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ERRATA TO EUREKA 35

We are grateful to Mr. P. H. A. Green, who points out that the answer to the problem on page 25 is not 65, as given, but 25. (Consider triangles (15, 20, 25)(25, 24, 7)).

We are grateful to Mr. T. C. Smyth and Mr K. Nilsen, who show that the 'Alphametric' has four solutions (A, P = 0, 6 or 6, 0; L, Y, E = 3, 5, 7 or 5, 7, 3).

FOOTNOTE TO "WELL-FOUNDED GAMES"

Let a position in Nim have k piles with $n_1, \ldots, n_k$ members. For each j, express n_j in binary form as $\Sigma_i d_{ji} 2^i$, where each d_{ji} is either 1 or 0. Now the position is a P-position for positive Nim iff $\Sigma_j d_{ji}$ is even for each $i \geq 0$.

A position is a P-position for negative Nim iff either there is a pile with more than one counter and it is a P-position for positive Nim or every pile has 0 or 1 counter and it is an N-position for positive Nim.

The 'Marienbad' game started with piles of 1, 3, 5 and 7 counters, which is a P-position for either game.

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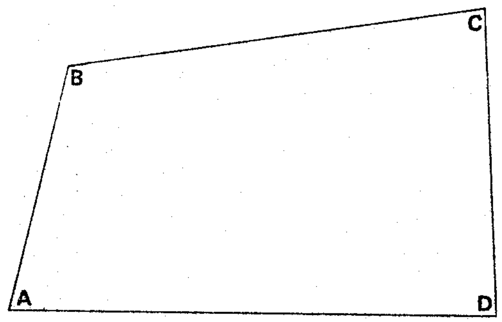
(a)



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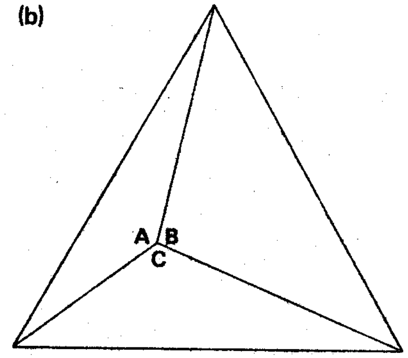

a set of 4 points, which will form either a convex quadrilateral (a) or else a triangle with 1 point inside it (b).

In (a) we have $A + B + C + D = 2\pi$; in (b) we have $A + B + C = 2\pi$. In either case there must be at least one non-acute angle, and so at least one of the four triangles must be non-acute. So we can take $\lambda_4 = 3/4$.

Let us now attack the recurrence relation. Put off by the [ ] function we consider its general properties. Clearly we have a non-increasing sequence, and numerical evidence suggests very strongly that it converges to 2/3. Encouraged by this we play with the first few numerical values and obtain: $\lambda_n = (2/3)(1 + (n - (n \text{ rem } 3))/n(n-1)(n-2))$ (where n rem 3 represents the remainder on division of n by 3).

By substituting this in (*) we verify that this is in fact its solution. So for the original problem we have: $\lambda_{100} = 3267/4900$.

Is this the best we can do? Can these upper bounds be attained? By drawing diagrams and doing a small computer search we obtained the following results:

n	$\lambda_n \cdot {}^nC_3$	largest number of acute triangles observed
4	3	3 (see (a))
5	7	7
6	14	14
7	24	22
8	38	32

So this is still an open question. If one found that in fact one of the upper bounds could be reduced, all the succeeding upper bounds could also be reduced using (*).

Alternatively, we can try to construct a sequence of diagrams containing increasing numbers of points, for which we have a general expression for the proportion of acute triangles in terms of the number of points. This will give us, so to speak, lower bounds for the upper bounds.

As an example, consider the regular polygons. Here we have a proportion of acute triangles which is $1/4 + O(1/n)$ as $n \to \infty$. This is not much use, but at least it disposes of the alarming possibility that the maximum proportion might $\to 0$ as $n \to \infty$.

Well-Founded Games

by David Fremlin

1. Most mathematicians are familiar with the game of Nim, but perhaps I should begin by briefly describing it. Two players face each other over a (finite) number of (finite) piles of counters. Each in turn must remove counters from a pile; he must remove at least one counter and he may touch only one pile. The player who takes the last counter loses. (A version of this game was prominent in a fashionable avant-garde film of a few years back, L'Année Dernière à Marienbad.)

2. Nim is one of a large family of games which are determinate in the sense that, for any given starting position, it is possible to predict which player will win if both play correctly. In the case of Nim, this is a consequence of the fact that there are only finitely many games possible following a given starting position. For we can define 'P-positions' and 'N-positions' inductively, thus. The 'empty' position, with no counters in no piles, is an N-position. Suppose we have classified all positions with a total of n counters or fewer, where $n \geqslant 0$. Now say that a position with $n + 1$ counters.

(a) is an N-position if there is a move by which it can be transformed into a P-position;

(b) otherwise, is a P-position.

It is now easy to see that if you have an N-position, you can force a win, while if you have a P-position, you will lose against an efficient opponent. For in the latter case your move will necessarily result in an N-position, and now your opponent can turn it into another P-position.

