Version of 9.6.16

Chapter 13

Complements

In this chapter I collect a number of results which do not lie in the direct line of the argument from 111A (the definition of 'σ-algebra') to 123C (Lebesgue's Dominated Convergence Theorem), but which nevertheless demand inclusion in this volume, being both relatively elementary, essential for further developments and necessary to a proper comprehension of what has already been done. The longest section is §134, dealing with a few of the elementary special properties of Lebesgue measure; in particular, its translation-invariance, the existence of non-measurable sets and functions, and the Cantor set. The other sections are comparatively lightweight. §131 discusses (measurable) subspaces and the interpretation of the formula $\int_E f$, generalizing the idea of an integral $\int_a^b f$ of a function over an interval. §132 introduces the outer measure associated with a measure, a kind of inverse of Carathéodory's construction of a measure from an outer measure. §§133 and 135 lay out suitable conventions for dealing with 'infinity' and complex numbers (separately! they don't mix well) as values either of integrands or of integrals; at the same time I mention 'upper' and 'lower' integrals. Finally, in §136, I give some theorems on σ-algebras of sets, describing how they can (in some of the most important cases) be generated by relatively restricted operations.

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131 Measurable subspaces

Very commonly we wish to integrate a function over a subset of a measure space; for instance, to form an integral $\int_a^b f(x)dx$, where $a < b$ in $\mathbb{R}$. As often as not, we wish to do this when the function is partly or wholly undefined outside the subset, as in such expressions as $\int_0^1 \ln x\,dx$. The natural framework in which to perform such operations is that of 'subspace measures'. If (X, Σ, μ) is a measure space and $H \in \Sigma$, there is a natural subspace measure μ_H on H, which I describe in this section. I begin with the definition of this subspace measure (131A-131C), with a description of integration with respect to it (131E-131H); this gives a solid foundation for the concept of 'integration over a (measurable) subset' (131D).

131A Proposition Let (X, Σ, μ) be a measure space, and $H \in \Sigma$. Set $\Sigma_H = \{E : E \in \Sigma,\ E \subseteq H\}$ and let μ_H be the restriction of μ to Σ_H. Then (H, Σ_H, μ_H) is a measure space.

131B Definition If (X, Σ, μ) is any measure space and $H \in \Sigma$, then μ_H, as defined in 131A, is the **subspace measure** on H.

When $X = \mathbb{R}^r$, where $r \geq 1$, and μ is Lebesgue measure on $\mathbb{R}^r$, I will call a subspace measure μ_H **Lebesgue measure on** H.

131C Lemma Let (X, Σ, μ) be a measure space, $H \in \Sigma$, and μ_H the subspace measure on H, with domain Σ_H. Then

(a) for any $A \subseteq H$, A is μ_H-negligible iff it is μ-negligible;

(b) if $G \in \Sigma_H$ then $(\mu_H)_G$, the subspace measure on G when G is regarded as a measurable subset of H, is identical to μ_G, the subspace measure on G when G is regarded as a measurable subset of X.

131D Integration over subsets: Definition Let (X, Σ, μ) be a measure space, $H \in \Sigma$ and f a real-valued function defined on a subset of X. By $\int_H f$ (or $\int_H f(x)\mu(dx)$, etc.) I shall mean $\int (f\restriction H)d\mu_H$, if this exists.

131E Proposition Let (X,Σ,μ) be a measure space, $H\in\Sigma$, and f a real-valued function defined on a subset $\operatorname{dom} f$ of H. Set $\tilde f(x)=f(x)$ if $x\in\operatorname{dom} f$, 0 if $x\in X\setminus H$. Then $\int f d\mu_H=\int\tilde f d\mu$ if either is defined in $\mathbb{R}$.

131F Corollary Let (X,Σ,μ) be a measure space and f a real-valued function defined on a subset $\operatorname{dom} f$ of X.

(a) If $H\in\Sigma$ and f is defined almost everywhere in X, then $f\restriction H$ is μ_H-integrable iff $f\times\chi H$ is μ-integrable, and in this case $\int_H f=\int f\times\chi H$.

(b) If f is μ-integrable, then $f\ge 0$ a.e. iff $\int_H f\ge 0$ for every $H\in\Sigma$.

(c) If f is μ-integrable, then $f=0$ a.e. iff $\int_H f=0$ for every $H\in\Sigma$.

131G Corollary Let (X,Σ,μ) be a measure space and $H\in\Sigma$ a conegligible set. If f is any real-valued function defined on a subset of X, $\int_H f=\int f$ if either is defined.

131H Corollary Let (X,Σ,μ) be a measure space and f, g two μ-integrable real-valued functions.

(a) If $\int_H f\ge\int_H g$ for every $H\in\Sigma$ then $f\ge g$ a.e.

(b) If $\int_H f=\int_H g$ for every $H\in\Sigma$ then $f=g$ a.e.

Version of 6.4.05

132 Outer measures from measures

The next topic I wish to mention is a simple construction with applications everywhere in measure theory. With any measure there is associated, in a straightforward way, a standard outer measure (132A-132B). If we start with Lebesgue measure we just return to Lebesgue outer measure (132C). I take the opportunity to introduce the idea of 'measurable envelope' (132D-132E).

