X_0,\dots,X_n$ are independent real-valued random variables such that

$$\mathbb{E}(X_k) = 0 \text{ for every } k \le n, \quad \sum_{k=0}^n \mathrm{Var}(X_k) = 1, \quad \sum_{k=0}^n \mathbb{E}(\psi_\delta(X_k)) \le \delta$$

(writing $\psi_\delta(x) = 0$ if $|x| \le \delta$, x^2 if $|x| > \delta$). Set $\gamma = \epsilon/\sqrt{\delta^2+\delta}$, and let Z be a standard normal random variable. Show that

$$|\varphi(y) - e^{-y^2/2}| \le \frac{1}{3}\epsilon|y|^3 + y^2(\delta + \mathbb{E}(\psi_\gamma(Z)))$$

for every $y \in \mathbb{R}$, writing φ for the characteristic function of $X = \sum_{k=0}^n X_k$.

(n) Show that for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $X_0,\dots,X_n$ are independent real-valued random variables such that

$$\mathbb{E}(X_k) = 0 \text{ for every } k \le n, \quad \sum_{k=0}^n \mathrm{Var}(X_k) = 1, \quad \sum_{k=0}^n \mathbb{E}(\psi_\delta(X_k)) \le \delta$$

(writing $\psi_\delta(x) = 0$ if $|x| \le \delta$, x^2 if $|x| > \delta$), then $|\varphi(y) - e^{-y^2/2}| \le \epsilon(y^2 + |y^3|)$ for every $y \in \mathbb{R}$, writing φ for the characteristic function of $X = X_0 + \ldots + X_n$.

(o) Use 285Yn to prove Lindeberg's theorem (274F).

(p) Let $r \ge 1$ and let P be the set of Radon probability measures on $\mathbb{R}^r$. Show that convolution, regarded as a map from $P \times P$ to P, is continuous when P is given the vague topology.

(q) Let $\mathfrak{S}$ be the topology on $\mathbb{R}$ defined by $\{\rho'_y : y \in \mathbb{R}\}$, where $\rho'_y(x,x') = |e^{iyx} - e^{iyx'}|$ (compare 285S). Show that addition and subtraction are continuous for $\mathfrak{S}$ in the sense of 2A5A.

(r) Let ν be a probability measure on $\mathbb{R}$. Show that $|\varphi_\nu(y) - \varphi_\nu(y')|^2 \le 2(1 - \mathcal{R}\mathrm{e}\,\varphi_\nu(y-y'))$ for any y, $y' \in \mathbb{R}$.

(s) Let $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$. Set $E = \{y : y \in \mathbb{R}, \lim_{n\to\infty} \varphi_{\nu_n}(y) = 1\}$. (i) Show that $E - E$ and $E + E$ are included in E. (ii) Show that if E is not Lebesgue negligible it is the whole of $\mathbb{R}$.

(t) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent sequence of real-valued random variables and set $S_n = \sum_{j=0}^{n} X_j$ for each $n \in \mathbb{N}$. Suppose that the sequence $\langle\nu_{S_n}\rangle_{n\in\mathbb{N}}$ of distributions is convergent for the vague topology to a distribution. Show that $\langle S_n\rangle_{n\in\mathbb{N}}$ converges in measure, therefore a.e.

285 Notes and comments Just as with Fourier transforms, the power of methods which use the characteristic functions of distributions is based on three points: (i) the characteristic function of a distribution determines the distribution (285M); (ii) the properties of interest in a distribution are reflected in accessible properties of its characteristic function (285G, 285I, 285J) (iii) these properties of the characteristic function are actually *different* from the corresponding properties of the distribution, and are amenable to different kinds of investigation. Above all, the fact that (for sequences!) convergence in the vague topology of distributions corresponds to pointwise convergence for characteristic functions (285L) provides us with a path to the classic limit theorems, as in 285Q and 285Xq. In 285S-285U I show that this result for sequences does not correspond immediately to any alternative characterization of the vague topology, though it can be adapted in more than one way to give such a characterization (see 285Ye-285Yi).

Concerning the Central Limit Theorem there is one conspicuous difference between the method suggested here and that of §274. The previous approach offered at least a theoretical possibility of giving an explicit formula for δ in 274F as a function of ϵ, and hence an estimate of the rate of convergence to be expected in the Central Limit Theorem. The arguments in the present chapter, involving as they do an entirely non-constructive compactness argument in 281A, leave us with no way of achieving such an estimate. But in fact the method of characteristic functions, suitably refined, is the basis of the best estimates known, such as the Berry-Esséen theorem (274Hc).

In 285D I try to show how the characteristic function φ_ν of a Radon probability measure can be related to a 'Fourier transform' $\hat\nu$ of ν which corresponds directly to the Fourier transforms of functions discussed in §§283-284. If f is a non-negative Lebesgue integrable function and we take ν to be the corresponding indefinite-integral measure, then $\hat\nu = \hat f$. Thus the concept of 'Fourier transform of a measure' is a natural extension of the Fourier transform of an integrable function. Looking at it from the other side, the formula of 285Dc shows that ν can be thought of as representing the inverse Fourier transform of $\hat\nu$ in the sense of 284H-284I. Taking ν to be the measure which assigns a mass 1 to the point 0, we get the Dirac delta function, with Fourier transform the constant function $\chi\mathbb{R}$. These ideas can be extended without difficulty to handle convolutions of measures (285R).

It is a striking fact that while there is no satisfactory characterization of the functions which are Fourier transforms of integrable functions, there is a characterization of the characteristic functions of probability distributions. This is 'Bochner's theorem'. I give the condition in 285Xu, asking you to prove its necessity as an exercise; we already have three-quarters of the machinery to prove its sufficiency, but the last step will have to wait for Volume 4.

Version of 30.3.16

286 Carleson's theorem

Carleson's theorem (Carleson 66) was the (unexpected) solution to a long-standing problem. Remarkably, it can be proved by 'elementary' methods. The hardest part of the work below, in 286J-286L, demands only the laborious verification of inequalities. How the inequalities were chosen is a different matter; for once, some of the ideas of the proof are embodied in the statements of the lemmas. The argument here is a greatly expanded version of Lacey & Thiele 00.

The Hardy-Littlewood Maximal Theorem (286A) is important, and worth learning even if you leave the rest of the section as an unexamined monument. I bring 286B-286D forward to the beginning of the section, even though they are little more than worked exercises, because they also have potential uses in other contexts.

The complexity of the argument is such that it is useful to introduce a substantial number of special notations. Rather than include these in the general index, I give a list in 286W. Among them are ten constants $C_1, \dots, C_{10}$. The values of these numbers are of no significance. The method of proof here is quite inappropriate if we want to estimate rates of convergence. I give recipes for the calculation of the C_n only for the sake of the linear logic in which this treatise is written, and because they occasionally offer clues concerning the tactics being used.

In this section all integrals are with respect to Lebesgue measure μ on $\mathbb{R}$ unless otherwise stated.

286A The Maximal Theorem Suppose that $1 < p < \infty$ and that $f \in \mathcal{L}^p_{\mathbb{C}}(\mu)$ (definition: 244P). Set

$$f^*(x) = \sup\{\frac{1}{b-a}\int_a^b |f| : a \le x \le b,\ a < b\}$$

for $x \in \mathbb{R}$. Then $\|f^*\|_p \le \dfrac{2^{1/p}p}{p-1}\|f\|_p$.

proof (a) It is enough to consider the case $f = |f|$. Note that if $E \subseteq \mathbb{R}$ has finite measure, then

$$\int_E f = \int (f \times \chi E) \times \chi E \le \|f \times \chi E\|_p (\mu E)^{1/q} \le \|f\|_p (\mu E)^{1/q}$$

is finite, where $q = \frac{p}{p-1}$, by Hölder's inequality (244Eb). Consequently, if $t > 0$ and $\int_E f \ge t\mu E$, we must have $t\mu E \le \|f \times \chi E\|_p (\mu E)^{1/q}$, $t(\mu E)^{1/p} \le \|f \times \chi E\|_p$ and

$$\mu E \le \frac{1}{t^p}\|f \times \chi E\|_p^p = \frac{1}{t^p}\int_E f^p.$$

(b) For $t > 0$, set

$$G_t = \{x : t(y - x) < \int_x^y f \text{ for some } y > x\}.$$

(i) G_t is an open set. **P** For any $y \in \mathbb{R}$,

$$G_{ty} = \{x : x < y,\ t(y - x) < \int_x^y f\}$$

is open, because $x \mapsto t(y - x)$ and $x \mapsto \int_x^y f$ are continuous (225A); so $G_t = \bigcup_{y\in\mathbb{R}} G_{ty}$ is open. **Q**

(ii) By 2A2I, there is a partition $\mathcal{C}$ of G_t into open intervals. Now C is bounded and $t\mu C \le \int_C f$ for every $C \in \mathcal{C}$.

P(α) For $x \in C$, consider $F_x = \{y : y \ge x,\ t(y - x) \le \int_x^y f\}$. $x \in F_x$ and $y - x \le \frac{1}{t^p}\int_{-\infty}^{\infty} f^p$ for every $y \in F_x$, by (a), so F_x is bounded above. Set $z_x = \sup F_x$. Because $y \mapsto t(y - x) - \int_x^y f$ is continuous, $z_x \in F_x$. **?** If $z_x \in G_t$, there is a $y > z_x$ such that $t(y - z_x) < \int_{z_x}^y f$; but now

$$t(y - x) \le \int_x^{z_x} f + \int_{z_x}^y f = \int_x^y f$$

and $y \in F_x$, which is impossible. **X** Thus $z_x \notin G_t$ and $z_x \notin C$, so that z_x is an upper bound of C.

(β) This shows that

$$\sup C \le z_x \le x + \frac{1}{t^p}\int_{-\infty}^{\infty} f^p$$

for every $x \in C$. So in fact C is bounded and is of the form $]a, b[$ where $a < b$ in $\mathbb{R}$. **?** If $t(b-a) > \int_a^b f$, there is an $x \in]a, b[$ such that $t(b-x) > \int_x^b f$. Now we know that $b \le z_x$ and $b \notin G_t$, so we have $t(z_x - b) \ge \int_b^{z_x} f$. Adding, $t(z_x - x) > \int_x^{z_x} f$ and $z_x \notin F_x$. **X**

(γ) Thus $t\mu C \le \int_C f$, as claimed. **Q**

(iii) Accordingly, because $\mathcal{C}$ is countable and f is non-negative, we can apply (a) in its full strength to see that

$$\mu G_t = \sum_{C\in\mathcal{C}} \mu C \le \sum_{C\in\mathcal{C}} \frac{1}{t^p}\int_C f^p \le \frac{1}{t^p}\int_{-\infty}^{\infty} f^p$$

is finite, and

$$\int_{G_t} f = \sum_{C\in\mathcal{C}} \int_C f \ge \sum_{C\in\mathcal{C}} t\mu C = t\mu G_t.$$

(c) All this is true for every $t > 0$. Now if we set

$$f_1^*(x) = \sup_{b>x} \frac{1}{b-x}\int_b^a f$$

for $x \in \mathbb{R}$, we have $\{x : f_1^*(x) > t\} = G_t$ for every $t > 0$.

For any $t > 0$,

$$\frac{1}{p}t\mu G_t = (1-\frac{1}{q})t\mu G_t \le \int_{G_t} f - \frac{1}{q}t\chi\mathbb{R} \le \int_{-\infty}^{\infty}(f - \frac{1}{q}t\chi\mathbb{R})^+.$$

So

$$\int_{-\infty}^{\infty}(f_1^*)^p = \int_0^{\infty} \mu\{x : f_1^*(x)^p > t\}dt$$

(see 252O)

$$= p\int_0^{\infty} u^{p-1}\mu\{x : f_1^*(x) > u\}du$$

(substituting $t = u^p$)

$$= p\int_0^{\infty} u^{p-1}\mu G_u du \le p^2\int_0^{\infty} u^{p-2}\Bigl(\int_{-\infty}^{\infty}(f - \frac{1}{q}u\chi\mathbb{R})^+\Bigr)du$$

$$= p^2\int_{-\infty}^{\infty}\int_0^{\infty} \max(0, f(x) - \frac{1}{q}u)u^{p-2}dudx$$

(by Fubini's theorem, 252B, because $(x,u) \mapsto u^{p-2}\max(0, f(x) - \frac{1}{q}u)$ is measurable and non-negative)

$$= p^2\int_{-\infty}^{\infty}\int_0^{qf(x)} u^{p-2}(f(x) - \frac{1}{q}u)dudx$$

$$= \frac{p^2q^{p-1}}{p(p-1)}\int_{-\infty}^{\infty} f^p = (\frac{p}{p-1})^p\|f\|_p^p.$$

(d) Similarly, setting $f_2^*(x) = \sup_{a<x}\frac{1}{x-a}\int_a^x f$ for $x \in \mathbb{R}$, $\int_{-\infty}^{\infty}(f_2^*)^p \le (\frac{p}{p-1})^p\|f\|_p^p$. But $f^* = \max(f_1^*, f_2^*)$. **P** Of course $f_1^* \le f^*$ and $f_2^* \le f^*$. But also, if $f^*(x) > t$, there must be a non-trivial interval I containing x such that $\int_I f > t\mu I$; if $a = \inf I$ and $b = \sup I$, then either $\int_a^x f > (x-a)t$ and $f_2^*(x) > t$, or $\int_x^b f > (b-x)t$ and $f_1^*(x) > t$. As x and t are arbitrary, $f^* = \max(f_1^*, f_2^*)$. **Q**

Accordingly

$$\|f^*\|_p^p = \int_{-\infty}^{\infty}(f^*)^p = \int_{-\infty}^{\infty}\max((f_1^*)^p, (f_2^*)^p)$$

$$\le \int_{-\infty}^{\infty}(f_1^*)^p + (f_2^*)^p \le 2(\frac{p}{p-1})^p\|f\|_p^p.$$

Taking pth roots, we have the inequality we seek.

286B Lemma Let $g : \mathbb{R} \to [0,\infty[$ be a function which is non-decreasing on $]-\infty,\alpha]$, non-increasing on $[\beta,\infty[$ and constant on $[\alpha,\beta]$, where $\alpha \le \beta$. Then for any measurable function $f : \mathbb{R} \to [0,\infty]$, $\int_{-\infty}^{\infty} f \times g \le \int_{-\infty}^{\infty} g \cdot \sup_{a\le\alpha,b\ge\beta,a<b}\frac{1}{b-a}\int_a^b f$.

proof Set $\gamma = \sup_{a\le\alpha,b\ge\beta,a<b}\frac{1}{b-a}\int_a^b f$. For $n, k \in \mathbb{N}$ set $E_{nk} = \{x : \alpha-2^n \le x \le \beta+2^n, g(x) \ge 2^{-n}(k+1)\}$, so that E_{nk} is either empty or a bounded interval including $[\alpha,\beta]$, and $\int_{E_{nk}} f \le \gamma\mu E_{nk}$. For $n \in \mathbb{N}$, set $g_n = 2^{-n}\sum_{k=0}^{4^n-1}\chi E_{nk}$; then $\langle g_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of functions with supremum g, and

$$\int_{-\infty}^{\infty} f \times g = \sup_{n\in\mathbb{N}} \int_{-\infty}^{\infty} f \times g_n = \sup_{n\in\mathbb{N}} 2^{-n} \sum_{k=0}^{4^n-1} \int_{E_{nk}} f$$

$$\leq \sup_{n\in\mathbb{N}} 2^{-n} \sum_{k=0}^{4^n-1} \gamma\mu E_{nk} = \sup_{n\in\mathbb{N}} \gamma \int_{-\infty}^{\infty} g_n = \gamma \int_{-\infty}^{\infty} g,$$

as claimed.

Remark Compare 224J.

286C Shift, modulation and dilation Some of the calculations below will be easier if we use the following formalism. For any function f with domain included in $\mathbb{R}$, and $\alpha \in \mathbb{R}$, we can define

$$(S_\alpha f)(x) = f(x+\alpha), \quad (M_\alpha f)(x) = e^{i\alpha x} f(x), \quad (D_\alpha f)(x) = f(\alpha x)$$

whenever the right-hand sides are defined. In the case of $S_\alpha f$ and $D_\alpha f$ it is sometimes convenient to allow $\pm\infty$ as a value of the function. We have the following elementary facts.

(a) $S_{-\alpha} S_\alpha f = f$, $D_{1/\alpha} D_\alpha f = f$ if $\alpha \neq 0$.

(b) $S_\alpha(f \times g) = S_\alpha f \times S_\alpha g$, $D_\alpha(f \times g) = D_\alpha f \times D_\alpha g$.

(c) $D_\alpha |f| = |D_\alpha f|$.

(d) If f is integrable, then

$$(M_\alpha f)^\wedge = S_{-\alpha} \hat{f}, \quad (S_\alpha f)^\wedge = M_\alpha \hat{f}, \quad (S_\alpha f)^\vee = M_{-\alpha} \check{f};$$

if moreover $\alpha > 0$, then

$$\alpha (D_\alpha f)^\wedge = D_{1/\alpha} \hat{f}, \quad \alpha (D_\alpha f)^\vee = D_{1/\alpha} \check{f}$$

(283Cc-283Ce).

(e) If f belongs to $\mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$, so do $S_\alpha f$, $M_\alpha f$ and (if $\alpha \neq 0$) $D_\alpha f$, and in this case

$$\|S_\alpha f\|_1 = \|M_\alpha f\|_1 = \|f\|_1, \quad \|D_\alpha f\|_1 = \frac{1}{|\alpha|} \|f\|_1.$$

(f) If f belongs to $\mathcal{L}^2_{\mathbb{C}}$ so do $S_\alpha f$, $M_\alpha f$ and (if $\alpha \neq 0$) $D_\alpha f$, and in this case

$$\|S_\alpha f\|_2 = \|M_\alpha f\|_2 = \|f\|_2, \quad \|D_\alpha f\|_2 = \frac{1}{\sqrt{|\alpha|}} \|f\|_2.$$

(g) If h is a rapidly decreasing test function (284A), so are $M_\alpha h$ and $S_\alpha h$ and (if $\alpha \neq 0$) $D_\alpha h$.

286D Lemma Suppose that $g : \mathbb{R} \to [0, \infty]$ is a measurable function such that, for some constant $C \geq 0$, $\int_E g \leq C\sqrt{\mu E}$ whenever $\mu E < \infty$. Then g is finite almost everywhere and $\int_{-\infty}^{\infty} \frac{1}{1+|x|} g(x) dx$ is finite.

proof For any $n \geq 1$, set $E_n = \{x : |x| \leq n, \, g(x) \geq n\}$; then

$$n\mu E_n \leq \int_{E_n} g \leq C\sqrt{\mu E_n},$$

so $\mu E_n \leq \frac{C^2}{n^2}$ and

$$\{x : g(x) = \infty\} = \bigcap_{n\geq 1} \bigcup_{m\geq n} E_m$$

has measure at most $\inf_{n\geq 1} \sum_{m=n}^{\infty} \mu E_m = 0$.

As for the integral, set $G(x) = \int_0^x g$ for $x \geq 0$. Then, for any $a \geq 0$,

$$\int_0^a \frac{g(x)}{1+x}dx = \frac{G(a)}{1+a} + \int_0^a \frac{G(x)}{(1+x)^2}dx$$

(225F)

$$\leq C\big(\frac{\sqrt{a}}{1+a} + \int_0^a \frac{\sqrt{x}}{(1+x)^2}dx\big) \leq C\big(1 + \int_0^\infty \frac{\sqrt{x}}{(1+x)^2}dx\big),$$

so

$$\int_0^\infty \frac{g(x)}{1+x}dx \leq C\Big(1 + \int_0^\infty \frac{\sqrt{x}}{(1+x)^2}dx\Big)$$

is finite. Similarly, $\int_{-\infty}^0 \frac{g(x)}{1-x}dx$ is finite, so we have the result.

286E The Lacey-Thiele construction (a) Let $\mathcal{I}$ be the family of all **dyadic intervals** of the form $\left[2^k n, 2^k(n+1)\right[$ where k, $n \in \mathbb{Z}$. The essential geometric property of $\mathcal{I}$ is that if I, $J \in \mathcal{I}$ then either $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$. Let Q be the set of all pairs $\sigma = (I_\sigma, J_\sigma) \in \mathcal{I}^2$ such that $\mu I_\sigma \cdot \mu J_\sigma = 1$. For $\sigma \in Q$, let $k_\sigma \in \mathbb{Z}$ be such that $\mu J_\sigma = 2^{k_\sigma}$ and $\mu I_\sigma = 2^{-k_\sigma}$; let x_σ be the midpoint of I_σ, y_σ the midpoint of J_σ, $J^l_\sigma \in \mathcal{I}$ the left-hand half-interval of J_σ, $J^r_\sigma \in \mathcal{I}$ the right-hand half-interval of J_σ, and y^l_σ the lower quartile of J_σ, that is, the midpoint of J^l_σ.