3. The finiteness of the whole game of Nim is not essential here; what is important is the fact that any particular game is bound to finish. Let us define a well-founded game to be a quadruple (G, R, T_N, T_P), where:

(i) G is a non-empty set.

(ii) R is a relation on G such that: if $A \subseteq G$ is non-empty, there is an $a \in A$ such that there is no $b \in A$ for which bRa. We interpret 'bRa' as 'there is a legal move transforming position a into position b', and say that b follows a. Thus the condition on R is that every non-empty subset A of G has a member which has no follower in A.

We see in particular (a) that there is no $a \in G$ for which aRa (for set $A = \{a\}$); (b) that there is no sequence (a_n) in G for which $a_{n+1} R a_n$ for every n (for set $A = \{a_n : n \in \mathbb{N}\}$); (c) that there exist terminal positions in G, i.e. positions with no followers (for set $A = G$). Using the axiom of choice, it can easily be shown that property (b) is equivalent to the condition on R. Thus, subject to the axiom of choice, a game is well-founded iff any particular sequence of moves must terminate.

(iii) Now finally (T_N, T_P) must be a partition of the set of terminal positions in G. We interpret T_N as the set of terminal positions for which the last player loses; T_P as the set of terminal positions for which the last player wins. Of course one of these may very well be empty. If $T_P = \emptyset$, as in ordinary Nim, we call the game negative; if $T_N = \emptyset$, the game is positive.

4. We can now prove the following.

Theorem Let (G, R, T_N, T_P) be a well-founded game in the sense of §3 above. Then there is a unique partition (G_N, G_P) of G such that:

$T_N \subseteq G_N$, $T_P \subseteq G_P$;

if $a \in G_P$, then there is no $b \in G_P$ such that bRa;

if $a \in G_N \setminus T_N$, then there is a $b \in G_P$ such that bRa.

Sketch of proof Say that a pair (U, V) of subsets of G is admissible if:

$T_N \subseteq U$, $T_P \subseteq V$;

$U \cap V = \emptyset$;

if $a \in V$, then every follower of a belongs to U;

if $a \in U \setminus T_N$, then there is a follower of a in V.

For instance, (T_N, T_P) is admissible, since the conditions are vacuously satisfied. If (U_1, V_1) and (U_2, V_2) are admissible, apply the well-foundedness condition (ii) of §3 to

$$A = (U_1 \cap V_2) \cup (U_2 \cap V_1)$$

to show that $A = \emptyset$; it follows that $(U_1 \cup U_2, V_1 \cup V_2)$ is admissible. Now set

$$G_N = \{a: \text{there is an admissible } (U, V) \text{ with } a \in U\}$$
$$G_P = \{a: \text{there is an admissible } (U, V) \text{ with } a \in V\}.$$

Show that (G_N, G_P) is admissible. Apply the well-foundedness condition again to

$$A = G \setminus (G_N \cup G_P)$$

to show that $A = \emptyset$ (for otherwise a bottom element of A could be added to one of G_N, G_P), and hence that G_N and G_P are the required sets. (This is a simple example of transfinite induction).

5. We can interpret the partition (G_N, G_P) by saying that G_N is the set of N-positions and G_P is the set of P-positions, just as in §2 above; every follower of a P-position is an N-position, while every non-terminal N-position is followed by at least one P-position.

It is clear that knowing G_N and G_P gives you a strategy: if you are faced with an N-position, change it to a P-position; if you are faced with a P-position, knock the board over. Unfortunately the theorem does not give an effective method of determining which are the crucial P-positions. In the case of a finite game like Nim or chess, there can in theory be found by enumeration of cases. (Of course, chess is a three-valued game, since draws are possible. Exercise for readers: adapt the theorem above to this case, showing that G is partitioned into 3 sets). The case of chess, of course, is very theoretical indeed.

6. However, for Nim, an effective algorithm for deciding whether a position is in G_N or G_P is known; it was published by Bouton (1). It is based on a curious fact. Let $(N^*, R, T, \emptyset)$ be the game of Nim in the terms of §3; $T_P = \emptyset$ because the only terminal position is an N-position. Now consider $(N^*, R, \emptyset, T)$, that is, the same game except that you win if you take the last counter. This is positive Nim, as opposed to the ordinary game, negative Nim. One would naturally suppose that they were quite different games. But in fact it is easy to see that positive and negative Nim have

nearly the same N- and P-positions. Now Bouton's analysis gives a very simple and elegant method of computing P-positions in positive Nim. (Hint: by direct calculation, as in §2, find all P-positions with ≤ 7 counters in each of ≤ 3 piles. Express your results in binary notation and look for a pattern. Answer on p. ii).