132A Proposition Let (X,Σ,μ) be a measure space. Define $\mu^*:\mathcal{P}X\to[0,\infty]$ by writing

$$\mu^*A=\inf\{\mu E: E\in\Sigma,\ A\subseteq E\}$$

for every $A\subseteq X$. Then

(a) for every $A\subseteq X$ there is an $E\in\Sigma$ such that $A\subseteq E$ and $\mu^*A=\mu E$;

(b) μ^* is an outer measure on X;

(c) $\mu^*E=\mu E$ for every $E\in\Sigma$;

(d) a set $A\subseteq X$ is μ-negligible iff $\mu^*A=0$;

(e) $\mu^*(\bigcup_{n\in\mathbb{N}}A_n)=\lim_{n\to\infty}\mu^*A_n$ for every non-decreasing sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X;

(f) $\mu^*A=\mu^*(A\cap F)+\mu^*(A\setminus F)$ whenever $A\subseteq X$ and $F\in\Sigma$.

132B Definition If (X,Σ,μ) is a measure space, I will call μ^*, as defined in 132A, **the outer measure defined from** μ.

132C Proposition If θ is Lebesgue outer measure on $\mathbb{R}^r$ and μ is Lebesgue measure, then μ^*, as defined in 132A, is equal to θ.

132D Measurable envelopes If (X,Σ,μ) is a measure space and $A\subseteq X$, a **measurable envelope** (or **measurable cover**) of A is a set $E\in\Sigma$ such that $A\subseteq E$ and $\mu(F\cap E)=\mu^*(F\cap A)$ for every $F\in\Sigma$.

132E Lemma Let (X,Σ,μ) be a measure space.

(a) If $A\subseteq E\in\Sigma$, then E is a measurable envelope of A iff $\mu F=0$ whenever $F\in\Sigma$ and $F\subseteq E\setminus A$.

(b) If $A\subseteq E\in\Sigma$ and $\mu E<\infty$ then E is a measurable envelope of A iff $\mu E=\mu^*A$.

(c) If E is a measurable envelope of A and $H\in\Sigma$, then $E\cap H$ is a measurable envelope of $A\cap H$.

(d) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of subsets of X. Suppose that each A_n has a measurable envelope E_n. Then $\bigcup_{n\in\mathbb{N}}E_n$ is a measurable envelope of $\bigcup_{n\in\mathbb{N}}A_n$.

(e) If $A\subseteq X$ can be covered by a sequence of sets of finite measure, then A has a measurable envelope.

(f) In particular, if μ is Lebesgue measure on $\mathbb{R}^r$, then every subset of $\mathbb{R}^r$ has a measurable envelope for μ.

132F Full outer measure If (X, Σ, μ) is a measure space, a set $A \subseteq X$ is **of full outer measure** or **thick** if X is a measurable envelope of A.

Version of 29.3.10

133 Wider concepts of integration

There are various contexts in which it is useful to be able to assign a value to the integral of a function which is not quite covered by the basic definition in 122M. In this section I offer suggestions concerning the assignment of the values $\pm\infty$ to integrals of real-valued functions (133A), the integration of complex-valued functions (133C-133H) and upper and lower integrals (133I-133L). In §135 below I will discuss a further elaboration of the ideas of Chapter 12.

133A Infinite integrals It is normal to restrict the phrase 'f is integrable' to functions f to which a finite integral $\int f$ can be assigned. But for non-negative functions it is sometimes convenient to write '$\int f = \infty$' if, in some sense, the only way in which f fails to be integrable is that the integral is too large; that is, f is defined almost everywhere, is μ-virtually measurable, and either

$$\{x : x \in \operatorname{dom} f,\ f(x) \ge \epsilon\}$$

includes a set of infinite measure for some $\epsilon > 0$, or

$$\sup\{\int h : h \text{ is simple, } h \le_{\text{a.e.}} f\} = \infty.$$

Under this rule,

$$\int f_1 + f_2 = \int f_1 + \int f_2, \quad \int cf = c\int f$$

whenever $c \in [0, \infty[$ and f_1, f_2, f are non-negative functions for which $\int f_1$, $\int f_2$, $\int f$ are defined in $[0, \infty]$.

We can therefore say that

$$\int f_1 - f_2 = \int f_1 - \int f_2$$

whenever f_1, f_2 are real-valued functions such that $\int f_1$, $\int f_2$ are defined in $[0, \infty]$ and are not both infinite.

We still have the rules that

$$\int f + g = \int f + \int g, \quad \int (cf) = c\int f, \quad \int |f| \ge |\int f|$$

at least when the right-hand-sides can be interpreted, allowing $0 \cdot \infty = 0$, but not allowing any interpretation of $\infty - \infty$; and $\int f \le \int g$ whenever both integrals are defined and $f \le_{\text{a.e.}} g$.

Setting $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$ for $x \in \operatorname{dom} f$, then

$$\int f = \infty \iff \int f^+ = \infty \text{ and } f^- \text{ is integrable,}$$

$$\int f = -\infty \iff f^+ \text{ is integrable and } \int f^- = \infty.$$

133B Functions with exceptional values It is also convenient to allow as 'integrable' functions f which take occasional values which are not real. For such a function I will write $\int f = \int \tilde{f}$ if $\int \tilde{f}$ is defined, where

$$\operatorname{dom} \tilde{f} = \{x : x \in \operatorname{dom} f,\ f(x) \in \mathbb{R}\}, \quad \tilde{f}(x) = f(x) \text{ for } x \in \operatorname{dom} \tilde{f}.$$

133D Definitions (a) Let X be a set and Σ a σ-algebra of subsets of X. If $D \subseteq X$ and $f : D \to \mathbb{C}$ is a function, then we say that f is **measurable** if its real and imaginary parts $\mathcal{R}\mathrm{e}\, f$, $\mathcal{I}\mathrm{m}\, f$ are measurable.