(b) There is a rapidly decreasing test function ϕ such that $\hat{\phi}$ is real-valued and $\chi[-\frac{1}{6}, \frac{1}{6}] \leq \hat{\phi} \leq \chi[-\frac{1}{5}, \frac{1}{5}]$. **P** Look at parts (b)-(d) of the proof of 284G. The process there can be used to provide us with a smooth function ψ_1 which is zero outside the interval $[\frac{1}{6}, \frac{1}{5}]$ and strictly positive on $\left]\frac{1}{6}, \frac{1}{5}\right[$; multiplying by a suitable factor, we can arrange that $\int_{-\infty}^{\infty} \psi_1 = 1$. So if we set $\psi_2(x) = 1 - \int_{-\infty}^x \psi_1$ for $x \in \mathbb{R}$, ψ_2 will be smooth, and $\chi\left]-\infty, \frac{1}{6}\right] \leq \psi_2 \leq \chi\left]-\infty, \frac{1}{5}\right]$. Now set $\psi_0(x) = \psi_2(x)\psi_2(-x)$ for $x \in \mathbb{R}$, and $\phi = \check{\psi}_0$; $\hat{\phi} = \psi_0$ (284C) will have the required property. **Q**

For $\sigma \in Q$, set $\phi_\sigma = 2^{k_\sigma/2} M_{y^l_\sigma} S_{-x_\sigma} D_{2^{k_\sigma}} \phi$, so that

$$\phi_\sigma(x) = \sqrt{\mu J_\sigma} e^{i y^l_\sigma x} \phi((x - x_\sigma)\mu J_\sigma).$$

Observe that ϕ_σ is a rapidly decreasing test function. Now $\hat{\phi}_\sigma = 2^{-k_\sigma/2} S_{-y^l_\sigma} M_{-x_\sigma} D_{2^{-k_\sigma}} \hat{\phi}$, that is,

$$\hat{\phi}_\sigma(y) = \sqrt{\mu I_\sigma} e^{-i x_\sigma (y - y^l_\sigma)} \hat{\phi}((y - y^l_\sigma)\mu I_\sigma),$$

which is zero unless $|y - y^l_\sigma| \leq \frac{1}{5}\mu J_\sigma$; since the length of J^l_σ is $\frac{1}{2}\mu J_\sigma$, this can be so only when $y \in J^l_\sigma$. We have the following simple facts.

(i) $\|\phi_\sigma\|_2 = \sqrt{\mu J_\sigma} \cdot \sqrt{\mu I_\sigma} \|\phi\|_2 = \|\phi\|_2$ for every $\sigma \in Q$.

(ii) $\|\hat{\phi}_\sigma\|_1 = \sqrt{\mu I_\sigma} \cdot \mu J_\sigma \|\hat{\phi}\|_1 = \sqrt{\mu J_\sigma} \|\hat{\phi}\|_1$ for every $\sigma \in Q$.

(iii) If σ, $\tau \in Q$ and $J^l_\sigma \cap J^l_\tau = \emptyset$ then

$$(\phi_\sigma | \phi_\tau) = (\hat{\phi}_\sigma | \hat{\phi}_\tau) = 0,$$

by 284Ob. (For f, $g \in \mathcal{L}^2_{\mathbb{C}}$, I write $(f|g)$ for $\int_{-\infty}^{\infty} f \times \bar{g}$.)

(iv) If σ, $\tau \in Q$ and $J_\sigma \neq J_\tau$ and $J^r_\sigma \cap J^r_\tau$ is non-empty, then $J^l_\sigma \cap J^l_\tau = \emptyset$ so $(\phi_\sigma | \phi_\tau) = 0$.

(c) Set $w(x) = \frac{1}{(1+|x|)^3}$ for $x \in \mathbb{R}$. For $\sigma \in Q$, set $w_\sigma = 2^{k_\sigma} S_{-x_\sigma} D_{2^{k_\sigma}} w$, so that

$$w_\sigma(x) = w((x - x_\sigma)\mu J_\sigma)\mu J_\sigma \leq \mu J_\sigma = 2^{k_\sigma}$$

for every x. Note that $w_\sigma = w_\tau$ whenever $I_\sigma = I_\tau$.

286F A partial order (a) For σ, $\tau \in Q$ say that $\tau \leq \sigma$ if $J_\tau \subseteq J_\sigma$ and $I_\sigma \subseteq I_\tau$. Then $\leq$ is a partial order on Q. We have the following elementary facts.

(i) If $\tau \leq \sigma$, then $k_\tau \leq k_\sigma$.

(ii) If σ and τ are incomparable (that is, $\sigma \not\leq \tau$ and $\tau \not\leq \sigma$), then $(I_\sigma \times J_\sigma) \cap (I_\tau \times J_\tau)$ is empty. **P** We may suppose that $k_\sigma \leq k_\tau$. If $J_\sigma \cap J_\tau \neq \emptyset$, then $J_\sigma \subseteq J_\tau$, because both are dyadic intervals, and J_σ is the shorter; but as $\sigma \not\leq \tau$, this means that $I_\tau \not\subseteq I_\sigma$ and $I_\sigma \cap I_\tau = \emptyset$. **Q**

(iii) If σ, σ' are incomparable and both greater than or equal to τ, then $I_\sigma \cap I_{\sigma'} = \emptyset$, because $J_\tau \subseteq J_\sigma \cap J_{\sigma'}$.

(iv) If $\tau \leq \sigma$ and $k_\tau \leq k \leq k_\sigma$, then there is a (unique) υ such that $\tau \leq \upsilon \leq \sigma$ and $k_\upsilon = k$. (The point is that there is a unique $I \in \mathcal{I}$ such that $I_\sigma \subseteq I \subseteq I_\tau$ and $\mu I = 2^{-k}$; and similarly there is just one candidate for J_υ.)

(b) It will be convenient to have a shorthand for the following: if $R \subseteq Q$, say that

$$R^+ = \bigcup_{\tau \in R} \{\sigma : \tau \leq \sigma \in Q\}.$$

(c) For $\tau \in Q$ set

$$T_\tau = \{\sigma : \sigma \in Q,\ \tau \leq \sigma,\ J^r_\tau \subseteq J^r_\sigma\}.$$

Note that if σ, $\sigma' \in T_\tau$ and $k_\sigma \neq k_{\sigma'}$ then $J_\sigma \neq J_{\sigma'}$ and $J^r_\sigma \cap J^r_{\sigma'} \neq \emptyset$, so $(\phi_\sigma|\phi_{\sigma'}) = 0$ (286E(b-iv)).

286G We shall need the results of some elementary calculations. The first four are nearly trivial.

Lemma (a) $\int_{-\infty}^{\infty} w_\sigma = \int_{-\infty}^{\infty} w = 1$ for every $\sigma \in Q$.

(b) For any $m \in \mathbb{N}$, $\sum_{n=m}^{\infty} w(n+\frac{1}{2}) \leq \frac{1}{2(1+m)^2}$.

(c) Suppose that $\sigma \in Q$ and that I is an interval not containing x_σ in its interior. Then $\int_I w_\sigma \geq w_\sigma(x)\mu I$, where x is the midpoint of I.

(d) For any $x \in \mathbb{R}$, $\sum_{n=-\infty}^{\infty} w(x-n) \leq 2$.

(e) There is a constant $C_1 \geq 0$ such that $|\phi(x)| \leq C_1 \min(w(3), w(x)^2)$ for every $x \in \mathbb{R}$ and

$$|\phi_\sigma(x)| \leq C_1\sqrt{\mu I_\sigma}\, w_\sigma(x) \min(1, w_\sigma(x)\mu I_\sigma)$$

for every $x \in \mathbb{R}$ and $\sigma \in Q$.

(f) There is a constant $C_2 \geq 0$ such that $\int_{-\infty}^{\infty} w(x)w(\alpha x+\beta)dx \leq C_2 w(\beta)$ whenever $0 \leq \alpha \leq 1$ and $\beta \in \mathbb{R}$.

(g) There is a constant $C_3 \geq 0$ such that $|(\phi_\sigma|\phi_\tau)| \leq C_3\sqrt{\mu I_\sigma}\sqrt{\mu J_\tau}\int_{I_\tau} w_\sigma$ whenever σ, $\tau \in Q$ and $k_\sigma \leq k_\tau$.

(h) There is a constant $C_4 \geq 0$ such that

$$\sum_{\sigma \in Q, \sigma \geq \tau, k_\sigma = k} \int_{\mathbb{R}\setminus I_\tau} w_\sigma \leq C_4$$

whenever $\tau \in Q$ and $k \in \mathbb{Z}$.

proof (a) Immediate from the definition in 286Ec, the formulae in 286Ce and the fact that $\int_0^\infty \frac{1}{(1+x)^3}dx = \frac{1}{2}$.

(b) The point is just that w is convex on $]-\infty, 0]$ and $[0, \infty[$. So we can apply 233Ib with $f(x) = x$, or argue directly from the fact that $w(n+\frac{1}{2}) \leq \frac{1}{2}(w(n+\frac{1}{2}+x) + w(n+\frac{1}{2}-x))$ for $|x| \leq \frac{1}{2}$, to see that $w(n+\frac{1}{2}) \leq \int_n^{n+1} w$ for every $n \geq 0$. Accordingly

$$\sum_{n=m}^{\infty} w(n+\tfrac{1}{2}) \leq \int_m^\infty w = \frac{1}{2(1+m)^2}.$$

(c) Similarly, because I lies all on the same side of x_σ, w_σ is convex on I, so the same inequality yields $w_\sigma(x)\mu I \leq \int_I w_\sigma$.

(d) Let m be such that $|x-m| \leq \frac{1}{2}$. Then, using the same inequalities as before to estimate $w(x-n)$ for $n \neq m$, we have

$$\sum_{n=-\infty}^{\infty} w(x-n) \le w(x-m) + \int_{-\infty}^{x-m-\frac{1}{2}} w + \int_{x-m+\frac{1}{2}}^{\infty} w$$
$$\le 1 + \int_{-\infty}^{\infty} w = 2.$$

(e) Because $\lim_{x\to\infty} x^6\phi(x) = \lim_{x\to-\infty} x^6\phi(x) = 0$, there is a $C_1 > 0$ such that $|\phi(x)| \le C_1 \min(w(3), w(x)^2)$ for every $x \in \mathbb{R}$. Now $|\phi(x)| \le C_1 w(x)^2 = C_1 w(x)\min(1, w(x))$ for every x, so

$$|\phi_\sigma(x)| = \sqrt{\mu J_\sigma}|\phi((x-x_\sigma)\mu J_\sigma| \le C_1\sqrt{\mu J_\sigma}w((x-x_\sigma)\mu J_\sigma)\min(1, w((x-x_\sigma)\mu J_\sigma))$$
$$= C_1\sqrt{\mu J_\sigma}w_\sigma(x)\mu I_\sigma \min(1, w_\sigma(x)\mu I_\sigma) = C_1\sqrt{\mu I_\sigma}w_\sigma(x)\min(1, w_\sigma(x)\mu I_\sigma)$$

whenever $\sigma \in Q$ and $x \in \mathbb{R}$.

(f)(i) The first step is to note that

$$\frac{w(\frac{1}{2}(1+\beta))}{w(\beta)} = \frac{8(1+\beta)^3}{(3+\beta)^3} \le 8$$

for every $\beta \ge 0$. Now $\alpha w(\alpha + \alpha\beta) \le 4w(\beta)$ whenever $\beta \ge 0$ and $\alpha \ge \frac{1}{2}$. **P** For $t \ge \frac{1}{2}$,

$$\frac{d}{dt}tw(t+t\beta) = \frac{1-2t(1+\beta)}{(1+t+t\beta)^4} \le 0,$$

so

$$\alpha w(\alpha+\alpha\beta) \le \tfrac{1}{2}w(\tfrac{1}{2}+\tfrac{1}{2}\beta) \le 4w(\beta). \text{ **Q**}$$

Of course this means that

$$\frac{1}{\alpha}w(\frac{1+\beta}{2\alpha}) \le 8w(\beta)$$

whenever $\beta \ge 0$ and $0 < \alpha \le 1$.

(ii) Try $C_2 = 16$. If $0 < \alpha \le 1$ and $\beta \ge 0$, set $\gamma = \frac{1+\beta}{2\alpha}$. Then, for any $x \ge -\gamma$,

$$1 + \alpha x + \beta = (1+\beta)(1 + \frac{\alpha x}{1+\beta}) \ge \frac{1}{2}(1+\beta),$$

so $w(\alpha x + \beta) \le 8w(\beta)$ and

$$\int_{-\gamma}^{\infty} w(x)w(\alpha x+\beta)dx \le 8w(\beta)\int_{-\gamma}^{\infty} w \le 8w(\beta).$$

On the other hand,

$$\int_{-\infty}^{-\gamma} w(x)w(\alpha x+\beta)dx \le w(\gamma)\int_{-\infty}^{\infty} w(\alpha x+\beta)dx$$
$$= \frac{1}{\alpha}w(\frac{1+\beta}{2\alpha})\int_{-\infty}^{\infty} w \le 8w(\beta).$$

Putting these together, $\int_{-\infty}^{\infty} w(x)w(\alpha x+\beta)dx \le 16w(\beta)$; and this is true whenever $0 < \alpha \le 1$ and $\beta \ge 0$.

(iii) If $\alpha = 0$, then

$$\int_{-\infty}^{\infty} w(x)w(\alpha x+\beta)dx = w(\beta)\int_{-\infty}^{\infty} w = w(\beta) \le C_2 w(\beta)$$

for any β. If $0 < \alpha \le 1$ and $\beta < 0$, then

$$\int_{-\infty}^{\infty} w(x)w(\alpha x+\beta)dx = \int_{-\infty}^{\infty} w(-x)w(-\alpha x-\beta)dx$$

(because w is an even function)

$$= \int_{-\infty}^{\infty} w(x)w(\alpha x - \beta)dx \le C_2 w(-\beta)$$

(by (ii) above)

$$= C_2 w(\beta).$$

So we have the required inequality in all cases.

(g) Set $C_3 = \max(C_1^2 C_2, \|\phi\|_2^2 / \int_{-1/2}^{1/2} w)$.

(i) It is worth disposing immediately of the case $\sigma = \tau$. In this case,

$$|(\phi_\sigma|\phi_\tau)| = \|\phi_\sigma\|_2^2 = \|\phi\|_2^2,$$

while

$$\int_{I_\tau} w_\sigma = \mu J_\sigma \int_{x_\sigma - \frac{1}{2}\mu I_\sigma}^{x_\sigma + \frac{1}{2}\mu I_\sigma} w((x - x_\sigma)\mu J_\sigma)dx = \int_{-1/2}^{1/2} w,$$

so certainly $|(\phi_\sigma|\phi_\tau)| \le C_3 \int_{I_\tau} w_\sigma$.

(ii) If $\sigma \ne \tau$ and $I_\sigma = I_\tau$ then $J_\sigma \cap J_\tau = \emptyset$ so $(\phi_\sigma|\phi_\tau) = 0$, by 286E(b-iii).

(iii) Now suppose that $I_\sigma \ne I_\tau$. In this case, because $\mu I_\tau \le \mu I_\sigma$, I_τ must lie all on the same side of x_σ, so $\int_{I_\tau} w_\sigma \ge w_\sigma(x_\tau)\mu I_\tau$, by (c). Accordingly

$$|(\phi_\sigma|\phi_\tau)| \le \int_{-\infty}^{\infty} |\phi_\sigma| \times |\phi_\tau| \le C_1^2 \sqrt{\mu I_\sigma}\sqrt{\mu I_\tau} \int_{-\infty}^{\infty} w_\sigma \times w_\tau$$

(using (e) twice)

$$= C_1^2 \sqrt{\mu J_\sigma}\sqrt{\mu J_\tau} \int_{-\infty}^{\infty} w((x - x_\sigma)\mu J_\sigma) w((x - x_\tau)\mu J_\tau)dx$$

$$= C_1^2 \sqrt{\mu J_\sigma}\sqrt{\mu I_\tau} \int_{-\infty}^{\infty} w(x\mu J_\sigma \mu I_\tau + (x_\tau - x_\sigma)\mu J_\sigma) w(x)dx$$

$$\le C_1^2 C_2 \sqrt{\mu J_\sigma}\sqrt{\mu I_\tau} w((x_\tau - x_\sigma)\mu J_\sigma)$$

(by (f), since $\mu J_\sigma \mu I_\tau \le 1$)

$$\le C_3 \sqrt{\mu I_\sigma}\sqrt{\mu I_\tau} w_\sigma(x_\tau) \le C_3 \sqrt{\mu I_\sigma}\sqrt{\mu J_\tau} \int_{I_\tau} w_\sigma,$$

as required.

(h) Set $C_4 = 2\sum_{j=0}^{\infty} \int_{j+\frac{1}{2}}^{\infty} w$; this is finite because $\int_\alpha^\infty w = \frac{1}{2(1+\alpha)^2}$ for every $\alpha \ge 0$.

If $k < k_\tau$ then $k_\sigma \ne k$ for any $\sigma \ge \tau$, so the result is trivial. If $k \ge k_\tau$, then for each dyadic subinterval I of I_τ of length 2^{-k} there is exactly one $\sigma \ge \tau$ such that $I_\sigma = I$, since J_σ must be the unique dyadic interval of length 2^k including J_τ. List these as $\sigma_0, \dots$ in ascending order of the centres x_{σ_j}, so that if $I_\tau = [m\mu I_\tau, (m+1)\mu I_\tau[$ then $x_{\sigma_j} = m\mu I_\tau + 2^{-k}(j + \frac{1}{2})$, for $j < 2^{k-k_\tau}$. Now

$$\sum_{j=0}^{2^{k-k_\tau}-1} \int_{-\infty}^{m\mu I_\tau} w_{\sigma_j} = 2^k \sum_{j=0}^{2^{k-k_\tau}-1} \int_{-\infty}^{m\mu I_\tau} w(2^k(x - m\mu I_\tau) - j - \tfrac{1}{2})dx$$

$$= \sum_{j=0}^{2^{k-k_\tau}-1} \int_{-\infty}^{0} w(x - j - \tfrac{1}{2})dx$$

$$\le \sum_{j=0}^{\infty} \int_{j+\frac{1}{2}}^{\infty} w = \frac{1}{2}C_4.$$

Similarly (since w is an even function, so the whole picture is symmetric about x_τ)

$$\sum_{j=0}^{2^{k-k_\tau}-1}\int_{(m+1)\mu I_\tau}^{\infty} w_{\sigma_j} \le \frac{1}{2}C_4,$$

and

$$\sum_{\sigma\ge\tau,k_\sigma=k}\int_{\mathbb{R}\setminus I_\tau} w_\sigma \le C_4,$$

as required.

286H 'Mass' and 'energy' (Lacey & Thiele 00) If P is a subset of Q, $E \subseteq \mathbb{R}$ is measurable, $g:\mathbb{R}\to\mathbb{R}$ is measurable, and $f \in \mathcal{L}^2_{\mathbb{C}}$, set

$$\mathrm{mass}_{Eg}(P) = \sup_{\sigma\in P,\tau\in Q,\tau\le\sigma}\int_{E\cap g^{-1}[J_\tau]} w_\tau \le \sup_{\tau\in Q}\int_{-\infty}^{\infty} w_\tau = 1,$$

$$\Delta_f(P) = \sum_{\sigma\in P}|(f|\phi_\sigma)|^2,$$

$$\mathrm{energy}_f(P) = \sup_{\tau\in Q}\sqrt{\mu J_\tau}\sqrt{\Delta_f(P\cap T_\tau)}.$$

If $P' \subseteq P$ then $\mathrm{mass}_{Eg}(P') \le \mathrm{mass}_{Eg}(P)$ and $\mathrm{energy}_f(P') \le \mathrm{energy}_f(P)$. Note that $\mathrm{energy}_f(\{\sigma\}) = \sqrt{\mu J_\sigma}|(f|\phi_\sigma)|$ for any $\sigma \in Q$, since if $\sigma \in T_\tau$ then $\mu J_\tau \le \mu J_\sigma$.

286I Lemma If $P \subseteq Q$ is finite and $f \in \mathcal{L}^2_{\mathbb{C}}$, then

(a) $\Delta_f(P) \le \|\sum_{\sigma\in P}(f|\phi_\sigma)\phi_\sigma\|_2\|f\|_2$,

(b) $\sum_{\sigma,\tau\in P,J_\sigma=J_\tau}\bigl|(f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\bigr| \le C_3\Delta_f(P)$.

proof (a)

$$\Delta_f(P) = \sum_{\sigma\in P}(f|\phi_\sigma)(\phi_\sigma|f) = \bigl(\sum_{\sigma\in P}(f|\phi_\sigma)\phi_\sigma\big|f\bigr) \le \|\sum_{\sigma\in P}(f|\phi_\sigma)\phi_\sigma\|_2\|f\|_2$$

by Cauchy's inequality (244Eb).