(b) Let (X, Σ, μ) be a measure space. If f is a complex-valued function defined on a conegligible subset of X, we say that f is **integrable** if its real and imaginary parts are integrable, and then

$$\int f = \int \mathcal{R}\mathrm{e}\, f + i\int \mathcal{I}\mathrm{m}\, f.$$

(c) Let (X,Σ,μ) be a measure space, $H\in\Sigma$ and f a complex-valued function defined on a subset of X. Then $\int_H f$ is $\int(f\restriction H)d\mu_H$ if this is defined in the sense of (b), taking the subspace measure μ_H to be that of 131A-131B.

133E Lemma (a) If X is a set, Σ is a σ-algebra of subsets of X, and f and g are measurable complex-valued functions with domains $\operatorname{dom} f$, $\operatorname{dom} g\subseteq X$, then

(i) $f+g:\operatorname{dom} f\cap\operatorname{dom} g\to\mathbb{C}$ is measurable;

(ii) $cf:\operatorname{dom} f\to\mathbb{C}$ is measurable, for every $c\in\mathbb{C}$;

(iii) $f\times g:\operatorname{dom} f\cap\operatorname{dom} g\to\mathbb{C}$ is measurable;

(iv) $f/g:\{x:x\in\operatorname{dom} f\cap\operatorname{dom} g,\ g(x)\neq 0\}\to\mathbb{C}$ is measurable;

(v) $|f|:\operatorname{dom} f\to\mathbb{R}$ is measurable.

(b) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable complex-valued functions defined on subsets of X, then $f=\lim_{n\to\infty} f_n$ is measurable, if we take $\operatorname{dom} f$ to be

$$\{x:x\in\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\operatorname{dom} f_m,\ \lim_{n\to\infty} f_n(x)\text{ exists in }\mathbb{C}\}$$
$$=\operatorname{dom}(\lim_{n\to\infty}\mathcal{R}\mathrm{e}\, f_n)\cap\operatorname{dom}(\lim_{n\to\infty}\mathcal{I}\mathrm{m}\, f_n).$$

133F Proposition Let (X,Σ,μ) be a measure space.

(a) If f and g are integrable complex-valued functions defined on conegligible subsets of X, then $f+g$ and cf are integrable, $\int f+g=\int f+\int g$ and $\int cf=c\int f$, for every $c\in\mathbb{C}$.

(b) If f is a complex-valued function defined on a conegligible subset of X, then f is integrable iff $|f|$ is integrable and f is μ-virtually measurable.

133G Lebesgue's Dominated Convergence Theorem Let (X,Σ,μ) be a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of integrable complex-valued functions on X such that $f(x)=\lim_{n\to\infty} f_n(x)$ exists in $\mathbb{C}$ for almost every $x\in X$. Suppose moreover that there is a real-valued integrable function g on X such that $|f_n|\le_{\text{a.e.}} g$ for each n. Then f is integrable and $\lim_{n\to\infty}\int f_n$ exists and is equal to $\int f$.

133H Corollary Let (X,Σ,μ) be a measure space and $]a,b[$ a non-empty open interval in $\mathbb{R}$. Let $f:X\times\,]a,b[\,\to\mathbb{C}$ be a function such that

(i) the integral $F(t)=\int f(x,t)dx$ is defined for every $t\in\,]a,b[$;

(ii) the partial derivative $\frac{\partial f}{\partial t}$ of f with respect to the second variable is defined everywhere in $X\times\,]a,b[$;

(iii) there is an integrable function $g:X\to[0,\infty[$ such that $|\frac{\partial f}{\partial t}(x,t)|\le g(x)$ for every $x\in X$, $t\in\,]a,b[$.

Then the derivative $F'(t)$ and the integral $\int\frac{\partial f}{\partial t}(x,t)dx$ exist for every $t\in\,]a,b[$, and are equal.

133I Upper and lower integrals Let (X,Σ,μ) be a measure space and f a real-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int} f=\inf\{\int g:\int g\text{ is defined in the sense of 133A and }f\le_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\{\infty\}$ or $\inf\emptyset$ and $-\infty$ for $\inf\mathbb{R}$. Similarly, the **lower integral** of f is

$$\underline{\int} f=\sup\{\int g:\int g\text{ is defined, }f\ge_{\text{a.e.}} g\},$$

allowing $-\infty$ for $\sup\{-\infty\}$ or $\sup\emptyset$ and ∞ for $\sup\mathbb{R}$.

133J Proposition Let (X,Σ,μ) be a measure space.

(a) Let f be a real-valued function defined almost everywhere in X.

(i) If $\overline{\int} f$ is finite, then there is an integrable g such that $f\le_{\text{a.e.}} g$ and $\int g=\overline{\int} f$. In this case,

$$\{x:x\in\operatorname{dom} f\cap\operatorname{dom} g,\ g(x)\le f(x)+g_0(x)\}$$

has full outer measure for every measurable function $g_0 : X \to]0, \infty[$.

(ii) If $\underline{\int} f$ is finite, then there is an integrable h such that $h \leq_{\text{a.e.}} f$ and $\int h = \underline{\int} f$. In this case,

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h,\ f(x) \leq h(x) + h_0(x)\}$$

has full outer measure for every measurable function $h_0 : X \to]0, \infty[$.

(b) For any real-valued functions f, g defined on conegligible subsets of X and any $c \geq 0$,

(i) $\underline{\int} f \leq \overline{\int} f$,

(ii) $\overline{\int} f + g \leq \overline{\int} f + \overline{\int} g$,

(iii) $\overline{\int} cf = c\overline{\int} f$,

(iv) $\underline{\int}(-f) = -\overline{\int} f$,

(v) $\underline{\int} f + g \geq \underline{\int} f + \underline{\int} g$,

(vi) $\underline{\int} cf = c\underline{\int} f$

whenever the right-hand-sides do not involve adding ∞ to $-\infty$.