(b)

$$\sum_{\substack{\sigma,\tau\in P\\ J_\sigma=J_\tau}}\bigl|(f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\bigr| \le \sum_{\substack{\sigma,\tau\in P\\ J_\sigma=J_\tau}}\frac{1}{2}\bigl(|(f|\phi_\sigma)|^2+|(f|\phi_\tau)|^2\bigr)|(\phi_\sigma|\phi_\tau)|$$

(because $|\xi\zeta| \le \frac{1}{2}(|\xi|^2+|\zeta|^2)$ for all complex numbers ξ, ζ)

$$= \sum_{\sigma\in P}\sum_{\substack{\tau\in P\\ J_\sigma=J_\tau}}|(f|\phi_\sigma)|^2|(\phi_\sigma|\phi_\tau)|$$

$$\le \sum_{\sigma\in P}|(f|\phi_\sigma)|^2\sum_{\substack{\tau\in P\\ J_\sigma=J_\tau}}C_3\int_{I_\tau} w_\sigma$$

(by 286Gg, since $k_\sigma = k_\tau$ if $J_\sigma = J_\tau$)

$$\le \sum_{\sigma\in P}|(f|\phi_\sigma)|^2 C_3\int_{-\infty}^{\infty} w_\sigma$$

(because if τ, τ' are distinct members of P and $J_\tau = J_{\tau'}$, then I_τ and $I_{\tau'}$ are disjoint)

$$= C_3\sum_{\sigma\in P}|(f|\phi_\sigma)|^2 = C_3\Delta_f(P).$$

286J Lemma Set $C_5 = 2^{12}$. If $P \subseteq Q$ is finite, $E \subseteq \mathbb{R}$ is measurable, $g:\mathbb{R}\to\mathbb{R}$ is measurable, and $\gamma \ge \mathrm{mass}_{Eg}(P)$, then we can find $R \subseteq Q$ such that $\gamma\sum_{\tau\in R}\mu I_\tau \le C_5\mu E$ and (in the notation of 286Fb) $\mathrm{mass}_{Eg}(P\setminus R^+) \le \frac{1}{4}\gamma$.

proof (a) If $\gamma=0$ we can take $R=\emptyset$. Otherwise, set $P_1=\{\sigma:\sigma\in P,\ \mathrm{mass}_{Eg}(\{\sigma\})>\frac{1}{4}\gamma\}$. For each $\sigma\in P_1$ let $\sigma'\in Q$ be such that $\sigma'\le\sigma$ and $\int_{E\cap g^{-1}[J_{\sigma'}]}w_{\sigma'}>\frac{1}{4}\gamma$. Let R be the set of elements of $\{\sigma':\sigma\in P_1\}$ which are minimal for $\le$. Then $P\setminus R^+\subseteq\{\sigma:\mathrm{mass}_{Eg}(\{\sigma\})\le\frac{1}{4}\gamma\}$ so $\mathrm{mass}_{Eg}(P\setminus R^+)\le\frac{1}{4}\gamma$.

(b) For $k\in\mathbb{N}$ set

$$R_k=\{\tau:\tau\in R,\ \mu J_\tau\cdot\mu(E\cap g^{-1}[J_\tau]\cap I_\tau^{(k)})\ge 2^{2k-9}\gamma\},$$

where $I_\tau^{(k)}$ is the half-open interval with the same centre as I_τ and 2^k times its length. Now $R=\bigcup_{k\in\mathbb{N}}R_k$. **P** Take $\tau\in R$. If $k\in\mathbb{N}$ and $x\in\mathbb{R}\setminus I_\tau^{(k)}$, then $|x-x_\tau|\ge\frac{1}{2}\mu I_\tau^{(k)}=2^{k-1}\mu I_\tau$, so

$$w_\tau(x)=w((x-x_\tau)\mu J_\tau)\mu J_\tau\le w(2^{k-1})\mu J_\tau=(1+2^{k-1})^{-3}\mu J_\tau.$$

Accordingly

$$\begin{aligned}\frac{1}{4}\gamma&<\int_{E\cap g^{-1}[J_\tau]}w_\tau=\int_{E\cap g^{-1}[J_\tau]\cap I_\tau}w_\tau+\sum_{k=0}^\infty\int_{E\cap g^{-1}[J_\tau]\cap I_\tau^{(k+1)}\setminus I_\tau^{(k)}}w_\tau\\&\le\mu J_\tau\cdot\mu(E\cap g^{-1}[J_\tau]\cap I_\tau)+\sum_{k=0}^\infty(1+2^{k-1})^{-3}\mu J_\tau\cdot\mu(E\cap g^{-1}[J_\tau]\cap I_\tau^{(k+1)}).\end{aligned}$$

It follows that either

$$\mu J_\tau\cdot\mu(E\cap g^{-1}[J_\tau]\cap I_\tau)\ge\frac{1}{8}\gamma$$

and $\tau\in R_0$, or there is some $k\in\mathbb{N}$ such that

$$(1+2^{k-1})^{-3}\mu J_\tau\cdot\mu(E\cap g^{-1}[J_\tau]\cap I_\tau^{(k+1)})\ge 2^{-k-4}\gamma$$

and

$$\mu J_\tau\cdot\mu(E\cap g^{-1}[J_\tau]\cap I_\tau^{(k+1)})\ge(1+2^{k-1})^3 2^{-k-4}\gamma\ge 2^{2k-7}\gamma,$$

so that $\tau\in R_{k+1}$. **Q**

(c) For every $k\in\mathbb{N}$, $\gamma\sum_{\tau\in R_k}\mu I_\tau\le 2^{11-k}\mu E$. **P** If $R_k=\emptyset$, this is trivial. Otherwise, enumerate R_k as $\langle\tau_j\rangle_{j\le n}$ in such a way that $k_{\tau_j}\le k_{\tau_l}$ if $j\le l\le n$. Define $q:\{0,\dots,n\}\to\{0,\dots,n\}$ inductively by the rule

$$q(l)=\min(\{l\}\cup\{j:j<l,\ q(j)=j,\ (I_{\tau_j}^{(k)}\times J_{\tau_j})\cap(I_{\tau_l}^{(k)}\times J_{\tau_l})\ne\emptyset\})$$

for each $l\le n$. Note that, for $l\le n$, $q(q(l))=q(l)\le l$ and $I_{\tau_{q(l)}}^{(k)}\cap I_{\tau_l}^{(k)}\ne\emptyset$, so that

$$I_{\tau_l}\subseteq I_{\tau_l}^{(k)}\subseteq I_{\tau_{q(l)}}^{(k+2)},$$

because $\mu I_{\tau_l}^{(k)}\le\mu I_{\tau_{q(l)}}^{(k)}$. Moreover, if $j<l\le n$ and $q(j)=q(l)$, then both J_{τ_j} and J_{τ_l} meet $J_{\tau_{q(j)}}$, therefore include it, and $J_{\tau_j}\subseteq J_{\tau_l}$. But as τ_j and τ_l are distinct members of R, $\tau_j\not\le\tau_l$ and $I_{\tau_j}\cap I_{\tau_l}$ must be empty.

Set $M=\{q(j):j\le n\}$. We have

$$\begin{aligned}\gamma\sum_{\tau\in R_k}\mu I_\tau&=\gamma\sum_{m\in M}\sum_{\substack{j\le n\\ q(j)=m}}\mu I_{\tau_j}\le\gamma\sum_{m\in M}\mu I_{\tau_m}^{(k+2)}=2^{k+2}\gamma\sum_{m\in M}\mu I_{\tau_m}\\&\le 2^{k+2}\sum_{m\in M}2^{9-2k}\mu(E\cap g^{-1}[J_{\tau_m}]\cap I_{\tau_m}^{(k)})\\&\le 2^{k+2}\cdot 2^{9-2k}\mu E=2^{11-k}\mu E\end{aligned}$$

because if l, $m\in M$ and $l<m$ then $I_{\tau_l}^{(k)}\times J_{\tau_l}$ and $I_{\tau_m}^{(k)}\times J_{\tau_m}$ are disjoint (since $q(m)=m$)), so that $g^{-1}[J_{\tau_l}]\cap I_{\tau_l}^{(k)}$ and $g^{-1}[J_{\tau_m}]\cap I_{\tau_m}^{(k)}$ are disjoint. **Q**

(d) Accordingly

$$\gamma\sum_{\tau\in R}\mu I_\tau\le\gamma\sum_{k=0}^\infty\sum_{\tau\in R_k}\mu I_\tau\le 2^{12}\mu E,$$

as required.

286K Lemma Set $C_6 = 4(C_3 + 4C_3\sqrt{2C_4})$. Suppose that $P \subseteq Q$ is a finite set, $f \in \mathcal{L}^2_{\mathbb{C}}$, $\|f\|_2 = 1$ and $\gamma \ge \text{energy}_f(P)$. Then we can find $R \subseteq Q$ such that $\gamma^2 \sum_{\tau\in R} \mu I_\tau \le C_6$ and $\text{energy}_f(P \setminus R^+) \le \frac{1}{2}\gamma$.

proof (a) We may suppose that $\gamma > 0$ and that $P \ne \emptyset$, since otherwise we can take $R = \emptyset$.

(i) There are only finitely many sets of the form $P \cap T_\tau$ for $\tau \in Q$; let $\tilde R \subseteq Q$ be a non-empty finite set such that whenever $\tau \in Q$ and $P \cap T_\tau$ is not empty, there is a $\tau' \in \tilde R$ such that $P \cap T_\tau = P \cap T_{\tau'}$ and $k_{\tau'} \ge k_\tau$; this is possible because if $A \subseteq P$ is not empty then $k_\tau \le \min_{\sigma\in A} k_\sigma$ whenever $T_\tau \supseteq A$.

(ii) Choose $\tau_0, \tau_1, \dots,\ P_0, P_1, \dots$ inductively, as follows. $P_0 = P$. Given that $P_j \subseteq P$ is not empty, consider

$$R_j = \{\tau : \tau \in \tilde R,\ \Delta_f(P_j \cap T_\tau) \ge \frac{1}{4}\gamma^2 \mu I_\tau\}.$$

If $R_j = \emptyset$, stop the induction and set $n = j$ and $R = \{\tau_l : l < j\}$. Otherwise, among the members of R_j take one with y_τ as far to the left as possible, and call it τ_j; set $P_{j+1} = P_j \setminus \{\tau_j\}^+$, and continue. Note that as $R_{j+1} \subseteq R_j$ for every j, $y_{\tau_{j+1}} \ge y_{\tau_j}$ for every j.

The induction must stop at a finite stage because if it does not stop with $n = j$ then $\Delta_f(P_j \cap T_{\tau_j}) > 0$, so $P_j \cap T_{\tau_j}$ is not empty and $P_{j+1} \subseteq P_j \setminus T_{\tau_j}$ is a proper subset of P_j, while $P_0 = P$ is finite. Since $R_n = \emptyset$,

$$\begin{aligned}\text{energy}_f(P \setminus R^+) = \text{energy}_f(P_n) &= \sup_{\tau\in Q} \sqrt{\mu J_\tau}\sqrt{\Delta_f(P_n \cap T_\tau)}\\ &= \max_{\tau\in\tilde R} \sqrt{\mu J_\tau}\sqrt{\Delta_f(P_n \cap T_\tau)} \le \frac{1}{2}\gamma.\end{aligned}$$

(iii) Set $P'_j = P_j \cap T_{\tau_j} \subseteq P_j \setminus P_{j+1}$ for $j < n$, so that $\langle P'_j\rangle_{j<n}$ is disjoint, and $P' = \bigcup_{j<n} P'_j \subseteq P$. Then if $\sigma \in P'$, $j < n$ and $J_{\tau_j} \subseteq J^l_\sigma$, $I_\sigma \cap I_{\tau_j} = \emptyset$. **P** Let $l < n$ be such that $\sigma \in P'_l$. Then $y_{\tau_j} \in J_{\tau_j} \subseteq J^l_\sigma$ and $y_{\tau_l} \in J^r_{\tau_l} \subseteq J^r_\sigma$, so $y_{\tau_j} < y_{\tau_l}$ and $j < l$. Accordingly $P_{j+1} \supseteq P_l$ contains σ, so $\sigma \not\ge \tau_j$; as $J_{\tau_j} \subseteq J_\sigma$, $I_\sigma \not\subseteq I_{\tau_j}$, while $\mu I_\sigma \le \mu I_{\tau_j}$, so I_σ is disjoint from I_{τ_j}. **Q**

It follows that if σ, $\tau \in P'$ are distinct and $J^l_\sigma \cap J^l_\tau$ is not empty, then $I_\sigma \cap I_\tau = \emptyset$. **P** If $J_\sigma = J_\tau$ this is true just because $\sigma \ne \tau$. Otherwise, since J_σ and J_τ intersect, one is included in the other; suppose that $J_\sigma \subset J_\tau$. Since J_σ meets J^l_τ, $J_\sigma \subseteq J^l_\tau$. Now let $j < n$ be such that $\sigma \in P'_j$; then $\sigma \ge \tau_j$, so $J_{\tau_j} \subseteq J_\sigma \subseteq J^l_\tau$, and $I_\sigma \cap I_\tau \subseteq I_{\tau_j} \cap I_\tau = \emptyset$ by the last remark. **Q**

(b) Now let us estimate

$$\gamma^2 \textstyle\sum_{j<n} \mu I_{\tau_j} \le 4 \sum_{j<n} \Delta_f(P'_j) = 4\Delta_f(P') = 4\alpha$$

say. Because $\|f\|_2 = 1$, we have $\alpha \le \|\sum_{\sigma\in P'} (f|\phi_\sigma)\phi_\sigma\|_2$ (286Ia). So

$$\begin{aligned}\alpha^2 \le \|\sum_{\sigma\in P'} (f|\phi_\sigma)\phi_\sigma\|_2^2 &= \sum_{\sigma,\tau\in P'} (f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\\ &= \sum_{\substack{\sigma,\tau\in P'\\ J_\sigma=J_\tau}} (f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\\ &\qquad + \sum_{\substack{\sigma,\tau\in P'\\ J_\sigma\subseteq J^l_\tau}} (f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f) + \sum_{\substack{\sigma,\tau\in P'\\ J_\tau\subseteq J^l_\sigma}} (f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\end{aligned}$$

because $(\phi_\sigma|\phi_\tau) = 0$ unless $J^l_\sigma \cap J^l_\tau \ne \emptyset$, as noted in 286E(b-iii).

Take these three terms separately. For the first, we have

$$\textstyle\sum_{\sigma,\tau\in P', J_\sigma=J_\tau} \bigl|(f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\bigr| \le C_3\alpha$$

by 286Ib. For the second term, we have

$$\sum_{\substack{\sigma,\tau\in P'\\ J_\sigma\subseteq J^l_\tau}}\bigl|(f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\bigr|\le\sum_{\sigma\in P'}|(f|\phi_\sigma)|\sum_{\substack{\tau\in P'\\ J_\sigma\subseteq J^l_\tau}}|(\phi_\sigma|\phi_\tau)(\phi_\tau|f)|$$

$$\le\sqrt{\sum_{\sigma\in P'}|(f|\phi_\sigma)|^2}\sqrt{\sum_{\sigma\in P'}\bigl(\sum_{\substack{\tau\in P'\\ J_\sigma\subseteq J^l_\tau}}|(\phi_\sigma|\phi_\tau)(\phi_\tau|f)|\bigr)^2}$$

$$=\sqrt{\alpha}\sqrt{\sum_{j<n}H_j},$$

where for $j<n$ I set

$$H_j=\sum_{\sigma\in P'_j}\bigl(\sum_{\substack{\tau\in P'\\ J_\sigma\subseteq J^l_\tau}}|(\phi_\sigma|\phi_\tau)(\phi_\tau|f)|\bigr)^2.$$

Now we can estimate H_j by observing that, for any $\tau\in P'$,

$$|(\phi_\tau|f)|=\sqrt{\mu I_\tau}\,\mathrm{energy}_f(\{\tau\})\le\gamma\sqrt{\mu I_\tau},$$

while if σ, $\tau\in P'$ and $J^l_\tau\supseteq J_\sigma$ then

$$|(\phi_\sigma|\phi_\tau)|\le C_3\sqrt{\mu I_\sigma}\sqrt{\mu J_\tau}\int_{I_\tau}w_\sigma$$

by 286Gg. We also need to know that if $\sigma\in P'_j$ and τ, τ' are distinct elements of P' such that $J_\sigma\subseteq J^l_\tau\cap J^l_{\tau'}$, then I_τ, $I_{\tau'}$ and I_{τ_j} are all disjoint, by (a-iii) above, because $J_{\tau_j}\subseteq J_\sigma$. So we have

$$H_j\le\sum_{\sigma\in P'_j}\bigl(\sum_{\substack{\tau\in P'\\ J_\sigma\subseteq J^l_\tau}}\gamma\sqrt{\mu I_\tau}\cdot C_3\sqrt{\mu I_\sigma}\sqrt{\mu J_\tau}\int_{I_\tau}w_\sigma\bigr)^2$$

$$=C_3^2\gamma^2\sum_{\sigma\in P'_j}\mu I_\sigma\bigl(\sum_{\substack{\tau\in P'\\ J_\sigma\subseteq J^l_\tau}}\int_{I_\tau}w_\sigma\bigr)^2\le C_3^2\gamma^2\sum_{\sigma\in P'_j}\mu I_\sigma\bigl(\int_{\mathbb{R}\setminus I_{\tau_j}}w_\sigma\bigr)^2$$

$$\le C_3^2\gamma^2\sum_{k=k_{\tau_j}}^\infty 2^{-k}\sum_{\substack{\sigma\in P'_j\\ k_\sigma=k}}\int_{\mathbb{R}\setminus I_{\tau_j}}w_\sigma\cdot\int_{-\infty}^\infty w_\sigma\le C_3^2\gamma^2\sum_{k=k_{\tau_j}}^\infty 2^{-k}C_4$$

(by 286Ga and 286Gh, since $\sigma\ge\tau_j$ for every $\sigma\in P'_j$)

$$=C_3^2\gamma^2 2^{-k_{\tau_j}+1}C_4=2C_3^2C_4\gamma^2\mu I_{\tau_j}.$$

Accordingly

$$\textstyle\sum_{j<n}H_j\le 2C_3^2\gamma^2C_4\sum_{j<n}\mu I_{\tau_j}\le 2C_3^2C_4\cdot 4\alpha,$$

and

$$\textstyle\sum_{\sigma,\tau\in P',J_\sigma\subseteq J^l_\tau}|(f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)|\le\sqrt{\alpha\sum_{j<n}H_j}\le 2C_3\alpha\sqrt{2C_4}.$$

Similarly,

$$\textstyle\sum_{\sigma,\tau\in P',J_\tau\subseteq J^l_\sigma}|(f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)|\le 2C_3\alpha\sqrt{2C_4};$$

putting these together,

$$\alpha^2\le\alpha(C_3+4C_3\sqrt{2C_4})=\frac{1}{4}\alpha C_6$$

and $\alpha\le\frac{1}{4}C_6$. But this means that

$$\gamma^2\sum_{j<n}\mu I_{\tau_j} \le 4\alpha \le C_6,$$

and $R = \{\tau_j : j < n\}$ has both the properties required.

286L Lemma Set

$$C_7 = C_1\big(\frac{7}{2} + \frac{8}{7} + \frac{28}{w(3/2)} + \frac{4\sqrt{14C_3}}{w(3/2)}\big).$$

Suppose that P is a finite subset of Q with a lower bound τ in Q for the ordering $\le$, $E \subseteq \mathbb{R}$ is measurable, $g : \mathbb{R} \to \mathbb{R}$ is measurable and $f \in \mathcal{L}^2_{\mathbb{C}}$. Then

$$\sum_{\sigma\in P} |(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]} \phi_\sigma| \le C_7 \operatorname{energy}_f(P) \operatorname{mass}_{Eg}(P)\mu I_\tau.$$

proof Set $\gamma = \operatorname{energy}_f(P)$, $\gamma' = \operatorname{mass}_{Eg}(P)$. If $P = \emptyset$ the result is trivial, so suppose that $P \ne \emptyset$.

(a)(i) Note that $\bigcup_{\sigma\in P} I_\sigma \subseteq I_\tau$, $J_\tau \subseteq \bigcap_{\sigma\in P} J_\sigma$ and $k_\tau \le \min_{\sigma\in P} k_\sigma$. So if σ, $\sigma' \in P$ are distinct and $\mu I_\sigma = \mu I_{\sigma'}$, then $J_\sigma = J_{\sigma'}$ and $I_\sigma \cap I_{\sigma'} = \emptyset$.

(ii) For a dyadic interval I let I^* be the half-open interval with the same centre as I and three times its length. Let $\mathcal{J}$ be the family of those $I \in \mathcal{I}$ such that $I_\sigma \not\subseteq I^*$ for any $\sigma \in P$ such that $\mu I_\sigma \le \mu I$. Because P is finite, all sufficiently small intervals belong to $\mathcal{J}$, and $\bigcup\mathcal{J} = \mathbb{R}$; let $\mathcal{K}$ be the set of maximal members of $\mathcal{J}$, so that $\mathcal{K}$ is disjoint. Then $\bigcup\mathcal{K} = \mathbb{R}$. **P** The point is that $P \ne \emptyset$; fix $\sigma \in P$ for the moment. If $I \in \mathcal{J}$, consider for each $n \in \mathbb{N}$ the interval $\tilde{I}^{(n)} \in \mathcal{I}$ including I with length $2^n\mu I$. Then there is some $n \in \mathbb{N}$ such that $\mu\tilde{I}^{(n)} \ge \mu I_\sigma$ and $I_\sigma \subseteq (\tilde{I}^{(n)})^*$, so that $\tilde{I}^{(k)} \notin \mathcal{J}$ for any $k \ge n$, and there must be some $k < n$ such that $\tilde{I}^{(k)} \in \mathcal{K}$. Thus $I \subseteq \tilde{I}^{(k)} \subseteq \bigcup\mathcal{K}$; as I is arbitrary, $\bigcup\mathcal{K} = \bigcup\mathcal{J} = \mathbb{R}$. **Q**

(iii) For $K \in \mathcal{K}$, let $l_K \in \mathbb{Z}$ be such that $\mu K = 2^{-l_K}$. If $l_K \ge k_\tau$, that is, $\mu K \le \mu I_\tau$, then K must lie within the half-open interval $\hat{I}$ with centre x_τ and length $7\mu I_\tau$, since otherwise we should have $I_\tau \cap \tilde{K}^* = \emptyset$, where $\tilde{K}$ is the dyadic interval of length $2\mu K$ including K, and $\tilde{K}$ would belong to $\mathcal{J}$. But this means that

$$\sum_{K\in\mathcal{K},\mu K\le\mu I_\tau} \mu K \le \mu\hat{I} = 7\mu I_\tau,$$

because $\mathcal{K}$ is disjoint.

(iv) For any $l < k_\tau$, there are just three members K of $\mathcal{K}$ such that $l_K = l$. **P** If $I \in \mathcal{I}$ and $\mu I > \mu I_\tau$, then either $I_\tau \subseteq I^*$ or $I_\tau \cap I^* = \emptyset$, and $I \in \mathcal{J}$ iff $I_\tau \cap I^*$ is empty. This means that if $K \in \mathcal{I}$ and $\mu K = 2^{-l}$, $K \in \mathcal{K}$ iff $I_\tau \cap K^*$ is empty and $I_\tau \subseteq \tilde{K}^*$. So if $I_\tau \subseteq \left[2^{-l}n, 2^{-l}(n+1)\right[$ and $K = \left[2^{-l}m, 2^{-l}(m+1)\right[$, we shall have $K \in \mathcal{K}$ iff

$$\textit{either } m = n-2 \textit{ or } m = n+2 \textit{ or } m = n-3 \text{ is even } \textit{or } m = n+3 \text{ is odd;}$$

which for any given n happens for just three values of m. **Q**

(b) For $\sigma \in P$, let ζ_σ be a complex number of modulus 1 such that $\zeta_\sigma(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]} \phi_\sigma$ is real and non-negaive. Set $W = P \times \mathcal{K}$. For $(\sigma, K) \in W$, set

$$\alpha_{\sigma K} = (f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]\cap K} \phi_\sigma.$$

The aim of the proof is to estimate

$$\sum_{\sigma\in P}\Big|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]} \phi_\sigma\Big| = \sum_{(\sigma,K)\in W} \zeta_\sigma\alpha_{\sigma K}.$$

It will be helpful to note straight away that

$$\sum_{(\sigma,K)\in W} |\alpha_{\sigma K}| \le \sum_{\sigma\in P} |(f|\phi_\sigma)|\int_{-\infty}^{\infty} |\phi_\sigma|$$

is finite.

Set

$$W_0 = \{(\sigma, K) : \sigma \in P,\ K \in \mathcal{K},\ \mu I_\sigma \le \mu K \le \mu I_\tau\},$$

$$W_1 = \{(\sigma, K) : \sigma \in P,\ K \in \mathcal{K},\ \mu I_\tau < \mu K\},$$

$$W_2 = \{(\sigma, K) : \sigma \in P \setminus T_\tau,\ K \in \mathcal{K},\ \mu K < \mu I_\sigma\},$$

$$W_3 = \{(\sigma, K) : \sigma \in P \cap T_\tau,\ K \in \mathcal{K},\ \mu K < \mu I_\sigma\}.$$

Because $\mu I_\sigma \le \mu I_\tau$ for every $\sigma \in P$, $W = W_0 \cup W_1 \cup W_2 \cup W_3$. I will give estimates for

$$\alpha_j = \sum\nolimits_{(\sigma,K)\in W_j} \zeta_\sigma \alpha_{\sigma K}$$

for each j; the four components in the expression for C_7 given above are bounds for $|\alpha_0|, |\alpha_1|$, $|\alpha_2|$ and $|\alpha_3|$ respectively.

(c)(i) If $K \in \mathcal{K}$ and $l_K = l$, then for any $k \ge l$

$$\sum\nolimits_{\sigma\in P, k_\sigma = k} |\alpha_{\sigma K}| \le 2^{-k} C_1 \gamma\gamma'(1 + 2^{k-l})^{-2} \le 2^{-k-2} C_1 \gamma\gamma'.$$

P For any $\sigma \in P$,

$$|(f|\phi_\sigma)| = \sqrt{\mu I_\sigma}\,\mathrm{energy}_f(\{\sigma\}) \le \gamma\sqrt{\mu I_\sigma}$$

as noted in 286H, and

$$\int_{E\cap g^{-1}[J_\sigma^r]\cap K} |\phi_\sigma| \le C_1 \mu I_\sigma \sqrt{\mu I_\sigma} \int_{E\cap g^{-1}[J_\sigma^r]\cap K} w_\sigma^2$$

(286Ge)

$$\le C_1 \mu I_\sigma \sqrt{\mu I_\sigma} \int_{E\cap g^{-1}[J_\sigma]} w_\sigma \cdot \sup_{x\in K} w_\sigma(x)$$

$$\le C_1 \mu I_\sigma \sqrt{\mu I_\sigma}\gamma' \sup_{x\in K} w_\sigma(x) = C_1 \sqrt{\mu I_\sigma}\gamma' w(\mu J_\sigma \rho(x_\sigma, K)),$$

where I write $\rho(x_\sigma, K)$ for $\inf_{x\in K} |x - x_\sigma|$. So, for $k \ge l$,

$$\sum_{\substack{\sigma\in P\\ k_\sigma = k}} |\alpha_{\sigma K}| \le \sum_{\substack{\sigma\in P\\ k_\sigma = k}} C_1 \gamma\gamma' \mu I_\sigma w(\mu J_\sigma \rho(x_\sigma, K))$$

$$= 2^{-k} C_1 \gamma\gamma' \sum_{\substack{\sigma\in P\\ k_\sigma = k}} w(2^k \rho(x_\sigma, K)) \le 2^{-k} C_1 \gamma\gamma' \cdot 2 \sum_{n=2^{k-l}}^{\infty} w(n + \tfrac{1}{2})$$

because the x_σ, for $\sigma \in P$ and $k_\sigma = k$, are all distinct (see (a-i) above) and all a distance at least $\mu K = 2^{k-l}2^{-k}$ from K (because $I_\sigma \not\subseteq K^*$); so there are at most two such σ with $\rho(x_\sigma, K) = 2^{-k}(n + \frac{1}{2})$ for each $n \ge 2^{k-l}$. So we have

$$\sum\nolimits_{\sigma\in P, k_\sigma = k} |\alpha_{\sigma K}| \le 2^{-k} C_1 \gamma\gamma'(1 + 2^{k-l})^{-2} \le 2^{-k-2} C_1 \gamma\gamma'$$

by 286Gb. **Q**

(ii) Now

$$|\alpha_0| \le \sum_{(\sigma,K)\in W_0} |\alpha_{\sigma K}| = \sum_{\substack{K\in\mathcal{K}\\ \mu K \le \mu I_\tau}} \sum_{\substack{\sigma\in P\\ \mu I_\sigma \le \mu K}} |\alpha_{\sigma K}|$$

$$= \sum_{\substack{K\in\mathcal{K}\\ \mu K \le \mu I_\tau}} \sum_{k=l_K}^{\infty} \sum_{\substack{\sigma\in P\\ k_\sigma = k}} |\alpha_{\sigma K}| \le \sum_{\substack{K\in\mathcal{K}\\ \mu K \le \mu I_\tau}} \sum_{k=l_K}^{\infty} 2^{-k-2} C_1 \gamma\gamma'$$

$$= C_1 \gamma\gamma' \sum_{\substack{K\in\mathcal{K}\\ \mu K \le \mu I_\tau}} 2^{-l_K-1} = \tfrac{1}{2} C_1 \gamma\gamma' \sum_{\substack{K\in\mathcal{K}\\ \mu K \le \mu I_\tau}} \mu K \le \tfrac{7}{2} C_1 \gamma\gamma' \mu I_\tau$$

by the formula in (a-iii). This deals with α_0.

(d) Next consider W_1. We have

$$|\alpha_1| \le \sum_{(\sigma,K)\in W_1} |\alpha_{\sigma K}| = \sum_{l=-\infty}^{k_\tau-1}\sum_{k=k_\tau}^{\infty}\sum_{\substack{K\in\mathcal{K}\\ l_K=l}}\sum_{\substack{\sigma\in P\\ k_\sigma=k}} |\alpha_{\sigma K}|$$

$$\le \sum_{l=-\infty}^{k_\tau-1}\sum_{k=k_\tau}^{\infty}\sum_{\substack{K\in\mathcal{K}\\ l_K=l}} 2^{-k}C_1\gamma\gamma'(1+2^{k-l})^{-2}$$

(by (c-i) above)

$$= 3C_1\gamma\gamma'\sum_{l=-\infty}^{k_\tau-1}\sum_{k=k_\tau}^{\infty} 2^{-k}(1+2^{k-l})^{-2}$$

(by (a-iv))

$$\le 3C_1\gamma\gamma'\sum_{l=-\infty}^{k_\tau-1}\sum_{k=k_\tau}^{\infty} 2^{-3k}2^{2l} = 3C_1\gamma\gamma' 2^{2(k_\tau-1)}2^{-3k_\tau}\sum_{l=0}^{\infty}2^{-2l}\sum_{k=0}^{\infty}2^{-3k}$$

$$= \frac{3}{4}C_1\gamma\gamma'2^{-k_\tau}\cdot\frac{4}{3}\cdot\frac{8}{7} = \frac{8}{7}C_1\gamma\gamma'\mu I_\tau.$$

This deals with α_1.

(e) For $K\in\mathcal{K}$, set $G_K = K\cap E\cap\bigcup_{\sigma\in P,\mu I_\sigma>\mu K} g^{-1}[J_\sigma]$. Then $\mu G_K \le 2\gamma'\mu K/w(\frac{3}{2})$. **P** If $\mu I_\tau \le \mu K$, then $G_K=\emptyset$, so we may suppose that $\mu K<\mu I_\tau$. Let $\tilde K\in\mathcal{I}$ be the dyadic interval containing K and with twice the length. Then $\tilde K\notin\mathcal{J}$, so there is a $\sigma\in P$ such that $\tilde K^*\supseteq I_\sigma$ and

$$\mu I_\sigma \le \mu\tilde K = 2\mu K \le \mu I_\tau.$$

Let $\upsilon\in Q$ be such that $\tau\le\upsilon\le\sigma$ and $\mu I_\upsilon = 2\mu K$ (286F(a-iv)). Then I_υ meets $\tilde K^*$, so $\tilde K$ is either equal to I_υ or adjacent to it, and $|x-x_\upsilon|\le\frac{3}{2}\cdot\mu I_\upsilon$ for every $x\in\tilde K$, therefore for every $x\in K$. Accordingly

$$w_\upsilon(x)\ge w(\tfrac{3}{2})\mu J_\upsilon = w(\tfrac{3}{2})/2\mu K$$

for every $x\in K$. On the other hand, because $\sigma\in P$ and $\upsilon\le\sigma$, $\int_{E\cap g^{-1}[J_\upsilon]} w_\upsilon\le\gamma'$. So

$$\mu(E\cap g^{-1}[J_\upsilon]\cap K)\le 2\gamma'\mu K/w(\tfrac{3}{2}).$$

Now suppose that $\sigma'\in P$ and $\mu I_{\sigma'}>\mu K$. Then $k_{\sigma'}\le k_\upsilon$ and $J_{\sigma'}$ is the dyadic interval of length $2^{k_{\sigma'}}$ including J_τ. But J_υ is the dyadic interval of length 2^{k_υ} including J_τ, so includes $J_{\sigma'}$, and $g^{-1}[J_{\sigma'}]\subseteq g^{-1}[J_\upsilon]$. As σ' is arbitrary, $G_K\subseteq E\cap g^{-1}[J_\upsilon]\cap K$ and $\mu G_K\le 2\gamma'\mu K/w(\frac{3}{2})$, as claimed. **Q**

(f)(i) If σ, $\upsilon\in P\setminus T_\tau$ and $k_\sigma\ne k_\upsilon$, then $J^r_\sigma\cap J^r_\upsilon=\emptyset$. **P** It is enough to consider the case $\mu J_\sigma<\mu J_\upsilon$, so that $\mu J_\sigma\le\mu J^r_\upsilon$. As J_σ includes J_τ, but J^r_υ does not, J_σ is disjoint from J^r_υ and we have the result. **Q**

(ii) For $x\in\mathbb{R}$, set

$$v_2(x) = \bigl|\textstyle\sum_{(\sigma,K)\in W_2}\zeta_\sigma(f|\phi_\sigma)\phi_\sigma(x)\chi(E\cap g^{-1}[J^r_\sigma]\cap K)(x)\bigr|.$$

(The sum is finite because there is at most one $K\in\mathcal{K}$ containing x.) Then for any $x\in\mathbb{R}$ there is a $k\ge k_\tau$ such that

$$v_2(x) = \bigl|\textstyle\sum_{\sigma\in P,k_\sigma=k}\zeta_\sigma(f|\phi_\sigma)\phi_\sigma(x)\bigr|.$$

P If $v_2(x)=0$, any sufficiently large k will serve. Otherwise, $x\in E$ and we have a pair $(\upsilon,L)\in W_2$ such that $x\in g^{-1}[J^r_\upsilon]\cap L$. Try $k=k_\upsilon$. L is the only member of $\mathcal{K}$ containing x, so

$$v_2(x) = \bigl|\textstyle\sum_{\sigma\in P_x}(f|\phi_\sigma)\phi_\sigma(x)\bigr|,$$

where $P_x=\{\sigma:\sigma\in P\setminus T_\tau,\ \mu I_\sigma>\mu L,\ g(x)\in J^r_\sigma\}$. Now if $\sigma\in P$ and $k_\sigma=k$, then $\mu I_\sigma=\mu I_\upsilon>\mu L$, $J_\sigma=J_\upsilon$ and $J^r_\sigma=J^r_\upsilon$ does not include J^r_τ, so that $\sigma\in P\setminus T_\tau$, $g(x)\in J^r_\sigma$ and $\sigma\in P_x$. On the other hand, (i) above tells us that $k_\sigma=k$ whenever $\sigma\in P\setminus T_\tau$ and $g(x)\in J^r_\sigma$. So $P_x=\{\sigma:\sigma\in P,\ k_\sigma=k\}$ and

$$v_2(x) = \bigl|\textstyle\sum_{\sigma\in P,k_\sigma=k}\zeta_\sigma(f|\phi_\sigma)\phi_\sigma(x)\bigr|.\ \mathbf{Q}$$

(iii) It follows that $v_2(x) \le 2C_1\gamma$ for every $x \in \mathbb{R}$. **P** If $v_2(x) = 0$ this is trivial. Otherwise, take k from (ii). Then

$$v_2(x) \le \sum_{\substack{\sigma\in P\\ k_\sigma=k}} |(f|\phi_\sigma)\phi_\sigma(x)| \le \sum_{\substack{\sigma\in P\\ k_\sigma=k}} \sqrt{\mu I_\sigma}\gamma\cdot\sqrt{\mu I_\sigma}C_1 w_\sigma(x)$$

(by 286H and 286Ge)

$$= C_1\gamma \sum_{\substack{\sigma\in P\\ k_\sigma=k}} w(2^k(x-x_\sigma)) \le C_1\gamma\sum_{n=-\infty}^{\infty} w(2^k x - n - \tfrac{1}{2})$$

(because the x_σ, for $\sigma \in P$ and $k_\sigma = k$, are all distinct and of the form $2^{-k}(n+\frac{1}{2})$)

$$\le 2C_1\gamma$$

by 286Gd. **Q**

(iv) Note also that, if $v_2(x) > 0$, there is a pair $(\sigma, K) \in W_2$ such that $x \in g^{-1}[J_\sigma] \cap K$, so that $\mu K < \mu I_\sigma \le \mu I_\tau$ and $x \in G_K$. But now we have

$$|\alpha_2| = \Bigl|\sum_{(\sigma,K)\in W_2} \zeta_\sigma(f|\phi_\sigma)\int_{-\infty}^{\infty}\phi_\sigma\times\chi(E\cap g^{-1}[J_\sigma^r]\cap K)\Bigr|$$

$$\le \int_{-\infty}^{\infty} v_2 \le \sum_{\substack{K\in\mathcal{K}\\ \mu K<\mu I_\tau}}\int_{G_K} v_2 \le \sum_{\substack{K\in\mathcal{K}\\ \mu K<\mu I_\tau}} \frac{4C_1\gamma\gamma'\mu K}{w(\frac{3}{2})}$$

(putting the estimates in (e) and (iii) just above together)

$$\le \frac{28\cdot C_1\gamma\gamma'\mu I_\tau}{w(\frac{3}{2})}$$

by (a-iii). This deals with α_2.

(g) Set $P' = P \cap T_\tau$ and $\tilde f = \sum_{\sigma\in P'}\zeta_\sigma(f|\phi_\sigma)\phi_\sigma$. Then

$$\|\tilde f\|_2^2 \le C_3\gamma^2\mu I_\tau.$$

P If σ, $\sigma' \in P'$ and $k_\sigma \ne k_{\sigma'}$, then $(\phi_\sigma|\phi_{\sigma'}) = 0$ (286Fc). While if $k_\sigma = k_{\sigma'}$, then $J_\sigma = J_{\sigma'}$, by (a-i). So

$$\|\tilde f\|_2^2 = \sum_{\sigma,\sigma'\in P'}\zeta_\sigma(f|\phi_\sigma)(\phi_\sigma|\phi_{\sigma'})(\phi_{\sigma'}|f)\bar\zeta_{\sigma'}$$

$$\le \sum_{\substack{\sigma,\sigma'\in P'\\ J_\sigma=J_{\sigma'}}}\bigl|(f|\phi_\sigma)(\phi_\sigma|\phi_{\sigma'})(\phi_{\sigma'}|f)\bigr| \le C_3\Delta_f(P')$$

(286Ib)

$$\le C_3\gamma^2\mu I_\tau$$

by the definition of 'energy', because $P' \subseteq T_\tau$. **Q**

(h) For $m \in \mathbb{N}$, set

$$\tilde f_m = \textstyle\sum_{\sigma\in P',k_\sigma\le m}\zeta_\sigma(f|\phi_\sigma)\phi_\sigma.$$

Then whenever x, $x' \in \mathbb{R}$ and $|x-x'| \le 2^{-m}$, $|\tilde f_m(x)| \le \frac{1}{2}C_1\tilde f^*(x')$, where

$$\tilde f^*(x') = \sup_{a\le x'\le b,a<b}\frac{1}{b-a}\int_a^b|\tilde f|$$

as in 286A. **P** **(i)** Since $k_\sigma \ge k_\tau$ for every $\sigma \in P'$, we may take it that $m \ge k_\tau$. Let $\hat{J}$ be the dyadic interval of length 2^m including J_τ, and $\hat{y}$ its midpoint. Set $\psi = S_{-\hat{y}} D_{2^{-m}/3} \overset{\wedge}{\phi}$, that is, $\psi(y) = \overset{\wedge}{\phi}(\frac{1}{3}2^{-m}(y-\hat{y}))$ for $y \in \mathbb{R}$.

(ii) If $\sigma \in P'$ and $k_\sigma \le m$ and $\overset{\wedge}{\phi}_\sigma(y) \ne 0$, then $y \in J^l_\sigma$. But $J_\sigma \cap \hat{J} \supseteq J_\tau$ is not empty, so $J_\sigma \subseteq \hat{J}$, $|y-\hat{y}| \le \frac{1}{2}2^m$, $|\frac{1}{3}2^{-m}(y-\hat{y})| \le \frac{1}{6}$ and $\psi(y) = 1$.

(iii) If $\sigma \in P'$ and $k_\sigma > m$ and $\overset{\wedge}{\phi}_\sigma(y) \ne 0$, then $J^r_\sigma \cap \hat{J} \supseteq J^r_\tau$ is non-empty, so $\hat{J} \subseteq J^r_\sigma$ and $y \le y_\sigma \le \hat{y}$; now

$$\hat{y} - y = (\hat{y} - y_\sigma) + (y_\sigma - y) \ge \frac{1}{2} \cdot 2^m + \frac{1}{20}\mu J_\sigma \ge \frac{3}{5} \cdot 2^m, \quad |\frac{1}{3} \cdot 2^{-m}(y-\hat{y})| \ge \frac{1}{5}$$

and $\psi(y) = 0$.