(c) If $f \leq_{\text{a.e.}} g$ then $\overline{\int} f \leq \overline{\int} g$ and $\underline{\int} f \leq \underline{\int} g$.

(d) A real-valued function f defined almost everywhere in X is integrable iff

$$\overline{\int} f = \underline{\int} f = a \in \mathbb{R},$$

and in this case $\int f = a$.

(e) $\mu^* A = \overline{\int} \chi A$ for every $A \subseteq X$.

133K Convergence theorems for upper integrals: Proposition Let (X, Σ, μ) be a measure space, and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of real-valued functions defined almost everywhere in X.

(a) If, for each n, $f_n \leq_{\text{a.e.}} f_{n+1}$, and $-\infty < \sup_{n \in \mathbb{N}} \overline{\int} f_n < \infty$, then $f(x) = \sup_{n \in \mathbb{N}} f_n(x)$ is defined in $\mathbb{R}$ for almost every $x \in X$, and $\overline{\int} f = \sup_{n \in \mathbb{N}} \overline{\int} f_n$.

(b) If, for each n, $f_n \geq 0$ a.e., and $\liminf_{n \to \infty} \overline{\int} f_n < \infty$, then $f(x) = \liminf_{n \to \infty} f_n(x)$ is defined in $\mathbb{R}$ for almost every $x \in X$, and $\overline{\int} f \leq \liminf_{n \to \infty} \overline{\int} f_n$.

***133L Proposition** Let (X, Σ, μ) be a measure space and f a real-valued function defined almost everywhere in X. Suppose that h_1, h_2 are non-negative virtually measurable functions defined almost everywhere in X. Then

$$\overline{\int} f \times (h_1 + h_2) = \overline{\int} f \times h_1 + \overline{\int} f \times h_2,$$

where here, for once, we can interpret $\infty + (-\infty)$ or $(-\infty) + \infty$ as ∞ if called for on the right-hand side.

Version of 7.1.04

134 More on Lebesgue measure

The special properties of Lebesgue measure will take up a substantial proportion of this treatise. In this section I present a miscellany of relatively easy basic results. In 134A-134F, r will be a fixed integer greater than or equal to 1, μ will be Lebesgue measure on $\mathbb{R}^r$ and μ^* will be Lebesgue outer measure; when I say that a set or a function is 'measurable', then it is to be understood that (unless otherwise stated) this means 'measurable with respect to the σ-algebra of Lebesgue measurable sets', while 'negligible' means 'negligible for Lebesgue measure'. Most of the results will be expressed in terms adapted to the multi-dimensional case; but if you are primarily interested in the real line, you will miss none of the ideas if you read the whole section as if $r = 1$.

134A Proposition Both Lebesgue outer measure and Lebesgue measure are translation-invariant; that is, setting $A + x = \{a + x : a \in A\}$ for $A \subseteq \mathbb{R}^r$, $x \in \mathbb{R}^r$, we have

(a) $\mu^*(A + x) = \mu^* A$ for every $A \subseteq \mathbb{R}^r$, $x \in \mathbb{R}^r$;

(b) whenever $E \subseteq \mathbb{R}^r$ is measurable and $x \in \mathbb{R}^r$, then $E + x$ is measurable, with $\mu(E + x) = \mu E$.

134B Theorem Not every subset of $\mathbb{R}^r$ is Lebesgue measurable.

***134D Proposition** There is a set $C \subseteq \mathbb{R}^r$ such that $F \cap C$ is not measurable for any measurable set F of non-zero measure; so that both C and its complement have full outer measure in $\mathbb{R}^r$.

134F Proposition (a) If $A \subseteq \mathbb{R}^r$ is any set, then

$$\mu^* A = \inf\{\mu G : G \text{ is open, } G \supseteq A\} = \min\{\mu H : H \text{ is Borel, } H \supseteq A\}.$$

(b) If $E \subseteq \mathbb{R}^r$ is measurable, then

$$\mu E = \sup\{\mu F : F \text{ is closed and bounded, } F \subseteq E\},$$

and there are Borel sets H_1, H_2 such that $H_1 \subseteq E \subseteq H_2$ and

$$\mu(H_2 \setminus H_1) = \mu(H_2 \setminus E) = \mu(E \setminus H_1) = 0.$$

(c) If $A \subseteq \mathbb{R}^r$ is any set, then A has a measurable envelope which is a Borel set.

(d) If f is a Lebesgue measurable real-valued function defined on a subset of $\mathbb{R}^r$, then there is a conegligible Borel set $H \subseteq \mathbb{R}^r$ such that $f \restriction H$ is Borel measurable.

134G The Cantor set(a) The 'Cantor set' $C \subseteq [0,1]$ is defined as the intersection of a sequence $\langle C_n \rangle_{n\in\mathbb{N}}$ of sets, constructed as follows. $C_0 = [0,1]$. Given that C_n consists of 2^n disjoint closed intervals each of length 3^{-n}, take each of these intervals and delete the middle third to produce two closed intervals each of length 3^{-n-1}; take C_{n+1} to be the union of the 2^{n+1} closed intervals so formed, and continue. Observe that $\mu C_n = (\frac{2}{3})^n$ for each n.

The **Cantor set** is $C = \bigcap_{n\in\mathbb{N}} C_n$. Its measure is

$$\mu C = \lim_{n\to\infty} \mu C_n = \lim_{n\to\infty} (\frac{2}{3})^n = 0.$$

(b) Each C_n can also be described as the set of real numbers expressible as $\sum_{j=1}^{\infty} 3^{-j}\epsilon_j$ where every ϵ_j is either 0, 1 or 2, and $\epsilon_j \neq 1$ for $j \leq n$. Consequently C itself is the set of numbers expressible as $\sum_{j=1}^{\infty} 3^{-j}\epsilon_j$ where every ϵ_j is either 0 or 2; that is, the set of numbers between 0 and 1 expressible in ternary form without 1's. The expression in each case will be unique, so we have a bijection $\phi : \{0,1\}^{\mathbb{N}} \to C$ defined by writing

$$\phi(z) = \frac{2}{3}\sum_{j=0}^{\infty} 3^{-j} z(j)$$

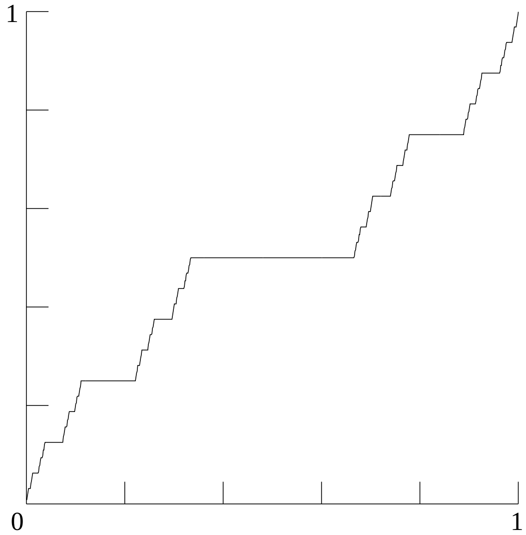
for every $z \in \{0,1\}^{\mathbb{N}}$.

134H The Cantor function(a) For each $n \in \mathbb{N}$ define a function $f_n : [0,1] \to [0,1]$ by setting

$$f_n(x) = (\frac{3}{2})^n \mu(C_n \cap [0,x])$$

for each $x \in [0,1]$. f_n is a polygonal function, with $f_n(0) = 0$, $f_n(1) = 1$; f_n is constant on each of the $2^n - 1$ open intervals composing $[0,1] \setminus C_n$, and rises with slope $(\frac{3}{2})^n$ on each of the 2^n closed intervals composing C_n.

$\langle f_n \rangle_{n\in\mathbb{N}}$ is uniformly convergent to a function $f : [0,1] \to [0,1]$, and f will be continuous. f is the **Cantor function** or **Devil's Staircase**.



The Cantor function

(b) Because every f_n is non-decreasing, so is f. f' is zero almost everywhere in $[0,1]$. $f:[0,1]\to[0,1]$ is surjective.

(c) Let $\phi:\{0,1\}^{\mathbb{N}}\to C$ be the function described in 134Gb. Then $f(\phi(z))=\frac{1}{2}\sum_{j=0}^{\infty}2^{-j}z(j)$ for every $z\in\{0,1\}^{\mathbb{N}}$. $f[C]=[0,1]$.

134I The Cantor function modified(a) Consider the formula

$$g(x)=\frac{1}{2}(x+f(x)),$$

where f is the Cantor function, as defined in 134H; this defines a continuous function $g:[0,1]\to[0,1]$ which is strictly increasing and has $g(0)=0$, $g(1)=1$; g is bijective, and its inverse $g^{-1}:[0,1]\to[0,1]$ is continuous. $g[C]$ is a closed set and $\mu g[C]=\frac{1}{2}$.

(b) By 134D there is a set $D\subseteq\mathbb{R}$ such that

$$\mu^*(g[C]\cap D)=\mu^*(g[C]\setminus D)=\mu g[C]=\frac{1}{2};$$

set $A=g[C]\cap D$. A cannot be measurable. However, $g^{-1}[A]\subseteq C$ must be measurable. This means that if we set $h=\chi(g^{-1}[A]):[0,1]\to\mathbb{R}$, then h is measurable; but hg^{-1} is not.

Thus **the composition of a measurable function with a continuous function need not be measurable**.

134J More examples(a) Let $\langle q_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathbb{Q}$, and for each $n\in\mathbb{N}$ set

$$I_n=\,]q_n-2^{-n},q_n+2^{-n}[,$$

$$G_n=\bigcup_{k\ge n}I_k.$$

Then G_n is an open set of measure at most $\sum_{k=n}^{\infty}2\cdot2^{-k}=4\cdot2^{-n}$, and it contains all but finitely many points of $\mathbb{Q}$, so is dense. Set $F_n=\mathbb{R}\setminus G_n$; then F_n is closed, $\mu(\mathbb{R}\setminus F_n)\le 4/2^n$, but F_n does not include any non-trivial interval. $\langle F_n\rangle_{n\in\mathbb{N}}$ is non-decreasing.

(b) There is a measurable set $E\subseteq\mathbb{R}$ such that $\mu(I\cap E)>0$ and $\mu(I\setminus E)>0$ for every non-trivial interval $I\subseteq\mathbb{R}$.

(c) E and its complement are measurable sets which are 'essentially' dense in that they meet every non-empty open interval in a set of positive measure, so that $E\setminus A$ is dense for every negligible set A.

***134K Riemann integration** If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, it is Lebesgue integrable, with the same integral.

***134L Proposition** If $a < b$ in $\mathbb{R}$, a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable iff it is continuous almost everywhere in $[a,b]$.

Version of 14.9.04/14.7.07

135 The extended real line

It is often convenient to allow '∞' into our formulae, and in the context of measure theory the appropriate manipulations are sufficiently consistent for it to be possible to develop a theory of the **extended real line**, the set $[-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}$, sometimes written $\overline{\mathbb{R}}$. I give a brief account without full proofs, as I hope that by the time this material becomes necessary to the arguments I use it will all appear thoroughly elementary.