(iv) What this means is that if $\sigma \in P'$ then

$$\begin{aligned} \overset{\wedge}{\phi}_\sigma \times \psi &= \overset{\wedge}{\phi}_\sigma \text{ if } k_\sigma \le m, \\ &= 0 \text{ if } k_\sigma > m, \end{aligned}$$

so that $\overset{\wedge}{\tilde{f}}_m = \psi \times \overset{\wedge}{\tilde{f}}$.

(v) By 283M, $\tilde{f}_m = \frac{1}{\sqrt{2\pi}} \tilde{f} * \overset{\vee}{\psi}$, where $\tilde{f} * \overset{\vee}{\psi}$ is the convolution of $\tilde{f}$ and the inverse Fourier transform $\overset{\vee}{\psi}$ of ψ. (Strictly speaking, 283M, with the help of 284C, tells us that $\tilde{f}_m$ and $\frac{1}{\sqrt{2\pi}}\tilde{f} * \overset{\vee}{\psi}$ have the same Fourier transforms. By 283G, they are equal almost everywhere; by 255K, the convolution is defined everywhere and is continuous; so in fact they are the same function.) Now

$$\overset{\vee}{\psi} = 3 \cdot 2^m M_{\hat{y}} D_{3\cdot 2^m} \overset{\wedge\vee}{\phi} = 3 \cdot 2^m M_{\hat{y}} D_{3\cdot 2^m} \phi,$$

that is,

$$\overset{\vee}{\psi}(x) = 3 \cdot 2^m e^{ix\hat{y}} \phi(3 \cdot 2^m x)$$

for $x \in \mathbb{R}$.

(vi) Set $w_1(x) = \min(w(3), w(x))$ for $x \in \mathbb{R}$, so that w_1 is non-decreasing on $]-\infty, -3]$, non-increasing on $[3, \infty[$, and constant on $[-3, 3]$, and $|\phi(x)| \le C_1 w_1(x)$ for every x, by the choice of C_1 (286Ge). Take x, $x' \in \mathbb{R}$ such that $|x - x'| \le 2^{-m}$. Then

$$\begin{aligned} |\tilde{f}_m(x)| &\le \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\tilde{f}(x-t)||\overset{\vee}{\psi}(t)|dt = \tfrac{3\cdot 2^m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\tilde{f}(x-t)||\phi(3 \cdot 2^m t)|dt \\ &\le \tfrac{3\cdot 2^m}{\sqrt{2\pi}} C_1 \int_{-\infty}^{\infty} |\tilde{f}(x-t)| w_1(3 \cdot 2^m t)dt = \tfrac{3\cdot 2^m}{\sqrt{2\pi}} C_1 \int_{-\infty}^{\infty} |\tilde{f}(x+t)| w_1(3 \cdot 2^m t)dt \end{aligned}$$

(because w_1 is an even function)

$$\le \tfrac{3\cdot 2^m}{\sqrt{2\pi}} C_1 \int_{-\infty}^{\infty} w_1(3 \cdot 2^m t)dt \cdot \sup_{\substack{a \le -2^{-m} \\ b \ge 2^{-m}}} \tfrac{1}{b-a} \int_a^b |\tilde{f}(x+t)|dt$$

(by 286B, because $t \mapsto w_1(3 \cdot 2^m t)$ is non-decreasing on $]-\infty, -2^{-m}]$, non-increasing on $[2^{-m}, \infty[$ and constant on $[-2^{-m}, 2^{-m}]$)

$$= \tfrac{1}{\sqrt{2\pi}} C_1 \int_{-\infty}^{\infty} w_1 \cdot \sup_{\substack{a \le x-2^{-m} \\ b \ge x+2^{-m}}} \tfrac{1}{b-a} \int_a^b |\tilde{f}| \le \tfrac{1}{2} C_1 \int_{-\infty}^{\infty} w \cdot \tilde{f}^*(x')$$

(because if $a \le x - 2^{-m}$ and $b \ge x + 2^{-m}$ then $a \le x' \le b$)

$$= \tfrac{1}{2} C_1 \tilde{f}^*(x')$$

(286Ga), as required. **Q**

(i) For $x \in \mathbb{R}$, set

$$v_3(x) = \bigl|\sum_{(\sigma,K)\in W_3} \zeta_\sigma (f|\phi_\sigma)\phi_\sigma(x)\chi(E \cap g^{-1}[J^r_\sigma] \cap K)(x)\bigr|.$$

Then whenever $L \in \mathcal{K}$ and x, $x' \in L$, $|v_3(x)| \le C_1 \tilde{f}^*(x')$. **P** We may suppose that $v_3(x) \ne 0$, so that, in particular, $x \in E$. The only pairs (σ, K) contributing to the sum forming $v_3(x)$ are those in which $x \in K$, so that $K = L$, and $g(x) \in J^r_\sigma$. Moreover, since we are looking only at $\sigma \in T_\tau$, so that $J^r_\tau \subseteq J^r_\sigma$, J^r_σ will always be the dyadic interval of length $2^{k_\sigma - 1}$ including J^r_τ. So these intervals are nested, and there will be some m such that (for $\sigma \in T_\tau$) $g(x) \in J^r_\sigma$ iff $k_\sigma \ge m$. Accordingly

$$v_3(x) = \bigl|\sum_{\sigma\in P', m\le k_\sigma < l_L} \zeta_\sigma (f|\phi_\sigma)\phi_\sigma(x)\bigr| = |\tilde{f}_{l_L-1}(x) - \tilde{f}_{m-1}(x)|$$

(we must have $m < l_L$ because $v_3(x) \ne 0$). Now $|x - x'| \le 2^{-l_L} \le 2^{-m}$, so (h) tells us that both $|\tilde{f}_{l_L-1}(x)|$ and $|\tilde{f}_{m-1}(x)|$ are at most $\frac{1}{2}C_1\tilde{f}^*(x')$, and $v_3(x) \le C_1\tilde{f}^*(x')$, as claimed. **Q**

It follows that $v_3(x) \le \frac{C_1}{\mu L}\int_L \tilde{f}^*$ for every $x \in L$.

(j) Now we are in a position to estimate

$$|\alpha_3| = |\sum_{(\sigma,K)\in W_3} \zeta_\sigma \alpha_{\sigma K}| \le \int_{-\infty}^{\infty} v_3 \le \sum_{\substack{K\in\mathcal{K}\\ \mu K<\mu I_\tau}} \int_{G_K} v_3$$

(because if $v_3(x) \ne 0$ there are $(\sigma, K) \in W_3$ such that $x \in K$, and in this case $x \in G_K$ and $\mu K < \mu I_\sigma \le \mu I_\tau$)

$$\le \sum_{\substack{K\in\mathcal{K}\\ \mu K<\mu I_\tau}} \mu G_K \cdot \frac{C_1}{\mu K}\int_K \tilde{f}^*$$

(by (i) above, because $G_K \subseteq K$)

$$\le C_1 \sum_{\substack{K\in\mathcal{K}\\ \mu K<\mu I_\tau}} \frac{2\gamma'}{w(\frac{3}{2})}\int_K \tilde{f}^*$$

(by (e))

$$\le \frac{2C_1\gamma'}{w(\frac{3}{2})}\int_{\hat{I}} \tilde{f}^*$$

(because if $\mu K < \mu I_\tau$ then $K \subseteq \hat{I}$, as noted in (a-iii))

$$\le \frac{2C_1\gamma'}{w(\frac{3}{2})}\sqrt{\mu\hat{I}} \cdot \|\tilde{f}^*\|_2$$

(by Cauchy's inequality)

$$\le \frac{2C_1\gamma'}{w(\frac{3}{2})}\sqrt{7\mu I_\tau} \cdot \sqrt{8}\|\tilde{f}\|_2$$

(by the Maximal Theorem, 286A)

$$\le \frac{4C_1\gamma'\sqrt{14}}{w(\frac{3}{2})}\sqrt{\mu I_\tau} \cdot \gamma\sqrt{C_3\mu I_\tau}$$

(by (g))

$$= \frac{4C_1\sqrt{14C_3}}{w(\frac{3}{2})}\gamma\gamma'\mu I_\tau.$$

(k) Assembling these,

$$\sum_{\sigma\in P}\bigl|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma\bigr| = \sum_{\substack{\sigma\in P\\ K\in\mathcal{K}}}\zeta_\sigma\alpha_{\sigma K} = \sum_{j=0}^{3}\sum_{(\sigma,K)\in W_j}\zeta_\sigma\alpha_{\sigma K} \le \sum_{j=0}^{3}|\alpha_j|$$

$$\le \frac{7}{2}\cdot C_1\gamma\gamma'\mu I_\tau + \frac{8}{7}\cdot C_1\gamma\gamma'\mu I_\tau + 28\cdot C_1\gamma\gamma'\mu I_\tau/w(\tfrac{3}{2})$$

$$+ 4\sqrt{14C_3}\cdot C_1\gamma\gamma'\mu I_\tau/w(\tfrac{3}{2})$$

$$= C_7\gamma\gamma'\mu I_\tau,$$

as claimed.

286M The Lacey-Thiele lemma Set $C_8 = 3C_7(C_5 + C_6)$. Then

$$\textstyle\sum_{\sigma\in Q}|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma| \le C_8$$

whenever $f \in \mathcal{L}^2_{\mathbb{C}}$, $\|f\|_2 = 1$, $\mu E \le 1$ and $g : \mathbb{R} \to \mathbb{R}$ is measurable.

proof (a) The first step is to combine 286J and 286K, as follows: if $P \subseteq Q$ is finite and $\max(\sqrt{\text{mass}_{Eg}(P)}, \text{energy}_f(P))$ $\le \gamma$, there is an $R \subseteq Q$ such that $\gamma^2\sum_{\tau\in R}\mu I_\tau \le C_5 + C_6$ and $\max(\sqrt{\text{mass}_{Eg}(P\setminus R^+)}, \text{energy}_f(P\setminus R^+)) \le \frac{1}{2}\gamma$. **P** Since $\text{mass}_{Eg}(P) \le \gamma^2$, 286J tells us that there is an $R_0 \subseteq Q$ such that $\gamma^2\sum_{\tau\in R_0}\mu I_\tau \le C_5$ and $\text{mass}_{Eg}(P\setminus R_0^+) \le \frac{1}{4}\gamma^2$. Turn to 286K: since $\text{energy}_f(P\setminus R_0^+) \le \text{energy}_f(P) \le \gamma$, we can find $R_1 \subseteq Q$ such that $\gamma^2\sum_{\tau\in R_1}\mu I_\tau \le C_6$ and $\text{energy}_f((P\setminus R_0^+)\setminus R_1^+) \le \frac{1}{2}\gamma$. Set $R = R_0 \cup R_1$. Then

$$\gamma^2\textstyle\sum_{\tau\in R}\mu I_\tau \le C_5 + C_6, \quad \text{mass}_{Eg}(P\setminus R^+) \le \text{mass}_{Eg}(P\setminus R_0^+) \le \frac{1}{4}\gamma^2$$

so $\max(\sqrt{\text{mass}_{Eg}(P\setminus R^+)}, \text{energy}_f(P\setminus R^+)) \le \frac{1}{2}\gamma$. **Q**

(b) Now take any finite $P \subseteq Q$. Let $k \in \mathbb{N}$ be such that $\max(\sqrt{\text{mass}_{Eg}(P)}, \text{energy}_f(P)) \le 2^k$. By (a), we can choose $\langle P_n\rangle_{n\in\mathbb{N}}$, $\langle R_n\rangle_{n\in\mathbb{N}}$ inductively such that $P_0 = P$ and, for each $n \in \mathbb{N}$,

$$P_{n+1} = P_n \setminus R_n^+,$$

$$2^{2k-2n}\textstyle\sum_{\tau\in R_n}\mu I_\tau \le C_5 + C_6, \quad \max(\sqrt{\text{mass}_{Eg}(P_n)}, \text{energy}_f(P_n)) \le 2^{k-n}.$$

Since $\text{energy}_f(\{\sigma\}) = \sqrt{\mu J_\sigma}|(f|\phi_\sigma)| > 0$ whenever $(f|\phi_\sigma) \ne 0$ (286H), $(f|\phi_\sigma) = 0$ whenever $\sigma \in \bigcap_{n\in\mathbb{N}} P_n$, and

$$\sum_{\sigma\in P}\bigl|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma\bigr| = \sum_{\sigma\in\bigcup_{n\in\mathbb{N}}P_n\setminus P_{n+1}}\bigl|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma\bigr|$$

$$= \sum_{n=0}^{\infty}\sum_{\sigma\in P_n\setminus P_{n+1}}\bigl|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma\bigr|$$

$$\le \sum_{n=0}^{\infty}\sum_{\tau\in R_n}\sum_{\substack{\sigma\in P_n\\ \sigma\ge\tau}}\bigl|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma\bigr|$$

$$\le \sum_{n=0}^{\infty}\sum_{\tau\in R_n}C_7\,\text{energy}_f(P_n)\,\text{mass}_{Eg}(P_n)\mu I_\tau$$

(by 286L)

$$\le C_7\sum_{n=0}^{\infty}2^{k-n}\min(1, 2^{2k-2n})\sum_{\tau\in R_n}\mu I_\tau$$

(because $\text{mass}_{Eg}(P_n) \le 1$ for every n, as noted in 286H)

$$\leq C_7 \sum_{n=0}^{\infty} 2^{k-n} \min(1, 2^{2k-2n}) 2^{2n-2k}(C_5 + C_6)$$

$$= C_7(C_5 + C_6) \sum_{n=0}^{\infty} \min(2^{n-k}, 2^{k-n})$$

$$\leq C_7(C_5 + C_6) \sum_{n=-\infty}^{\infty} \min(2^n, 2^{-n}) = 3C_7(C_5 + C_6).$$

(c) Since this true for every finite $P \subseteq Q$,

$$\sum_{\sigma\in Q} |(f|\phi_\sigma)\int_{E\cap g^{-1}[J_\sigma^r]} \phi_\sigma| \leq 3C_7(C_5 + C_6) = C_8,$$

as claimed.

286N Lemma Set $C_9 = C_8\sqrt{2}$. Suppose that $f \in \mathcal{L}^2_{\mathbb{C}}$, $g : \mathbb{R} \to \mathbb{R}$ is measurable and $\mu F < \infty$. Then

$$\sum_{\sigma\in Q} |(f|\phi_\sigma)\int_{F\cap g^{-1}[J_\sigma^r]} \phi_\sigma| \leq C_9\|f\|_2\sqrt{\mu F}.$$

proof This is trivial if $\|f\|_2 = 0$, that is, $f = 0$ a.e. So we may take it that $\|f\|_2 > 0$. Dividing both sides by $\|f\|_2$, we may suppose that $\|f\|_2 = 1$.

Let $k \in \mathbb{Z}$ be such that $2^{k-1} < \mu F \leq 2^k$. We have a permutation $\sigma \mapsto \sigma^* : Q \to Q$ defined by saying that $\sigma^* = (2^{-k}I_\sigma, 2^k J_\sigma)$; so that $k_{\sigma^*} = k_\sigma + k$, $x_{\sigma^*} = 2^{-k}x_\sigma$, $y^l_{\sigma^*} = 2^k y^l_\sigma$, $J^r_{\sigma^*} = 2^k J^r_\sigma$, and for every $x \in \mathbb{R}$

$$\begin{aligned}\phi_\sigma(2^k x) &= \sqrt{\mu J_\sigma} e^{2^k i y^l_\sigma x}\phi((2^k x - x_\sigma)\mu J_\sigma)\\ &= \sqrt{\mu J_\sigma} e^{i y^l_{\sigma^*} x}\phi(2^{k_\sigma + k}(x - 2^{-k}x_\sigma))\\ &= 2^{-k/2}\sqrt{\mu J_{\sigma^*}} e^{i y^l_{\sigma^*} x}\phi((x - x_{\sigma^*})\mu J_{\sigma^*}) = 2^{-k/2}\phi_{\sigma^*}(x).\end{aligned}$$

Write $\tilde F = 2^{-k}F$, so that $\mu\tilde F \leq 1$, and $\tilde g(x) = 2^k g(2^k x)$ for every x. Then, for $\sigma \in Q$,

$$\begin{aligned}F \cap g^{-1}[J^r_\sigma] &= \{x : x \in F,\, g(x) \in J^r_\sigma\} = \{x : 2^{-k}x \in \tilde F,\, 2^{-k}\tilde g(2^{-k}x) \in J^r_\sigma\}\\ &= \{x : 2^{-k}x \in \tilde F,\, \tilde g(2^{-k}x) \in J^r_{\sigma^*}\} = 2^k\{x : x \in \tilde F,\, \tilde g(x) \in J^r_{\sigma^*}\}.\end{aligned}$$

Write $\tilde f(x) = 2^{k/2} f(2^k x)$, so that

$$\|\tilde f\|_2 = 2^{k/2}\|D_{2^k} f\|_2 = \|f\|_2 = 1,$$

while

$$(f|\phi_\sigma) = \int_{-\infty}^{\infty} f \times \bar\phi_\sigma = 2^k \int_{-\infty}^{\infty} f(2^k x)\overline{\phi_\sigma(2^k x)}dx = (\tilde f|\phi_{\sigma^*})$$

for every $\sigma \in Q$. Putting all these together,

$$\begin{aligned}\sum_{\sigma\in Q}\Big|(f|\phi_\sigma)\int_{F\cap g^{-1}[J^r_\sigma]}\phi_\sigma\Big| &= 2^k\sum_{\sigma\in Q}\Big|(f|\phi_\sigma)\int_{2^{-k}(F\cap g^{-1}[J^r_\sigma])}\phi_\sigma(2^k x)dx\Big|\\ &= 2^{k/2}\sum_{\sigma\in Q}\Big|(\tilde f|\phi_{\sigma^*})\int_{\tilde F\cap\tilde g^{-1}[J^r_{\sigma^*}]}\phi_{\sigma^*}\Big|\\ &= 2^{k/2}\sum_{\tau\in Q}\Big|(\tilde f|\phi_\tau)\int_{\tilde F\cap\tilde g^{-1}[J^r_\tau]}\phi_\tau\Big|\\ &\leq 2^{k/2}C_8\end{aligned}$$

(by the Lacey-Thiele lemma, applied to $\tilde g$, $\tilde F$ and $\tilde f$)

$$\leq C_9\sqrt{\mu F} = C_9\|f\|_2\sqrt{\mu F}.$$

286O Lemma (a) For $z \in \mathbb{R}$, define $\theta_z : \mathbb{R} \to [0,1]$ by setting

$$\theta_z(y) = \hat{\phi}(2^{-k}(y - \hat{y}))^2$$

whenever there is a dyadic interval $J \in \mathcal{I}$ of length 2^k such that z belongs to the right-hand half of J and y belongs to the left-hand half of J and $\hat{y}$ is the lower quartile of J, and zero if there is no such J. Then $(y,z) \mapsto \theta_z(y)$ is Borel measurable, $0 \le \theta_z(y) \le 1$ for all y, $z \in \mathbb{R}$, and $\theta_z(y) = 0$ if $y \ge z$.

(b) For $k \in \mathbb{Z}$, set $Q_k = \{\sigma : \sigma \in Q,\ k_\sigma = k\}$. Let $[Q]^{<\omega}$ be the set of finite subsets of Q, $[\mathbb{Z}]^{<\omega}$ the set of finite subsets of $\mathbb{Z}$ and $\mathcal{L}$ the set of subsets L of Q such that $L \cap Q_k$ is finite for every k. If $K \in [\mathbb{Z}]^{<\omega}$ and $L \in \mathcal{L}$, set

$$\mathcal{P}_{KL} = \{P : P \in [Q]^{<\omega},\ P \cap Q_k \supseteq L \cap Q_k \text{ whenever } k \in \mathbb{Z} \text{ and either } k \in K \text{ or } P \cap Q_k \ne \emptyset\};$$

set

$$\mathcal{F} = \{\mathcal{P} : \mathcal{P} \subseteq [Q]^{<\omega} \text{ and there are } K \in [\mathbb{Z}]^{<\omega},\ L \in \mathcal{L} \text{ such that } \mathcal{P} \supseteq \mathcal{P}_{KL}\}.$$

Then $\mathcal{F}$ is a filter on $[Q]^{<\omega}$ and

$$2\pi \int_F (\hat{h} \times \theta_z)^\vee = \lim_{P \to \mathcal{F}} \sum_{\sigma \in P, z \in J_\sigma^r} (h|\phi_\sigma) \int_F \phi_\sigma$$

for every $z \in \mathbb{R}$ and rapidly decreasing test function h and measurable set $F \subseteq \mathbb{R}$ of finite measure.

proof (a)(i) I had better start by explaining why the recipe above defines a function θ_z. Let M be the set of those $k \in \mathbb{Z}$ such that z belongs to the right-hand half of the dyadic interval $\hat{J}_k$ of length 2^k containing z. For $k \in M$, let $\hat{y}_k$ be the midpoint of the left-hand half $\hat{J}_k^l$ of $\hat{J}_k$, and set $\psi_k(y) = \hat{\phi}(2^{-k}(y - \hat{y}_k))^2$ for $y \in \mathbb{R}$; then ψ_k is smooth and zero outside $\hat{J}_k^l$. But now observe that if k, k' are distinct members of M, then $\hat{J}_k^l$ and $\hat{J}_{k'}^l$ are disjoint, as remarked in 286E(b-iv). So θ_z is just the sum $\sum_{k \in M} \psi_k$. Because $\hat{\phi}$ takes values in $[0,1]$, so does θ_z. If $y \ge z$, then of course $y \notin \hat{J}_k^l$ for any $k \in M$, so $\theta_z(y) = 0$.