135A The algebraic structure of $[-\infty,\infty]$ **(a)** If we write

$$a + \infty = \infty + a = \infty, \quad a + (-\infty) = (-\infty) + a = -\infty$$

for every $a \in \mathbb{R}$, and

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty,$$

but refuse to define $\infty + (-\infty)$ or $(-\infty) + \infty$, we obtain a partially-defined binary operation on $[-\infty,\infty]$, extending ordinary addition on $\mathbb{R}$. This is *associative* in the sense that

> if u, v, $w \in [-\infty,\infty]$ and one of $u + (v + w)$, $(u + v) + w$ is defined, so is the other, and they are then equal,

and *commutative* in the sense that

> if u, $v \in [-\infty,\infty]$ and one of $u + v$, $v + u$ is defined, so is the other, and they are then equal.

It has an *identity* 0 such that $u + 0 = 0 + u = u$ for every $u \in [-\infty,\infty]$; but ∞ and $-\infty$ lack inverses.

(b) If we define

$$a \cdot \infty = \infty \cdot a = \infty, \quad a \cdot (-\infty) = (-\infty) \cdot a = -\infty$$

for real $a > 0$,

$$a \cdot \infty = \infty \cdot a = -\infty, \quad a \cdot (-\infty) = (-\infty) \cdot a = \infty$$

for real $a < 0$,

$$\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty, \quad (-\infty) \cdot \infty = \infty \cdot (-\infty) = -\infty,$$

$$0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$$

then we obtain a binary operation on $[-\infty,\infty]$ extending ordinary multiplication on $\mathbb{R}$, which is associative and commutative and has an identity 1; 0, ∞ and $-\infty$ lack inverses.

(c) We have a *distributive law*:

> if u, v, $w \in [-\infty,\infty]$ and both $u(v + w)$ and $uv + uw$ are defined, then they are equal.

(d) While ∞ and $-\infty$ do not have inverses in the semigroup $([-\infty,\infty],\cdot)$, there seems no harm in writing $a/\infty = a/(-\infty) = 0$ for every $a \in \mathbb{R}$.

135B The order structure of $[-\infty,\infty]$ **(a)** If we write

$$-\infty \le u \le \infty \text{ for every } u \in [-\infty,\infty],$$

we obtain a relation on $[-\infty,\infty]$, extending the usual ordering of $\mathbb{R}$, which is a *total* ordering, that is,

> for any u, v, $w \in [-\infty,\infty]$, if $u \le v$ and $v \le w$ then $u \le w$,

$u \leq u$ for every $u \in [-\infty, \infty]$,

for any u, $v \in [-\infty, \infty]$, if $u \leq v$ and $v \leq u$ then $u = v$,

for any u, $v \in [-\infty, \infty]$, either $u \leq v$ or $v \leq u$.

Moreover, every subset of $[-\infty, \infty]$ has a supremum and an infimum, if we write $\sup\emptyset = -\infty$, $\inf\emptyset = \infty$.

(b) The ordering is 'translation-invariant' in the weak sense that

if u, v, $w \in [-\infty, \infty]$ and $v \leq w$ and $u + v$, $u + w$ are both defined, then $u + v \leq u + w$.

It is preserved by non-negative multiplications in the sense that

if u, v, $w \in [-\infty, \infty]$ and $0 \leq u$ and $v \leq w$, then $uv \leq uw$,

while it is reversed by non-positive multiplications in the sense that

if u, v, $w \in [-\infty, \infty]$ and $u \leq 0$ and $v \leq w$, then $uw \leq uv$.

135C The Borel structure of $[-\infty, \infty]$ We say that a set $E \subseteq [-\infty, \infty]$ is a **Borel set** in $[-\infty, \infty]$ if $E \cap \mathbb{R}$ is a Borel subset of $\mathbb{R}$. It is easy to check that the family of such sets is a σ-algebra of subsets of $[-\infty, \infty]$.

135D Convergent sequences in $[-\infty, \infty]$ We can say that a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $[-\infty, \infty]$ **converges** to $u \in [-\infty, \infty]$ if

whenever $v < u$ there is an $n_0 \in \mathbb{N}$ such that $v \leq u_n$ for every $n \geq n_0$, and whenever $u < v$ there is an $n_0 \in \mathbb{N}$ such that $u_n \leq v$ for every $n \geq n_0$;

alternatively,

either $u \in \mathbb{R}$ and for every $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that $u_n \in [u - \delta, u + \delta]$ for every $n \geq n_0$

or $u = -\infty$ and for every $a \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}$ such that $u_n \leq a$ for every $n \geq n_0$

or $u = \infty$ and for every $a \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}$ such that $u_n \geq a$ for every $n \geq n_0$.

135E Measurable functions Let X be any set and Σ a σ-algebra of subsets of X.

(a) Let D be a subset of X and Σ_D the subspace σ-algebra (121A). For any function $f : D \to [-\infty, \infty]$, the following are equiveridical:

(i) $\{x : f(x) < u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;
(ii) $\{x : f(x) \leq u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;
(iii) $\{x : f(x) > u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;
(iv) $\{x : f(x) \geq u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;
(v) $\{x : f(x) \leq q\} \in \Sigma_D$ for every $q \in \mathbb{Q}$.

(b) We may therefore say that a function taking values in $[-\infty, \infty]$ is **measurable** if it satisfies these equivalent conditions.