(ii) To see that $(y,z) \mapsto \theta_z(y)$ is Borel measurable, observe that

$$\{(y,z) : \theta_z(y) \ge \gamma\} = \bigcup_{\sigma \in Q} \{(y,z) : \hat{\phi}((y - y_\sigma^l)\mu I_\sigma)^2 \ge \gamma,\ z \in J_\sigma^r\}$$

for every $\gamma \in \mathbb{R}$.

(b)(i) $\emptyset$ belongs to both $[\mathbb{Z}]^{<\omega}$ and $\mathcal{L}$ and $[Q]^{<\omega} = \mathcal{P}_{\emptyset\emptyset}$ belongs to $\mathcal{F}$. If $K \in [\mathbb{Z}]^{<\omega}$ and $L \in \mathcal{L}$ then $\bigcup_{k \in K} L \cap Q_k$ belongs to $\mathcal{P}_{KL}$. So no $\mathcal{P}_{KL}$ is empty and $\emptyset \notin \mathcal{F}$.

If $\mathcal{P}$, $\mathcal{P}' \in \mathcal{F}$, there are K, $K' \in [\mathbb{Z}]^{<\omega}$ and L, $L' \in \mathcal{L}$ such that $\mathcal{P}_{KL} \subseteq \mathcal{P}$ and $\mathcal{P}_{K'L'} \subseteq \mathcal{P}'$. Now $K \cup K' \in [\mathbb{Z}]^{<\omega}$, $L \cup L' \in \mathcal{L}$ and

$$\mathcal{P}_{K \cup K', L \cup L'} \subseteq \mathcal{P}_{KL} \cap \mathcal{P}_{K'L'} \subseteq \mathcal{P} \cap \mathcal{P}',$$

so $\mathcal{P} \cap \mathcal{P}' \in \mathcal{F}$.

If $\mathcal{P} \in \mathcal{F}$ and $\mathcal{P} \subseteq \mathcal{P}' \subseteq [Q]^{<\omega}$, then of course $\mathcal{P}' \in \mathcal{F}$. So $\mathcal{F}$ is a filter on $[Q]^{<\omega}$.

(ii) Now fix on $z \in \mathbb{R}$, a rapidly decreasing test function h and a set F of finite measure. Take M and ψ_k, $\hat{J}_k$, $\hat{y}_k$ for $k \in M$ from (a-i) above; it will be convenient to set $\psi_k = 0$ for $k \in \mathbb{Z} \setminus M$, so that $\theta_z = \sum_{k \in \mathbb{Z}} \psi_k$.

For $k \in \mathbb{Z}$,

$$2\pi \int_F (\hat{h} \times \psi_k)^\vee = \sum_{\sigma \in Q_k, z \in J_\sigma^r} (h|\phi_\sigma) \int_F \phi_\sigma.$$

P If $k \notin M$, then $z \notin J_\sigma^r$ for any $\sigma \in Q_k$, while $\psi_k = 0$, so the result is trivial. So I will suppose that $k \in M$ and that $\hat{y}_k$ is defined. If $\sigma \in Q_k$ and $z \in J_\sigma^r$, $y_\sigma^l = \hat{y}_k$ and x_σ is of the form $2^{-k}(n + \frac{1}{2})$ for some $n \in \mathbb{Z}$. So

$$(h|\phi_\sigma) = \int_{-\infty}^{\infty} \hat{h} \times \bar{\hat{\phi}}_\sigma$$

(284O)

$$= \int_{-\infty}^{\infty} \hat{h}(t) \cdot 2^{-k/2} e^{2^{-k} i(n+\frac{1}{2})(t-\hat{y}_k)} \hat{\phi}(2^{-k}(t-\hat{y}_k))dt$$

(by the formula in 286Eb, because $\hat{\phi}$ is real-valued)

$$= 2^{k/2} \int_{-\infty}^{\infty} \hat{h}(2^k t + \hat{y}_k) e^{i(n+\frac{1}{2})t} \hat{\phi}(t)dt = 2^{k/2} \int_{-\pi}^{\pi} \hat{h}(2^k t + \hat{y}_k) e^{i(n+\frac{1}{2})t} \hat{\phi}(t)dt$$

(because $\hat{\phi}(t) = 0$ if $|t| \ge \frac{1}{5}$)

$$= 2^{k/2} \int_{-\pi}^{\pi} g(t) e^{int} dt,$$

where $g(t) = \hat{h}(2^k t + \hat{y}_k) e^{it/2} \hat{\phi}(t)$ for $-\pi < t \le \pi$. So if we set $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt$, as in 282A, we have

$$(h|\phi_\sigma) = 2^{k/2} \cdot 2\pi c_{-n}$$

when $\sigma \in Q_k$, $z \in J_\sigma^r$ and $x_\sigma = 2^{-k}(n+\frac{1}{2})$. Note that as g is smooth and zero outside $[-\frac{1}{5}, \frac{1}{5}]$, $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ (282Rb).

Now, for any $y \in \hat{J}_k^l$, writing R_k for

$$\{\sigma : \sigma \in Q_k,\, z \in J_\sigma^r\} = \{\sigma : \sigma \in Q_k,\, J_\sigma = \hat{J}_k\} = \{(I, \hat{J}_k) : I \in \mathcal{I},\, \mu I = 2^{-k}\},$$

we have

$$\begin{aligned}
\sum_{\sigma \in R_k} (h|\phi_\sigma)\hat{\phi}_\sigma(y) &= \sum_{n=-\infty}^{\infty} 2^{k/2} \cdot 2\pi c_{-n} \cdot 2^{-k/2} e^{-2^{-k} i(n+\frac{1}{2})(y-\hat{y}_k)} \hat{\phi}(2^{-k}(y-\hat{y}_k)) \\
&= 2\pi \hat{\phi}(2^{-k}(y-\hat{y}_k)) e^{-2^{-k-1} i(y-\hat{y}_k)} \sum_{n=-\infty}^{\infty} c_{-n} e^{-2^{-k} in(y-\hat{y}_k)} \\
&= 2\pi \hat{\phi}(2^{-k}(y-\hat{y}_k)) e^{-2^{-k-1} i(y-\hat{y}_k)} \sum_{n=-\infty}^{\infty} c_n e^{in2^{-k}(y-\hat{y}_k)} \\
&= 2\pi \hat{\phi}(2^{-k}(y-\hat{y}_k)) e^{-2^{-k-1} i(y-\hat{y}_k)} g(2^{-k}(y-\hat{y}_k))
\end{aligned}$$

(by 282L(i), because $2^{-k}|y-\hat{y}_k| \le \frac{1}{4} < \pi$ and g is smooth)

$$\begin{aligned}
&= 2\pi e^{-2^{-k-1} i(y-\hat{y}_k)} \hat{\phi}(2^{-k}(y-\hat{y}_k)) \hat{h}(y) e^{2^{-k-1} i(y-\hat{y}_k)} \hat{\phi}(2^{-k}(y-\hat{y}_k)) \\
&= 2\pi \hat{h}(y) \psi_k(y).
\end{aligned}$$

On the other hand, if $y \in \mathbb{R} \setminus \hat{J}_k^l$, $\psi_k(y) = \hat{\phi}_\sigma(y) = 0$ for every $\sigma \in R_k$, so again $\sum_{\sigma \in R_k} (h|\phi_\sigma)\hat{\phi}_\sigma(y) = 2\pi \hat{h}(y)\psi_k(y)$.

Next,

$$\sum_{\sigma \in R_k} |(h|\phi_\sigma)| = 2\pi \cdot 2^{k/2} \sum_{n=-\infty}^{\infty} |c_n|$$

and

$$\sup_{\sigma \in R_k} \int_{-\infty}^{\infty} |\hat{\phi}_\sigma| = 2^{k/2} \int_{-\infty}^{\infty} |\hat{\phi}|$$

are finite, while of course $\widehat{\chi F}$ is bounded. So

$$2\pi \int_F (\hat{h} \times \psi_k)^\vee = 2\pi (\hat{h} \times \psi_k)^\vee | \chi F) = 2\pi ((\hat{h} \times \psi_k)^\vee)^\wedge | \widehat{\chi F})$$

(284Ob again)

$$= 2\pi(\hat{h}\times\psi_k|\chi\hat{F}) = \int_{-\infty}^{\infty}2\pi\hat{h}\times\psi_k\times\overline{\chi\hat{F}}$$

$$= \int_{-\infty}^{\infty}\sum_{\sigma\in R_k}(h|\phi_\sigma)\hat{\phi}_\sigma\times\overline{\chi\hat{F}} = \sum_{\sigma\in R_k}(h|\phi_\sigma)\int_{-\infty}^{\infty}\hat{\phi}_\sigma\times\overline{\chi\hat{F}}$$

(226E)

$$= \sum_{\sigma\in R_k}(h|\phi_\sigma)\int_F\phi_\sigma = \sum_{\sigma\in Q_k,z\in J^r_\sigma}(h|\phi_\sigma)\int_F\phi_\sigma. \text{ **Q**}$$

(iii) In the last sentence of the argument just above, I quoted B.Levi's theorem in the form 226E, even though R_k has a natural enumeration, because I shall specifically want to say later that

for every $\epsilon > 0$ there is a finite $L_0 \subseteq R_k$ such that

$$|2\pi\int_F(\hat{h}\times\psi_k)^\vee - \sum_{\sigma\in L}(h|\phi_\sigma)\int_F\phi_\sigma| \le \epsilon$$

whenever $L \subseteq R_k$ is finite and $L \supseteq L_0$;

it follows at once that

for every $\epsilon > 0$ there is a finite $L_0 \subseteq Q_k$ such that

$$|2\pi\int_F(\hat{h}\times\psi_k)^\vee - \sum_{\sigma\in L,z\in J^r_\sigma}(h|\phi_\sigma)\int_F\phi_\sigma| \le \epsilon$$

whenever $L \subseteq Q_k$ is finite and $L \supseteq L_0$.

(iv) Now let us consider $(\hat{h}\times\theta_z)^\vee$. Because every ψ_k is non-negative, $\theta_z = \sum_{k\in\mathbb{Z}}\psi_k$ is bounded above by 1, and $\hat{h}$ is integrable,

$$\int_F(\hat{h}\times\theta_z)^\vee = \int_{-\infty}^{\infty}\hat{h}\times\theta_z\times\chi\hat{F}$$

$$= \sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}\hat{h}\times\psi_k\times\chi\hat{F} = \sum_{k\in\mathbb{Z}}\int_F(\hat{h}\times\psi_k)^\vee.$$

So here we can say

for every $\epsilon > 0$ there is a $K_0 \in [\mathbb{Z}]^{<\omega}$ such that

$$|\int_F(\hat{h}\times\theta_z)^\vee - \sum_{k\in K}\int_F(\hat{h}\times\psi_k)^\vee| \le \epsilon$$

whenever $K \in [\mathbb{Z}]^{<\omega}$ and $K \supseteq K_0$.

(v) To express the facts above in terms of a limit along the filter $\mathcal{F}$, we can argue as follows. Take any $\epsilon > 0$. For each $k \in \mathbb{Z}$, (iii) tells us that there is a finite set $L_k \subseteq Q_k$ such that

$$|2\pi\int_F(\hat{h}\times\psi_k)^\vee - \sum_{\sigma\in L',z\in J^r_\sigma}(h|\phi_\sigma)\int_F\phi_\sigma| \le 2^{-|k|}\epsilon$$

whenever $L' \subseteq Q_k$ is finite and $L' \supseteq L_k$; of course we can suppose that every L_k is non-empty. Set $L = \bigcup_{k\in\mathbb{Z}}L_k$, so that $L\cap Q_k = L_k$ is finite for each k, and $L \in \mathcal{L}$. Next, there is a $K \in [\mathbb{Z}]^{<\omega}$ such that

$$|\int_F(\hat{h}\times\theta_z)^\vee - \sum_{k\in K'}\int_F(\hat{h}\times\psi_k)^\vee| \le \epsilon$$

whenever $K' \in [\mathbb{Z}]^{<\omega}$ and $K' \supseteq K$. Take any $P \in \mathcal{P}_{KL}$. Setting $K' = \{k : P\cap Q_k \ne \emptyset\}$, we have $K' \supseteq K$ and $P\cap Q_k \supseteq L\cap Q_k$ for every $k \in K'$. Accordingly

$$|2\pi\int_F(\hat{h}\times\theta_z)^\vee-\sum_{\sigma\in P,z\in J^r_\sigma}(h|\phi_\sigma)\int_F\phi_\sigma|$$
$$=|2\pi\int_F(\hat{h}\times\theta_z)^\vee-\sum_{k\in K'}\sum_{\substack{\sigma\in P\cap Q_k\\ z\in J^r_\sigma}}(h|\phi_\sigma)\int_F\phi_\sigma|$$
$$\le 2\pi|\int_F(\hat{h}\times\theta_z)^\vee-\sum_{k\in K'}\int_F(\hat{h}\times\psi_k)^\vee|$$
$$+\sum_{k\in K'}|2\pi\int_F(\hat{h}\times\psi_k)^\vee-\sum_{\substack{\sigma\in P\cap Q_k\\ z\in J^r_\sigma}}(h|\phi_\sigma)\int_F\phi_\sigma|$$
$$\le 2\pi\epsilon+\sum_{k\in K'}2^{-|k|}\epsilon\le(2\pi+3)\epsilon.$$

As $\mathcal{P}_{KL}\in\mathcal{F}$, and ϵ was arbitrary,

$$2\pi\int_F(\hat{h}\times\theta_z)^\vee=\lim_{P\to\mathcal{F}}\sum_{\sigma\in P,z\in J^r_\sigma}(h|\phi_\sigma)\int_F\phi_\sigma$$

as claimed.

286P Lemma Suppose that h is a rapidly decreasing test function. For $x\in\mathbb{R}$, set

$$Ah(x)=\sup_{z\in\mathbb{R}}|2\pi(\hat{h}\times\theta_z)^\vee(x)|.$$

Then $Ah:\mathbb{R}\to[0,\infty]$ is Borel measurable, and $\int_F Ah\le 4C_9\|h\|_2\sqrt{\mu F}$ whenever $\mu F<\infty$.

proof (a) As $(\hat{h}\times\theta_z)^\vee$ is continuous for every z, Ah is lower semi-continuous, therefore Borel measurable, by 256Ma. By 256Mb,

$$\int_F Ah=\sup\{\int_F\sup_{i\le n}|2\pi(\hat{h}\times\theta_{z_i})^\vee|:z_0,\dots,z_n\in\mathbb{R}\}.$$

(b) Fix $z_0,\dots,z_n\in\mathbb{R}$ for the moment.

(i) Set $v_i=2\pi(\hat{h}\times\theta_{z_i})^\vee$ for $i\le n$, and $v=\sup_{i\le n}|v_i|$. Set $E_i=\{x:v(x)=|v_i(x)|\}\setminus\bigcup_{j<i}\{x:v(x)=|v_j(x)|\}$ for $i\le n$, so that $(E_0,\dots,E_n)$ is a partition of $\mathbb{R}$ into Borel sets, and

$$\int_F v=\int_F\sum_{i=0}^n|v_i|\times\chi E_i=\int_F|\sum_{i=0}^n v_i\times\chi E_i|\le 4|\int_{F'}\sum_{i=0}^n v_i\times\chi E_i|$$

for a suitable measurable $F'\subseteq F$ (246K). Setting $g(x)=z_i$ for $x\in E_i$, $g:\mathbb{R}\to\mathbb{R}$ is Borel measurable.

(ii) For each $i\le n$,

$$\int_{F'}v_i\times\chi E_i=\int_{F'\cap E_i}v_i=\lim_{P\to\mathcal{F}}\sum_{\sigma\in P,z_i\in J^r_\sigma}(h|\phi_\sigma)\int_{F'\cap E_i}\phi_\sigma$$

(where $\mathcal{F}$ is the filter on $[Q]^{<\omega}$ described in 286O)

$$=\lim_{P\to\mathcal{F}}\int_{F'\cap E_i}\sum_{\sigma\in P,z_i\in J^r_\sigma}(h|\phi_\sigma)\phi_\sigma$$
$$=\lim_{P\to\mathcal{F}}\int_{F'\cap E_i}\sum_{\sigma\in P,g(x)\in J^r_\sigma}(h|\phi_\sigma)\phi_\sigma(x)dx.$$

So

$$\int_{F'}\sum_{i=0}^{n}v_i\times\chi E_i=\lim_{P\to\mathcal{F}}\sum_{i=0}^{n}\int_{F'\cap E_i}\sum_{\sigma\in P,g(x)\in J^r_\sigma}(h|\phi_\sigma)\phi_\sigma(x)dx$$
$$=\lim_{P\to\mathcal{F}}\int_{F'}\sum_{\sigma\in P,g(x)\in J^r_\sigma}(h|\phi_\sigma)\phi_\sigma(x)dx.$$

Now for any finite set $P\subseteq Q$,

$$\int_{F'}\sum_{\sigma\in P,g(x)\in J^r_\sigma}(h|\phi_\sigma)\phi_\sigma(x)dx=\sum_{\sigma\in P}\int_{F'\cap g^{-1}[J^r_\sigma]}(h|\phi_\sigma)\phi_\sigma;$$

if you like, you can think of this as an application of Fubini's theorem, if you give counting measure to Q and look at the function

$$(x,\sigma)\mapsto(h|\phi_\sigma)\phi_\sigma(x)\text{ if }x\in F',\ \sigma\in P\text{ and }g(x)\in J^r_\sigma,$$
$$\mapsto 0\text{ otherwise.}$$

But this means that

$$|\int_{F'}\sum_{\sigma\in P,g(x)\in J^r_\sigma}(h|\phi_\sigma)\phi_\sigma(x)dx|\le\sum_{\sigma\in P}|(h|\phi_\sigma)\int_{F'\cap g^{-1}[J^r_\sigma]}\phi_\sigma|\le C_9\|h\|_2\sqrt{\mu F'}$$

by 286N. Taking the limit as $P\to\mathcal{F}$,

$$|\int_{F'}\sum_{i=0}^{n}v_i\times\chi E_i|\le C_9\|h\|_2\sqrt{\mu F'}.$$

(iii) Thus we have

$$\int_F\sup_{i\le n}|2\pi(\hat{h}\times\theta_{z_i})^\vee|=\int_F v\le 4|\int_{F'}\sum_{i=0}^{n}v_i\times\chi E_i|$$
$$\le 4C_9\|h\|_2\sqrt{\mu F'}\le 4C_9\|h\|_2\sqrt{\mu F}.$$

(c) As $z_0,\dots,z_n$ were arbitrary, $\int_F Ah\le 4C_9\|h\|_2\sqrt{\mu F}$, as claimed.

286Q Lemma For $\alpha>0$ and $y,\ z,\ \beta\in\mathbb{R}$, set $\theta'_{z\alpha\beta}(y)=\theta_{\alpha z+\beta}(\alpha y+\beta)$. Then
(a) the function $(\alpha,\beta,y,z)\mapsto\theta'_{z\alpha\beta}(y):\,]0,\infty[\,\times\mathbb{R}^3\to[0,1]$ is Borel measurable;
(b) $\theta'_{z\alpha\beta}(y)=0$ whenever $y\ge z$;
(c) for any rapidly decreasing test function h, and any $z\in\mathbb{R}$,

$$2\pi|(\hat{h}\times\theta'_{z\alpha\beta})^\vee|\le D_{1/\alpha}AM_\beta D_\alpha h$$

(in the notation of 286C) at every point.

proof (a) We need only recall that $(y,z)\mapsto\theta_z(y):\mathbb{R}^2\to\mathbb{R}$ is Borel measurable (286Oa), and that $(\alpha,\beta,y,z)\mapsto\theta'_{z\alpha\beta}(y)$ is built up from this, $+$ and $\times$.

(b) Again, this is immediate from 286Oa, because $\alpha>0$.