(c) Note that if $f : D \to [-\infty, \infty]$ is Σ-measurable, then

$$E_\infty(f) = f^{-1}[\{\infty\}] = \{x : f(x) \geq \infty\}, \quad E_{-\infty}(f) = f^{-1}[\{-\infty\}] = \{x : f(x) \leq -\infty\}$$

must belong to Σ_D, while $f_\mathbb{R} = f\restriction D \setminus (E_\infty(f) \cup E_{-\infty}(f))$, the 'real-valued part of f', is measurable in the sense of 121C.

(d) Conversely, if E_∞ and $E_{-\infty}$ belong to Σ_D, and $f_\mathbb{R} : D \setminus (E_\infty \cup E_{-\infty}) \to \mathbb{R}$ is measurable, then $f : D \to [-\infty, \infty]$ will be measurable, where $f(x) = \infty$ if $x \in E_\infty$, $f(x) = -\infty$ if $x \in E_{-\infty}$ and $f(x) = f_\mathbb{R}(x)$ for other $x \in D$.

(e) It follows that if f, g are measurable functions from subsets of X to $[-\infty, \infty]$, then $f + g$, $f \times g$ and f/g are measurable.

(f) We can say that a function h from a subset D of $[-\infty, \infty]$ to $[-\infty, \infty]$ is **Borel measurable** if it is measurable with respect to the Borel σ-algebra of $[-\infty, \infty]$. Now if X is a set, Σ is a σ-algebra of subsets of X, f is a measurable function from a subset of X to $[-\infty, \infty]$ and h is a Borel measurable function from a subset of $[-\infty, \infty]$ to $[-\infty, \infty]$, then hf is measurable.

(g) Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions from subsets of X to $[-\infty,\infty]$. Then $\lim_{n\to\infty} f_n$, $\sup_{n\in\mathbb{N}} f_n$ and $\inf_{n\in\mathbb{N}} f_n$ are measurable, if we take their domains to be

$$\{x : x \in \bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n} \operatorname{dom} f_m,\ \lim_{n\to\infty} f_n(x) \text{ exists in } [-\infty,\infty]\},$$

$$\bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n.$$

135F $[-\infty,\infty]$-valued integrable functions (a) We are surely not going to admit a function as 'integrable' unless it is finite almost everywhere, and for such functions the remarks in 133B are already adequate.

(b) However, it is possible to make a consistent extension of the idea of an infinite integral, elaborating slightly the ideas of 133A. If (X,Σ,μ) is a measure space and f is a function, defined almost everywhere in X, taking values in $[0,\infty]$, and virtually measurable (that is, such that $f{\restriction}E$ is measurable for some conegligible set E), then we can safely write '$\int f = \infty$' whenever f is not integrable. We shall find that for such functions we have $\int f+g = \int f + \int g$ and $\int cf = c\int f$ for every $c \in [0,\infty]$, using the definitions given above for addition and multiplication on $[0,\infty]$. Consequently, as in 122M-122O, we can say that for a general virtually measurable function f, defined almost everywhere in X, taking values in $[-\infty,\infty]$, $\int f = \int f_1 - \int f_2$ whenever f is expressible as a difference $f_1 - f_2$ of non-negative functions such that $\int f_1$ and $\int f_2$ are both defined and not both infinite. Now we have the basic formulae

$$\int f+g = \int f + \int g, \quad \int cf = c\int f, \quad \int |f| \ge |\int f|$$

whenever the right-hand-sides are defined, and $\int f \le \int g$ whenever $f \le_{\text{a.e.}} g$ and both integrals are defined. $\int f$ can be finite, on this definition, only when f is finite almost everywhere.

135G Proposition Let (X,Σ,μ) be a measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of $[-\infty,\infty]$-valued functions defined almost everywhere in X which have integrals defined in $[-\infty,\infty]$.

(a) If $f_n \le_{\text{a.e.}} f_{n+1}$ for every n and $-\infty < \sup_{n\in\mathbb{N}}\int f_n$, then $\int \sup_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}}\int f_n$.

(b) If, for each n, $f_n \ge 0$ a.e., then $\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty}\int f_n$.

135H Upper and lower integrals again (a) To handle functions taking values in $[-\infty,\infty]$ we need to adapt the definitions in 133I. Let (X,Σ,μ) be a measure space and f a $[-\infty,\infty]$-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int} f = \inf\{\int g : \int g \text{ is defined in the sense of 135F and } f \le_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\{\infty\}$ and $-\infty$ for $\inf\,]-\infty,\infty]$ or $\inf[-\infty,\infty]$. Similarly, the **lower integral** of f is

$$\underline{\int} f = \sup\{\int g : \int g \text{ is defined, } f \ge_{\text{a.e.}} g\}.$$

With this modification, all the results of 133J are valid for functions taking values in $[-\infty,\infty]$ rather than in $\mathbb{R}$.

(b) Let (X,Σ,μ) be a measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of $[-\infty,\infty]$-valued functions defined almost everywhere in X.

(i) If $f_n \le_{\text{a.e.}} f_{n+1}$ for every n and $\sup_{n\in\mathbb{N}}\overline{\int} f_n > -\infty$, then $\overline{\int}\sup_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}}\overline{\int} f_n$.

(ii) If, for each n, $f_n \ge 0$ a.e., then $\overline{\int}\liminf_{n\to\infty} f_n \le \liminf_{n\to\infty}\overline{\int} f_n$.

135I Subspace measures: Proposition Let (X,Σ,μ) be a measure space, and $H \in \Sigma$; write Σ_H for the subspace σ-algebra on H and μ_H for the subspace measure. For any $[-\infty,\infty]$-valued function f defined on a subset of H, write $\tilde f$ for the extension of f defined by saying that $\tilde f(x) = f(x)$ if $x \in \operatorname{dom} f$, 0 if $x \in X\setminus H$.