(c) Set $v=\alpha z+\beta$, so that $\theta'_{z\alpha\beta}=D_\alpha S_\beta\theta_v$. Then

$$\hat{h}\times\theta'_{z\alpha\beta}=\hat{h}\times D_\alpha S_\beta\theta_v=D_\alpha S_\beta(S_{-\beta}D_{1/\alpha}\hat{h}\times\theta_v)$$
$$=\alpha D_\alpha S_\beta(S_{-\beta}(D_\alpha h)^\wedge\times\theta_v)=\alpha D_\alpha S_\beta((M_\beta D_\alpha h)^\wedge\times\theta_v),$$

so

$$(\hat{h}\times\theta'_{z\alpha\beta})^\vee=\alpha\big(D_\alpha S_\beta((M_\beta D_\alpha h)^\wedge\times\theta_v)\big)^\vee$$
$$=D_{1/\alpha}\big(S_\beta((M_\beta D_\alpha h)^\wedge\times\theta_v)\big)^\vee=D_{1/\alpha}M_{-\beta}\big((M_\beta D_\alpha h)^\wedge\times\theta_v\big)^\vee$$

and

$$2\pi|(\hat{h}\times\theta'_{z\alpha\beta})^\vee|=2\pi D_{1/\alpha}\big|\big((M_\beta D_\alpha h)^\wedge\times\theta_v\big)^\vee\big|\le D_{1/\alpha}A(M_\beta D_\alpha h).$$

286R Lemma For any y, $z \in \mathbb{R}$,

$$\tilde{\theta}_z(y) = \int_1^2 \frac{1}{\alpha}\Bigl(\lim_{n\to\infty} \frac{1}{n}\int_0^n \theta'_{z\alpha\beta}(y)d\beta\Bigr)d\alpha$$

is defined, and

$$\begin{aligned}\tilde{\theta}_z(y) &= \tilde{\theta}_1(0) > 0 \text{ if } y < z,\\ &= 0 \text{ if } y \ge z.\end{aligned}$$

proof (a) The case $y \ge z$ is trivial, because if $y \ge z$ then $\theta'_{z\alpha\beta}(y) = 0$ for all $\alpha > 0$ and $\beta \in \mathbb{R}$ (286Qb) and $\tilde{\theta}_z(y) = 0$. For the rest of the proof, therefore, I look at the case $y < z$.

(b)(i) Given $y < z \in \mathbb{R}$ and $\alpha > 0$, set $l = \lfloor \log_2(20\alpha(z-y))\rfloor$. Then $\theta'_{z,\alpha,\beta+2^l}(y) = \theta'_{z\alpha\beta}(y)$ for every $\beta \in \mathbb{R}$. **P** If $\theta'_{z\alpha\beta}(y) = \theta_{\alpha z+\beta}(\alpha y+\beta)$ is non-zero, there must be k, $m \in \mathbb{Z}$ such that

$$2^k(m+\tfrac{1}{2}) \le \alpha z+\beta < 2^k(m+1)$$

and

$$\hat{\phi}(2^{-k}(\alpha y+\beta)-(m+\tfrac{1}{4}))^2 = \theta'_{z\alpha\beta}(y) \ne 0,$$

so

$$2^k m \le \alpha y+\beta \le 2^k(m+\tfrac{9}{20})$$

because $\hat{\phi}$ is zero outside $[-\frac{1}{5}, \frac{1}{5}]$. In this case, $\dfrac{1}{20}\cdot 2^k < \alpha(z-y)$, so that $k \le l$. We therefore have

$$2^k(m+2^{l-k}+\tfrac{1}{2}) \le \alpha z+\beta+2^l < 2^k(m+2^{l-k}+1),$$

$$2^k(m+2^{l-k}) \le \alpha y+\beta+2^l < 2^k(m+2^{l-k}+\tfrac{1}{2}),$$

so

$$\theta'_{z,\alpha,\beta+2^l}(y) = \hat{\phi}(2^{-k}(\alpha y+\beta+2^l)-(m+2^{l-k}+\tfrac{1}{4}))^2 = \theta'_{z\alpha\beta}(y).$$

Similarly,

$$2^k(m-2^{l-k}+\tfrac{1}{2}) \le \alpha z+\beta-2^l < 2^k(m-2^{l-k}+1),$$

$$2^k(m-2^{l-k}) \le \alpha y+\beta-2^l < 2^k(m-2^{l-k}+\tfrac{1}{2}),$$

so

$$\theta'_{z,\alpha,\beta-2^l}(y) = \hat{\phi}(2^{-k}(\alpha y+\beta-2^l)-(m-2^{l-k}+\tfrac{1}{4}))^2 = \theta'_{z\alpha\beta}(y).$$

What this shows is that $\theta'_{z,\alpha,\beta+2^l}(y) = \theta'_{z\alpha\beta}(y)$ if either is non-zero, so we have the equality in any case. **Q**

(ii) It follows that $g(\alpha,y,z) = \lim_{b\to\infty} \frac{1}{b}\int_0^b \theta'_{z\alpha\beta}(y)d\beta$ is defined. **P** Set

$$\gamma = 2^{-l}\int_0^{2^l} \theta'_{z\alpha\beta}(y)d\beta.$$

From (i) we see that

$$\gamma = 2^{-l}\int_{2^l m}^{2^l(m+1)} \theta'_{z\alpha\beta}(y)d\beta$$

for every $m \in \mathbb{Z}$, and therefore that

$$\gamma = \frac{1}{2^l m}\int_0^{2^l m} \theta'_{z\alpha\beta}(y)d\beta$$

for every $m \ge 1$. Now $\theta'_{z\alpha\beta}(y)$ is always greater than or equal to 0, so if $2^l m \le b \le 2^l(m+1)$ then

$$\frac{m}{m+1}\gamma = \frac{1}{2^l(m+1)}\int_0^{2^l m}\theta'_{z\alpha\beta} \le \frac{1}{b}\int_0^b \theta'_{z\alpha\beta} \le \frac{1}{2^l m}\int_0^{2^l(m+1)}\theta'_{z\alpha\beta} = \frac{m+1}{m}\gamma,$$

which approach γ as $b \to \infty$. **Q**

(c) Because $(\alpha, y) \mapsto \theta'_{z\alpha\beta}(y)$ is always Borel measurable, each of the functions $\alpha \mapsto \frac{1}{n}\int_0^n \theta'_{z\alpha\beta}$, for $n \ge 1$, is Borel measurable (putting 251M and 252P together), and $\alpha \mapsto g(\alpha, y, z) :]0,\infty[\to \mathbb{R}$ is Borel measurable; at the same time, since $0 \le \theta'_{z\alpha\beta}(y) \le 1$ for all α and β, $0 \le g(\alpha, y, z) \le 1$ for every α, and $\tilde\theta_z(y) = \int_1^2 \frac{1}{\alpha} g(\alpha, y, z)d\alpha$ is defined in $[0, 1]$.

(d) For any $y < z$, $\gamma \in \mathbb{R}$ and $\alpha > 0$, $g(\alpha, y+\gamma, z+\gamma) = g(\alpha, y, z)$. **P** It is enough to consider the case $\gamma \ge 0$. In this case

$$\begin{aligned} g(\alpha, y+\gamma, z+\gamma) &= \lim_{b\to\infty}\frac{1}{b}\int_0^b \theta'_{z+\gamma,\alpha,\beta}(y+\gamma)d\beta \\ &= \lim_{b\to\infty}\frac{1}{b}\int_0^b \theta_{\alpha z+\alpha\gamma+\beta}(\alpha y+\alpha\gamma+\beta)d\beta \\ &= \lim_{b\to\infty}\frac{1}{b}\int_{\alpha\gamma}^{b+\alpha\gamma} \theta_{\alpha z+\beta}(\alpha y+\beta)d\beta = \lim_{b\to\infty}\frac{1}{b}\int_{\alpha\gamma}^{b+\alpha\gamma}\theta'_{z\alpha\beta}(y)d\beta, \end{aligned}$$

so

$$\begin{aligned} |g(\alpha, y+\gamma, z+\gamma) - g(\alpha, y, z)| &= \lim_{b\to\infty}\frac{1}{b}\Bigl|\int_b^{b+\alpha\gamma}\theta'_{z\alpha\beta}(y)d\beta - \int_0^{\alpha\gamma}\theta'_{z\alpha\beta}(y)d\beta\Bigr| \\ &\le \lim_{b\to\infty}\frac{2\alpha\gamma}{b} = 0. \text{ } \mathbf{Q} \end{aligned}$$

It follows that whenever $y < z$ and $\gamma \in \mathbb{R}$,

$$\tilde\theta_{z+\gamma}(y+\gamma) = \int_0^1 \frac{1}{\alpha} g(\alpha, y+\gamma, z+\gamma)d\alpha = \int_0^1 \frac{1}{\alpha} g(\alpha, y, z)d\alpha = \tilde\theta_z(y).$$

(e) The next essential fact to note is that $\theta_{2z}(2y)$ is always equal to $\theta_z(y)$. **P** If $\theta_z(y) \ne 0$, then (as in (b) above) there are k, $m \in \mathbb{Z}$ such that

$$2^k(m+\tfrac{1}{2}) \le z < 2^k(m+1), \quad 2^k m \le y < 2^k(m+\tfrac{1}{2}), \quad \theta_z(y) = \hat\phi(2^{-k}y - (m+\tfrac{1}{4}))^2.$$

In this case,

$$2^{k+1}(m+\tfrac{1}{2}) \le 2z < 2^{k+1}(m+1), \quad 2^{k+1}m \le 2y < 2^{k+1}(m+\tfrac{1}{2}),$$

so

$$\theta_{2z}(2y) = \hat\phi(2^{-k-1}\cdot 2y - (m+\tfrac{1}{4}))^2 = \theta_z(y).$$

Similarly,

$$2^{k-1}(m+\tfrac{1}{2}) \le \tfrac{1}{2}z < 2^{k-1}(m+1), \quad 2^{k-1}m \le \tfrac{1}{2}y < 2^{k-1}(m+\tfrac{1}{2}),$$

so

$$\theta_{\frac{1}{2}z}(\tfrac{1}{2}y) = \hat\phi(2^{-k+1}\cdot\tfrac{1}{2}y - (m+\tfrac{1}{4}))^2 = \theta_z(y).$$

This shows that $\theta_{2z}(2y) = \theta_z(y)$ if either is non-zero, and therefore in all cases. **Q**

Accordingly

$$\theta'_{z,2\alpha,2\beta}(y) = \theta_{2\alpha z+2\beta}(2\alpha y+2\beta) = \theta_{\alpha z+\beta}(\alpha y+\beta) = \theta'_{z\alpha\beta}(y)$$

for all y, z, $\beta\in\mathbb{R}$ and all $\alpha>0$.

(f) Consequently

$$\begin{aligned}g(2\alpha,y,z)&=\lim_{b\to\infty}\frac{1}{b}\int_0^b\theta'_{z,2\alpha,\beta}(y)d\beta=\lim_{b\to\infty}\frac{2}{b}\int_0^{b/2}\theta'_{z,2\alpha,2\beta}(y)d\beta\\&=\lim_{b\to\infty}\frac{2}{b}\int_0^{b/2}\theta'_{z\alpha\beta}(y)d\beta=\lim_{b\to\infty}\frac{1}{b}\int_0^b\theta'_{z\alpha\beta}(y)d\beta=g(\alpha,y,z)\end{aligned}$$

whenever $\alpha>0$ and y, $z\in\mathbb{R}$. It follows that

$$\int_\gamma^\delta\frac{1}{\alpha}g(\alpha,y,z)d\alpha=\int_\gamma^\delta\frac{1}{\alpha}g(2\alpha,y,z)d\alpha=\int_{2\gamma}^{2\delta}\frac{1}{\alpha}g(\alpha,y,z)d\alpha$$

whenever $0<\gamma\le\delta$, and therefore that

$$\int_\gamma^{2\gamma}\frac{1}{\alpha}g(\alpha,y,z)d\alpha=\int_1^2\frac{1}{\alpha}g(\alpha,y,z)d\alpha$$

for every $\gamma>0$. **P** Take $k\in\mathbb{Z}$ such that $2^k\le\gamma<2^{k+1}$. Then

$$\begin{aligned}\int_\gamma^{2\gamma}\frac{1}{\alpha}g(\alpha,y,z)d\alpha&=\int_{2^k}^{2^{k+1}}\frac{1}{\alpha}g(\alpha,y,z)d\alpha-\int_{2^k}^{\gamma}\frac{1}{\alpha}g(\alpha,y,z)d\alpha+\int_{2^{k+1}}^{2\gamma}\frac{1}{\alpha}g(\alpha,y,z)d\alpha\\&=\int_{2^k}^{2^{k+1}}\frac{1}{\alpha}g(\alpha,y,z)d\alpha=\int_1^2\frac{1}{\alpha}g(\alpha,y,z)d\alpha.\ \mathbf{Q}\end{aligned}$$

(g) Now if α, $\gamma>0$ and $y<z$,

$$g(\alpha,\gamma y,\gamma z)=\lim_{b\to\infty}\frac{1}{b}\int_0^b\theta_{\alpha\gamma z+\beta}(\alpha\gamma y+\beta)d\beta=g(\alpha\gamma,y,z).$$

So if $\gamma>0$ and $y<z$,

$$\begin{aligned}\tilde\theta_{\gamma z}(\gamma y)&=\int_1^2\frac{1}{\alpha}g(\alpha,\gamma y,\gamma z)d\alpha=\int_1^2\frac{1}{\alpha}g(\alpha\gamma,y,z)d\alpha\\&=\int_\gamma^{2\gamma}\frac{1}{\alpha}g(\alpha,y,z)d\alpha=\int_1^2\frac{1}{\alpha}g(\alpha,y,z)d\alpha=\tilde\theta_z(y).\end{aligned}$$

Putting this together with (d), we see that if $y<z$ then

$$\tilde\theta_z(y)=\tilde\theta_{z-y}(0)=\tilde\theta_1(0).$$

(h) I have still to check that $\tilde\theta_1(0)$ is not zero. But suppose that $1\le\alpha<\frac{7}{6}$ and that there is some $m\in\mathbb{Z}$ such that $2(m+\frac{1}{12})\le\beta\le2(m+\frac{5}{12})$. Then $2(m+\frac{1}{2})\le\alpha+\beta<2(m+1)$, while $|\frac{1}{2}\beta-(m+\frac{1}{4})|\le\frac{1}{6}$, so

$$\theta_{\alpha+\beta}(\beta)=\hat\phi(\frac{1}{2}\beta-(m+\frac{1}{4}))^2=1.$$

What this means is that, for $1\le\alpha<\frac{7}{6}$,

$$\begin{aligned}g(\alpha,0,1)&=\lim_{m\to\infty}\frac{1}{2m}\int_0^{2m}\theta_{\alpha+\beta}(\beta)d\beta\\&\ge\lim_{m\to\infty}\frac{1}{2m}\sum_{j=0}^{m-1}\mu[2(j+\frac{1}{12}),2(j+\frac{5}{12})]=\frac{1}{3}.\end{aligned}$$

So

$$\tilde\theta_1(0)=\int_1^2\frac{1}{\alpha}g(\alpha,0,1)d\alpha\ge\frac{1}{3}\int_1^{7/6}\frac{1}{\alpha}d\alpha>0.$$

This completes the proof.

286S Lemma Suppose that h is a rapidly decreasing test function.

(a) For every $x \in \mathbb{R}$,

$$(\tilde{A}h)(x) = \liminf_{n\to\infty} \frac{1}{n}\int_1^2 \frac{1}{\alpha}\int_0^n (D_{1/\alpha}AM_\beta D_\alpha h)(x)d\beta d\alpha$$

is defined in $[0,\infty]$, and $\tilde{A}h : \mathbb{R} \to [0,\infty]$ is Borel measurable.

(b) $\int_F \tilde{A}h \le 3C_9\|h\|_2\sqrt{\mu F}$ whenever $\mu F < \infty$.

(c) If $z \in \mathbb{R}$, $2\pi|(\hat{h} \times \tilde{\theta}_z)^\vee| \le \tilde{A}h$ at every point.

proof (a) The point here is that the function

$$(\alpha,\beta,x) \mapsto (D_{1/\alpha}AM_\beta D_\alpha h)(x) :]0,\infty[\times \mathbb{R}^2 \to [0,\infty]$$

is Borel measurable. **P**

$$\begin{aligned}(D_{1/\alpha}AM_\beta D_\alpha h)(x) &= (AM_\beta D_\alpha h)(\frac{x}{\alpha})\\ &= \sup_{z\in\mathbb{R}} |2\pi((M_\beta D_\alpha h)^\wedge \times \theta_z)^\vee(\frac{x}{\alpha})|\\ &= \frac{2\pi}{\alpha}\sup_{z\in\mathbb{R}} |(S_{-\beta}D_{1/\alpha}\hat{h} \times \theta_z)^\vee(\frac{x}{\alpha})|.\end{aligned}$$

Now, for any $z \in \mathbb{R}$,

$$(S_{-\beta}D_{1/\alpha}\hat{h} \times \theta_z)^\vee(\tfrac{x}{\alpha}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy/\alpha}\hat{h}(\tfrac{y-\beta}{\alpha})\theta_z(y)dy.$$

We know that $\hat{h}$ is a rapidly decreasing test function, so there is some $\gamma \ge 0$ such that $|\hat{h}(t)| \le \frac{\gamma}{1+t^2}$ for every $t \in \mathbb{R}$. This means that if $\alpha > 0$ and $\beta \in \mathbb{R}$ and $\langle\alpha_n\rangle_{n\in\mathbb{N}}$, $\langle\beta_n\rangle_{n\in\mathbb{N}}$ are sequences in $]0,2\alpha]$ and $[\beta-1,\beta+1]$, converging to α, β respectively, and we set $g(t) = \sup_{n\in\mathbb{N}} |\hat{h}(\frac{t-\beta_n}{\alpha_n})\theta_z(t)|$, then

$$\begin{aligned}g(t) &\le \tfrac{4\gamma\alpha^2}{(|t|-|\beta|-1)^2} \text{ if } |t| \ge |\beta|+2,\\ &\le \gamma \text{ otherwise,}\end{aligned}$$

and g is integrable. (Remember that $0 \le \theta_z(y) \le 1$ for every y, as noted in 286Oa.) So Lebesgue's Dominated Convergence Theorem tells us that if $\langle\alpha_n\rangle_{n\in\mathbb{N}} \to \alpha$ and $\langle\beta_n\rangle_{n\in\mathbb{N}} \to \beta$ and $\langle x_n\rangle_{n\in\mathbb{N}} \to x$, then

$$\int_{-\infty}^{\infty} e^{ix_ny/\alpha_n}\hat{h}(\tfrac{y-\beta_n}{\alpha_n})\theta_z(y)dy \to \int_{-\infty}^{\infty} e^{ixy/\alpha}\hat{h}(\tfrac{y-\beta}{\alpha})\theta_z(y)dy.$$

Thus $(\alpha,\beta,x) \mapsto (S_{-\beta}D_{1/\alpha}\hat{h} \times \theta_z)^\vee(\frac{x}{\alpha})$ is continuous; and this is true for every $z \in \mathbb{R}$. Consequently

$$(\alpha,\beta,x) \mapsto \sup_{z\in\mathbb{R}} |(S_{-\beta}D_{1/\alpha}\hat{h} \times \theta_z)^\vee(\frac{x}{\alpha})|$$

and $(\alpha,\beta,x) \mapsto (D_{1/\alpha}AM_\beta D_\alpha h)(x)$ are lower semi-continuous, therefore Borel measurable, by 256Ma again. **Q**

It follows that the repeated integrals

$$\int_1^2 \frac{1}{\alpha}\int_0^n (D_{1/\alpha}AM_\beta D_\alpha h)(x)d\beta d\alpha$$

are defined in $[0,\infty]$ and are Borel measurable functions of x (252P again), so that $\tilde{A}f$ is Borel measurable.

(b) For any $n \in \mathbb{N}$,

$$\int_F \frac{1}{n}\int_1^2 \frac{1}{\alpha}\int_0^n (D_{1/\alpha}AM_\beta D_\alpha h)(x)d\beta d\alpha dx$$

$$= \frac{1}{n}\int_1^2 \frac{1}{\alpha}\int_0^n \int_F (D_{1/\alpha}AM_\beta D_\alpha h)(x)dxd\beta d\alpha$$

(by Fubini's theorem, 252H)

$$= \frac{1}{n}\int_1^2 \int_0^n \int_F \frac{1}{\alpha}(AM_\beta D_\alpha h)(\frac{x}{\alpha})dxd\beta d\alpha$$

$$= \frac{1}{n}\int_1^2 \int_0^n \int_{\alpha^{-1}F}(AM_\beta D_\alpha h)(x)dxd\beta d\alpha$$

$$\le \frac{1}{n}\int_1^2 \int_0^n 4C_9\|M_\beta D_\alpha h\|_2\sqrt{\mu(\alpha^{-1}F)}d\beta d\alpha$$

(286P)

$$= 4C_9\cdot\frac{1}{n}\int_1^2 \int_0^n \frac{1}{\sqrt{\alpha}}\|h\|_2\cdot\frac{1}{\sqrt{\alpha}}\sqrt{\mu F}d\beta d\alpha$$

$$= 4C_9\|h\|_2\sqrt{\mu F}\cdot\frac{1}{n}\int_1^2 \frac{1}{\alpha}\int_0^n d\beta d\alpha$$

$$= 4C_9\|h\|_2\sqrt{\mu F}\ln 2 \le 3C_9\|h\|_2\sqrt{\mu F}.$$

So

$$\int_F \tilde{A}h = \int_F \liminf_{n\to\infty}\frac{1}{n}\int_1^2 \frac{1}{\alpha}\int_0^n (D_{1/\alpha}AM_\beta D_\alpha h)(x)d\beta d\alpha dx$$

$$\le \liminf_{n\to\infty}\int_F \frac{1}{n}\int_1^2 \frac{1}{\alpha}\int_0^n (D_{1/\alpha}AM_\beta D_\alpha h)(x)d\beta d\alpha dx$$

(by Fatou's lemma)

$$\le 3C_9\|h\|_2\sqrt{\mu F}.$$

(c) For any $x\in\mathbb{R}$,

$$\int_{-\infty}^{\infty}|\hat{h}(y)|\int_1^2\frac{1}{\alpha}\Bigl(\sup_{n\in\mathbb{N}}\frac{1}{n}\int_0^n\theta'_{z\alpha\beta}(y)d\beta\Bigr)d\alpha dy \le \ln 2\cdot\int_{-\infty}^{\infty}|\hat{h}|$$

is finite. So

$$(\hat{h}\times\tilde{\theta}_z)^\vee(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixy}\hat{h}(y)\tilde{\theta}_z(y)dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixy}\hat{h}(y)\int_1^2\frac{1}{\alpha}\lim_{n\to\infty}\frac{1}{n}\int_0^n\theta'_{z\alpha\beta}(y)d\beta d\alpha dy$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{ixy}\hat{h}(y)\int_1^2\frac{1}{\alpha n}\int_0^n\theta'_{z\alpha\beta}(y)d\beta d\alpha dy$$

(by Lebesgue's Dominated Convergence Theorem)

$$= \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty}\int_1^2\frac{1}{\alpha n}\int_0^n\int_{-\infty}^{\infty}e^{ixy}\hat{h}(y)\theta'_{z\alpha\beta}(y)dyd\beta d\alpha$$

(by Fubini's theorem)

$$= \lim_{n\to\infty}\int_1^2\frac{1}{\alpha n}\int_0^n(\hat{h}\times\theta'_{z\alpha\beta})^\vee(x)d\beta d\alpha,$$

and

$$
\begin{aligned}
2\pi|(\hat{h}\times\tilde{\theta}_z)^\vee(x)| &= 2\pi\Bigl|\lim_{n\to\infty}\int_1^2\frac{1}{\alpha n}\int_0^n(\hat{h}\times\theta'_{z\alpha\beta})^\vee(x)d\beta d\alpha\Bigr| \\
&\le 2\pi\liminf_{n\to\infty}\int_1^2\frac{1}{\alpha n}\int_0^n|(\hat{h}\times\theta'_{z\alpha\beta})^\vee(x)|d\beta d\alpha \\
&\le \liminf_{n\to\infty}\int_1^2\frac{1}{\alpha n}\int_0^n(D_{1/\alpha}AM_\beta D_\alpha h)(x)d\beta d\alpha
\end{aligned}
$$

(286Qb)

$$= (\tilde{A}h)(x).$$

286T Lemma Set $C_{10}=3C_9/\pi\tilde{\theta}_1(0)$. For $f\in\mathcal{L}^2_{\mathbb{C}}$, define $\hat{A}f:\mathbb{R}\to[0,\infty]$ by setting

$$(\hat{A}f)(y)=\sup_{a\le b}\frac{1}{\sqrt{2\pi}}|\int_a^b e^{-ixy}f(x)dx|$$

for each $y\in\mathbb{R}$. Then $\int_F\hat{A}f\le C_{10}\|f\|_2\sqrt{\mu F}$ whenever $\mu F<\infty$.

proof (a) As usual, the first step is to confirm that $\hat{A}f$ is measurable. **P** For $a\le b$, $y\mapsto|\frac{1}{\sqrt{2\pi}}\int_a^b e^{-ixy}f(x)dx|$ is continuous (by 283Cf, since $f\times\chi[a,b]$ is integrable), so $\hat{A}f$ is lower semi-continuous, therefore Borel measurable (256Ma once more). **Q**

(b) Suppose that h is a rapidly decreasing test function. Then

$$(\hat{A}h)(y)\le\frac{1}{\pi\tilde{\theta}_1(0)}(\tilde{A}\check{h})(-y)$$

for every $y\in\mathbb{R}$. **P** If $a\in\mathbb{R}$ then

$$\frac{1}{\sqrt{2\pi}}|\int_{-\infty}^a e^{-ixy}h(x)dx|=\frac{1}{\tilde{\theta}_1(0)\sqrt{2\pi}}|\int_{-\infty}^\infty e^{-ixy}\tilde{\theta}_a(x)h(x)dx|$$

(286R)

$$=\frac{1}{\tilde{\theta}_1(0)}|(h\times\tilde{\theta}_a)^\vee(-y)|=\frac{1}{\tilde{\theta}_1(0)}|(\hat{\check{h}}\times\tilde{\theta}_a)^\vee(-y)|$$

(284C once more)

$$\le\frac{1}{2\pi\tilde{\theta}_1(0)}(\tilde{A}\check{h})(-y)$$

(286Sc). So if $a\le b$ in $\mathbb{R}$,

$$\frac{1}{\sqrt{2\pi}}|\int_a^b e^{-ixy}h(x)dx|\le\frac{1}{\pi\tilde{\theta}_1(0)}(\tilde{A}\check{h})(-y);$$

taking the supremum over a and b, we have the result. **Q**

It follows that

$$\int_F\hat{A}h\le\frac{1}{\pi\tilde{\theta}_1(0)}\int_{-F}\tilde{A}\check{h}\le\frac{3}{\pi\tilde{\theta}_1(0)}C_9\|h\|_2\sqrt{\mu(-F)}$$

(286Sb, 284Oa)

$$=C_{10}\|h\|_2\sqrt{\mu F}.$$

(c) For general square-integrable f, take any $\epsilon>0$ and any $n\in\mathbb{N}$. Set

$$(\hat{A}_n f)(y) = \sup_{-n\le a\le b\le n} \frac{1}{\sqrt{2\pi}}\Bigl|\int_a^b e^{-ixy} f(x)dx\Bigr|$$

for each $y \in \mathbb{R}$. Let h be a rapidly decreasing test function such that $\|f-h\|_2 \le \epsilon$ (284N). Then

$$\hat{A}h \ge \hat{A}_n h \ge \hat{A}_n f - \frac{\sqrt{2n}}{\sqrt{2\pi}}\epsilon$$

(using Cauchy's inequality), so

$$\int_F \hat{A}_n f \le \int_F \hat{A}h + \sqrt{\tfrac{n}{\pi}}\epsilon\mu F \le C_{10}(\|f\|_2+\epsilon)\sqrt{\mu F} + \sqrt{\tfrac{n}{\pi}}\epsilon\mu F.$$

As ϵ is arbitrary, $\int_F \hat{A}_n f \le C_{10}\|f\|_2\sqrt{\mu F}$; letting $n\to\infty$, we get $\int_F \hat{A}f \le C_{10}\|f\|_2\sqrt{\mu F}$.

286U Theorem If $f \in \mathcal{L}^2_{\mathbb{C}}$ then

$$g(y) = \lim_{a\to-\infty,b\to\infty} \frac{1}{\sqrt{2\pi}}\int_a^b e^{-ixy} f(x)dx$$

is defined in $\mathbb{C}$ for almost every $y \in \mathbb{R}$, and g represents the Fourier transform of f.

proof (a) For $n \in \mathbb{N}$, $y \in \mathbb{R}$ set

$$\gamma_n(y) = \sup_{a\le -n,b\ge n} \frac{1}{\sqrt{2\pi}}\Bigl|\int_a^b e^{-ixy} f(x)dx - \int_{-n}^n e^{-ixy} f(x)dx\Bigr|.$$

Then $g(y)$ is defined whenever $\inf_{n\in\mathbb{N}}\gamma_n(y) = 0$. **P** If $\inf_{n\in\mathbb{N}}\gamma_n(y) = 0$ and $\epsilon > 0$, take $m \in \mathbb{N}$ such that $\gamma_m(y) \le \frac{1}{2}\epsilon$; then $\frac{1}{\sqrt{2\pi}}|\int_a^b e^{-ixy}f(x)dx - \int_{-n}^n e^{-ixy}f(x)dx| \le \epsilon$ whenever $n \ge m$, $a \le -n$ and $b \ge n$. But this means, first, that $\langle\int_{-n}^n e^{-ixy}f(x)dx\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence, so has a limit ζ say, and, second, that $\zeta = \lim_{a\to-\infty,b\to\infty}\int_a^b e^{-ixy}f(x)dx$, so that $g(y) = \frac{\zeta}{\sqrt{2\pi}}$ is defined. **Q**

Also each γ_n is lower-semicontinuous (cf. part (a) of the proof of 286T).

(b) ? Suppose, if possible, that $\{y : \inf_{n\in\mathbb{N}}\gamma_n(y) > 0\}$ is not negligible. Then

$$\lim_{m\to\infty}\mu\{y : |y| \le m,\ \inf_{n\in\mathbb{N}}\gamma_n(y) \ge \frac{1}{m}\} > 0,$$

so there is an $\epsilon > 0$ such that

$$F = \{y : |y| \le \frac{1}{\epsilon},\ \inf_{n\in\mathbb{N}}\gamma_n(y) \ge \epsilon\}$$

has measure greater than ϵ. Let $n \in \mathbb{N}$ be such that

$$4C_{10}^2\Bigl(\int_{-\infty}^\infty |f(x)|^2dx - \int_{-n}^n |f(x)|^2dx\Bigr) \le \epsilon^3,$$

and set $f_1 = f - f\times\chi[-n,n]$; then $2C_{10}\|f_1\|_2 \le \epsilon^{3/2}$.

We have

$$\gamma_n(y) = \sup_{a\le -n,b\ge n}\frac{1}{\sqrt{2\pi}}\Bigl|\int_a^b e^{-ixy}f_1(x)dx - \int_{-n}^n e^{-ixy}f_1(x)dx\Bigr|$$
$$\le 2\sup_{a\le b}\frac{1}{\sqrt{2\pi}}\Bigl|\int_a^b e^{-ixy}f_1(x)dx\Bigr| \le 2(\hat{A}f_1)(y),$$

so that

$$\epsilon\mu F \le \int_F \gamma_n \le 2\int_F \hat{A}f_1 \le 2C_{10}\|f_1\|_2\sqrt{\mu F}$$

(286T)

$$\le \epsilon^{3/2}\sqrt{\mu F}$$

and $\mu F \le \epsilon$; but we chose ϵ so that μF would be greater than ϵ. **X**

(c) Thus $g(y)$ is defined for almost every $y \in \mathbb{R}$. Now g represents the Fourier transform of f. **P** Let h be a rapidly decreasing test function. The restriction of $\hat{A}f$ to the set on which it is finite is a tempered function, by 286D, so $\int_{-\infty}^{\infty}(\hat{A}f) \times |h|$ is finite, by 284F. Now

$$\int_{-\infty}^{\infty} g \times h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bigl(\lim_{n\to\infty} \int_{-n}^{n} e^{-ixy} f(x)dx\bigr) h(y)dy$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-\infty}^{\infty} \int_{-n}^{n} e^{-ixy} f(x)h(y)dxdy$$

(because $\frac{1}{\sqrt{2\pi}} |\int_{-n}^{n} e^{-ixy} f(x)dx| \le \hat{A}f(y)$ for every n and y, so we can use Lebesgue's Dominated Convergence Theorem)

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-n}^{n} \int_{-\infty}^{\infty} e^{-ixy} f(x)h(y)dydx$$

(because $\int_{-\infty}^{\infty} \int_{-n}^{n} |f(x)h(y)|dxdy$ is finite for each n)

$$= \lim_{n\to\infty} \int_{-n}^{n} f \times \hat{h} = \int_{-\infty}^{\infty} f \times \hat{h}$$

because $f \times \hat{h}$ is certainly integrable. As h is arbitrary, g represents the Fourier transform of f. **Q**

286V Theorem For any square-integrable complex-valued function on $]-\pi, \pi]$, its sequence of Fourier sums converges to it almost everywhere.

proof Suppose that $f \in \mathcal{L}^2_{\mathbb{C}}(\mu_{]-\pi,\pi]})$. Set $f_1(x) = f(x)$ for $x \in \operatorname{dom} f$, 0 for $x \in \mathbb{R} \setminus {]-\pi, \pi]}$; then $f_1 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$. Let $g \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ represent the inverse Fourier transform of f_1 (284O). Then 286U tells us that $f_2(x) = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ixy} g(y)dy$ is defined for almost every x, and that f_2 represents the Fourier transform of g, so is equal almost everywhere to f_1 (284Ib).

Now, for any $a \ge 0$, $x \in \mathbb{R}$,

$$\int_{-a}^{a} e^{-ixy} g(y)dy = (g|h_{ax})$$

(where $h_{ax}(y) = e^{ixy}$ if $|y| \le a$, 0 otherwise)

$$= (f_2|\hat{h}_{ax})$$

(284Ob)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t) \overline{\int_{-\infty}^{\infty} e^{-ity} h_{ax}(y)dy}\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin(x-t)a}{x-t} f_2(t)dt = \frac{2}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{\sin(x-t)a}{x-t} f(t)dt.$$

So

$$f(x) = f_2(x) = \lim_{a\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin(x-t)a}{x-t} f(t)dt$$

for almost every $x \in {]-\pi, \pi]}$.

On the other hand, writing $\langle s_n \rangle_{n\in\mathbb{N}}$ for the sequence of Fourier sums of f, we have, for any $x \in {]-\pi, \pi[}$,

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin(n+\frac{1}{2})(x-t)}{\sin \frac{1}{2}(x-t)} dt$$

for each n, by 282Da. Now

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}dt - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{x-t}dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Bigl(\frac{\sin(n+\frac{1}{2})(x-t)}{2\sin\frac{1}{2}(x-t)} - \frac{\sin(n+\frac{1}{2})(x-t)}{x-t}\Bigr)dt$$
$$= \frac{1}{\pi}\int_{x-\pi}^{x+\pi} \Bigl(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\Bigr)f(x-t)\sin(n+\tfrac{1}{2})t\,dt.$$

But if we look at the function

$$p_x(t) = \Bigl(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\Bigr)f(x-t) \text{ if } x-\pi < t < x+\pi \text{ and } t \ne 0,$$
$$= 0 \text{ otherwise,}$$

p_x is integrable, because f is integrable over $]-\pi,\pi]$ and $\lim_{t\to 0}\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t} = 0$, so $\sup_{t\ne 0, x-\pi\le t\le x+\pi}|\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}|$ is finite. (This is where we need to know that $|x| < \pi$.) So

$$\lim_{n\to\infty} s_n(x) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})(x-t)}{x-t}dt = \lim_{n\to\infty}\int_{-\infty}^{\infty} p_x(t)\sin(n+\tfrac{1}{2})t\,dt = 0$$

by the Riemann-Lebesgue lemma (282Fb). But this means that $\lim_{n\to\infty} s_n(x) = f(x)$ for any $x \in \,]-\pi,\pi[$ such that $f(x) = \lim_{a\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(x-t)a}{x-t}f(t)dt$, which is almost every $x \in \,]-\pi,\pi]$.

286W Glossary The following special notations are used in more than one paragraph of this section:

μ for Lebesgue measure on $\mathbb{R}$.
286A: f^*.
286C: $S_\alpha f$, $M_\alpha f$, $D_\alpha f$.
286Ea: $\mathcal{I}$, Q, I_σ, J_σ, k_σ, x_σ, y_σ, J^l_σ, J^r_σ, y^l_σ.
286Eb: ϕ, ϕ_σ, $(f|g)$.
286Ec: w, w_σ.
286F: $\preceq$, R^+, T_τ.
286G: C_1, C_2, C_3, C_4.
286H: mass, Δ_f, energy.
286J: C_5.
286K: C_6.
286L: C_7.
286M: C_8.
286N: C_9.
286O: θ_z, $\mathcal{F}$.
286P: Ah.
286Q: $\theta'_{z\alpha\beta}$.
286R: $\tilde\theta_z$.
286S: $\tilde A h$.
286T: C_{10}, $\hat A f$.

286X Basic exercises (a) Use 284Oa and 284Xg to shorten part (c) of the proof of 286U.

(b) Show that if $\langle c_k\rangle_{k\in\mathbb{N}}$ is a sequence of complex numbers such that $\sum_{k=0}^{\infty}|c_k|^2$ is finite, then $\sum_{k=0}^{\infty} c_k e^{ikx}$ is defined in $\mathbb{C}$ for almost all $x \in \mathbb{R}$.

286Y Further exercises (a) Show that if f is a square-integrable function on $\mathbb{R}^r$, where $r \ge 2$, then

$$g(y) = \frac{1}{(\sqrt{2\pi})^r}\lim_{\alpha_1,\dots,\alpha_r\to-\infty,\beta_1,\dots,\beta_r\to\infty}\int_a^b e^{-iy\cdot x}f(x)dx$$

is defined in $\mathbb{C}$ for almost every $y \in \mathbb{R}^r$, and that g represents the Fourier transform of f.

286 Notes and comments This is not the longest single section in this treatise as a whole, but it is by a substantial margin the longest in the present volume, and thirty pages of sub-superscripts must tax the endurance of the most enthusiastic. You will easily understand why Carleson's theorem is not usually presented at this level. But I am trying in this book to present complete proofs of the principal theorems, there is no natural place for Carleson's theorem in later volumes as at present conceived, and it is (just) accessible at this point; so I take the space to do it here.

The proof here divides naturally into two halves: the 'combinatorial' part in 286E-286M, up to the Lacey-Thiele lemma, followed by the 'analytic' part in 286N-286V, in which the averaging process

$$\int_1^2 \frac{1}{\alpha} \lim_{b\to\infty} \frac{1}{b}\int_0^b \ldots d\beta d\alpha$$

is used to transform the geometrically coherent, but analytically irregular, functions θ_z into the indicator functions $\frac{1}{\tilde\theta_1(0)}\tilde\theta_z$. From the standpoint of ordinary Fourier analysis, this second part is essentially routine; there are many paths we could follow, and we have only to take the ordinary precautions against illegitimate operations.[2]

Carleson (Carleson 66) stated his theorem in the Fourier-series form of 286V; but it had long been understood that this was equiveridical with the Fourier-transform version in 286U. There are of course many ways of extending the theorem. In particular, there are corresponding results for functions in $\mathcal{L}^p$ for any $p > 1$, and even for functions f such that $f \times \ln(1 + |f|) \times \ln\ln\ln(16 + |f|)$ is integrable (Antonov 96). The methods here do not seem to reach so far. I ought also to remark that if we define $\hat{A}f$ as in 286T, then there is for every $p > 1$ a constant C such that $\|\hat{A}f\|_p \le C\|f\|_p$ for every $f \in \mathcal{L}^p_{\mathbb{C}}$ (Hunt 67, Mozzochi 71, Jørsboe & Mejlbro 82, Arias de Reyna 02, Lacey 05).

Note that the point of Carleson's theorem, in either form, is that we take special limits. In the formulae

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \lim_{a\to-\infty,b\to\infty} \int_a^b e^{-ixy} f(x)dx,$$

$$f(x) = \lim_{n\to\infty} \sum_{-n}^{n} c_k e^{ikx},$$

valid almost everywhere for square-integrable functions f, we are not taking the ordinary integral $\int_{-\infty}^{\infty} e^{-ixy} f(x)dx$ or the unconditional sum $\sum_{k\in\mathbb{Z}} c_k e^{ikx}$. If f is not integrable, or $\sum_{k=-\infty}^{\infty} |c_k|$ is infinite, these will not be defined at even one point. Carleson's theorem makes sense only because we have a natural preference for particular kinds of improper integral and conditional sum. So when we return, in Chapter 44 of Volume 4, to Fourier analysis on general topological groups, there will simply be no language in which to express the theorem, and while versions have been proved for other groups (e.g., Schipp 78), they necessarily depend on some structure beyond the simple notion of 'locally compact Hausdorff abelian topological group'. Even in $\mathbb{R}^2$, I understand that it is still unknown whether

$$\lim_{a\to\infty} \frac{1}{2\pi}\int_{B(\mathbf{0},a)} e^{-iy\,.\,x} f(x)dx$$

will be defined a.e. for any square-integrable function f, if we use ordinary Euclidean balls $B(\mathbf{0}, a)$ in place of the rectangles in 286Ya.

[2] I ought at this point to confess that I blundered badly in the 2001 edition of this volume, and failed to notice my error until it was brought to my attention by A.Derighetti at the end of 2013. I hope that the version presented here is essentially correct.

Version of 6.1.10

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

285Xm Cauchy distribution The exercise introducing the Cauchy distribution, referred to in the 2002, 2004 and 2012 printings of Volume 3, is now 285Xp.

285Xo Poisson distribution The exercise naming the Poisson distribution, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 285Xr.

285Xr Bochner's theorem The exercise on a special case of Bochner's theorem, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 285Xu.

286U Carleson's theorem The sequential form, referred to in Bogachev 07, is now in 286V.

References

Bogachev V.I. [07] *Measure theory.* Springer, 2007.

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