(a) Suppose that f is a $[-\infty,\infty]$-valued function defined on a subset of H.

(i) $\operatorname{dom} f$ is μ_H-conegligible iff $\operatorname{dom}\tilde f$ is μ-conegligible.

(ii) f is μ_H-virtually measurable iff $\tilde{f}$ is μ-virtually measurable.

(iii) $\int_H f d\mu_H = \int_X \tilde{f} d\mu$ if either is defined in $[-\infty, \infty]$.

(b) Suppose that h is a $[-\infty, \infty]$-valued function defined almost everywhere in X. Then $\int_H (h \restriction H) d\mu_H = \int h \times \chi H \, d\mu$ if either is defined in $[-\infty, \infty]$.

(c) If h is a $[-\infty, \infty]$-valued function and $\int_X h \, d\mu$ is defined in $[-\infty, \infty]$, then $\int_H (h \restriction H) d\mu_H$ is defined in $[-\infty, \infty]$.

(d) Suppose that h is a $[-\infty, \infty]$-valued function defined almost everywhere in X. Then

$$\overline{\int}_H (h \restriction H) d\mu_H = \overline{\int}_X h \times \chi H d\mu.$$

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*136 The Monotone Class Theorem

For the final section of this volume, I present two theorems on σ-algebras, with some simple corollaries. They are here because I find no natural home for them in Volume 2. While they (especially 136B) are part of the basic technique of measure theory, and have many and widespread applications, they are not central to the particular approach I have chosen, and can if you wish be left on one side until they come to be needed.

136A Lemma Let X be a set, and $\mathcal{A}$ a family of subsets of X. Then the following are equiveridical:

(i) $X \in \mathcal{A}$, $B \setminus A \in \mathcal{A}$ whenever A, $B \in \mathcal{A}$ and $A \subseteq B$, and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$;

(ii) $\emptyset \in \mathcal{A}$, $X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$, and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$.

Definition If $\mathcal{A} \subseteq \mathcal{P}X$ satisfies the conditions of (i) and/or (ii) above, it is called a **Dynkin class** of subsets of X.

136B Monotone Class Theorem Let X be a set and $\mathcal{A}$ a Dynkin class of subsets of X. Suppose that $\mathcal{I} \subseteq \mathcal{A}$ is such that $I \cap J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$. Then $\mathcal{A}$ includes the σ-algebra of subsets of X generated by $\mathcal{I}$.

136C Corollary Let X be a set, and μ, ν two measures defined on X with domains Σ, T respectively. Suppose that $\mu X = \nu X < \infty$, and that $\mathcal{I} \subseteq \Sigma \cap \mathrm{T}$ is a family of sets such that $\mu I = \nu I$ for every $I \in \mathcal{I}$ and $I \cap J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$. Then $\mu E = \nu E$ for every E in the σ-algebra of subsets of X generated by $\mathcal{I}$.

136D Corollary Let μ, ν be two measures on $\mathbb{R}^r$, where $r \geq 1$, both defined, and agreeing, on all intervals of the form

$$]-\infty, a] = \{x : x \leq a\} = \{(\xi_1, \dots, \xi_r) : \xi_i \leq \alpha_i \text{ for every } i \leq r\}$$

for $a = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$. Suppose further that $\mu\mathbb{R}^r < \infty$. Then μ and ν agree on all the Borel subsets of $\mathbb{R}^r$.

136E Algebras of sets: Definition Let X be a set. A family $\mathcal{E} \subseteq \mathcal{P}X$ is an **algebra** or **field** of subsets of X if

(i) $\emptyset \in \mathcal{E}$;

(ii) for every $E \in \mathcal{E}$, its complement $X \setminus E$ belongs to $\mathcal{E}$;

(iii) for every E, $F \in \mathcal{E}$, $E \cup F \in \mathcal{E}$.

136F Remarks (b) If $\mathcal{E}$ is an algebra of subsets of X, then

$$E \cap F, \quad E \setminus F,$$

$$E_0 \cup E_1 \cup \ldots \cup E_n, \quad E_0 \cap E_1 \cap \ldots \cap E_n$$

belong to $\mathcal{E}$ for all E, F, $E_0, \ldots, E_n \in \mathcal{E}$.

(c) A σ-algebra of subsets of X is an algebra of subsets of X.

136G Theorem Let X be a set and $\mathcal{E}$ an algebra of subsets of X. Suppose that $\mathcal{A} \subseteq \mathcal{P}X$ is a family of sets such that

(α) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-decreasing sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$,
(β) $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-increasing sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$,
(γ) $\mathcal{E} \subseteq \mathcal{A}$.

Then $\mathcal{A}$ includes the σ-algebra of subsets of X generated by $\mathcal{E}$.

***136H Proposition** Let (X, Σ, μ) be a measure space such that $\mu X < \infty$, and $\mathcal{E}$ a subalgebra of Σ; let Σ' be the σ-algebra of subsets of X generated by $\mathcal{E}$. If $F \in \Sigma'$ and $\epsilon > 0$, there is an $E \in \mathcal{E}$ such that $\mu(E \cap F) \leq \epsilon$.

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Concordance for Chapter 13

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

132E Measurable envelopes Parts (d) and (e) of 132E in the 2000 and 2001 editions, referred to in the 2001 edition of Volume 2 and the 2002 edition of Volume 3, are now parts (e) and (f).

132G Pull-back measures Proposition 132G, referred to in the 2006 edition of Volume 4, has been moved to 234F.