*7. The same arguments can be used for 'transfinite Nim'. In this case, the finite piles of counters are replaced by ordinals, possibly infinite; the number of piles, however, must remain finite. The trick is to work out what the binary expansion of an ordinal is. What you do is to identify the binary expansion of a finite integer k with a finite set $A \subseteq \underline{N}$; A is the set of places where 1's occur, i.e.

$$k = \sum_{n \in A} 2^n.$$

If $k = 0$, $A = \emptyset$. Now we can order the class of all finite sets of ordinals by writing

$$A < B \Leftrightarrow \exists\, \beta \in B \setminus A \text{ such that for } \alpha > \beta,\ \alpha \in A \Leftrightarrow \alpha \in B.$$

This is a well-ordering and consequently the class of finite sets of ordinals is canonically isomorphic to the class of all ordinals; thus each ordinal is naturally associated with a finite set of ordinals, and it is this finite set which behaves like a binary expansion. The Nim analysis is now easy to apply directly to these finite sets.

8. A whole class of games can now be tackled; this was done by Sprague (2). If G and H are positive finite games, their disjoint sum $G \oplus H$ is the game with position set $G \times H$ and follower relation given by

$$(a_1, a_2)R(b_1, b_2) \Leftrightarrow a_1 = a_2 \ \&\ b_1 R b_2 \text{ or } a_1 R a_2 \ \&\ b_1 = b_2.$$

The terminal positions of $G \oplus H$ are just the pairs (a, b) where a is terminal in G and b is terminal in H; they are all P-positions, as this is to be a positive game. It is easy to see that $G \oplus H$ is still finite. We observe that positive Nim is just a disjoint sum of copies of the trivial game Nim_1, positive-Nim-with-one-pile.

Sprague's work applies to any disjoint sum of positive finite games. He uses the notion of the rank of a position. If a is a position in a positive finite game, then r(a) is defined inductively by

$r(a) = 0$ if a is terminal;

$r(a) =$ least non-negative integer not equal to r(b) for any follower b of a, if a is not terminal.

It is easy to see that r is well-defined; that a is a P-position iff $r(a) = 0$ (this is where we use the fact that the game is positive); and that r(a) is that unique integer such that the pair (a, r(a)) is a P-position in the disjoint sum $G \oplus Nim_1$.

The point is that if G and H are positive finite games, then (a, b) is a P-position in the disjoint sum $G \oplus H$ iff $r(a) = r(b)$. From this we see that (a, b) is a P-position in $G \oplus H$ iff (a, r(b)) is a P-position in $G \oplus Nim_1$. Now a simple induction shows that if $G_1, \ldots, G_n$ are positive finite games, a position $(a_1, \ldots, a_n)$ in $G_1 \oplus \ldots \oplus G_n$ is a P-position iff $(r(a_1), \ldots, r(a_n))$ is a P-position in positive Nim; and we know all about P-positions in Nim.

From the arguments above it is clear that r(a, b) is a function of r(a) and r(b). I leave it to the reader to describe this function.

*9. This analysis works just as well for arbitrary positive well-founded games; but the rank function must now be allowed to take infinite ordinal values, and its definition requires a true transfinite induction. We then use the analysis of transfinite positive Nim (§7 above).

10. Grundy & Smith (3) made a determined assault on the problem of disjoint sums of negative games, like ordinary Nim. They shought a new kind of rank function which would be such that (i) $\theta(a, b)$ would depend only on $\theta(a)$ and $\theta(b)$ (ii) the value of $\theta(a)$ would determine whether a was an N- or a P-position. Their arguments are interesting and subtle but the most natural conclusion to draw from them is that the problem is very hard.

11. I will conclude this essay with a description of two particular finite games. The first is taken from (2). Its positions are the same as those of ordinary positive Nim, but there is a new kind of move; as an alternative to removing counters, you may break one of the piles into two, each part, of course, not empty. Calculate the Sprague rank of a single pile of n counters; the answer may surprise you.

12. The second game I shall call Nim-squared; I saw it in the Observer in 1961 or 1962. Imagine counters placed, not in piles, but in a rectangular array, as on the squares of a chessboard. For a move, you take counters away; as usual, you must remove at least one; and if you take more than one, they must either all belong to the same column or all belong to the same row. Thus

```
XXXX        XXXX        XXX         XXX
XXXX  -->    X    -->    X    -->    X
XXXX        XXXX        XXXX        XXXX
XXXX        XXXX        XXX         XX
```

is a legitimate sequence. What are the N- and P-positions (a) for the positive game (b) for the negative game? I have no idea how to find them in general, but I cajoled the PDP-10 computer at the University of Essex into giving me a list of all P-positions that can be got into a 4 × 4 square. Observe that (i) permuting the columns (ii) permuting the rows (iii) reflecting about the diagonal, do not change the value of a position; so I give only one example of each type. The first list (Table 1) gives the

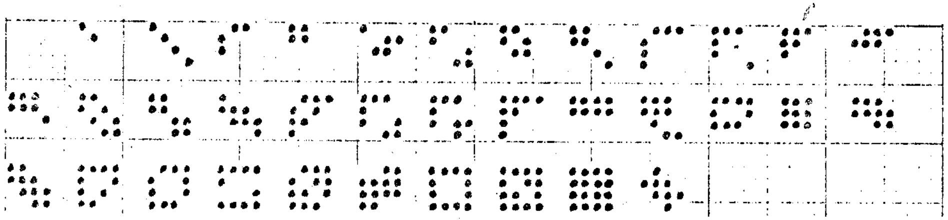

Table 1

P-positions for positive Nim-squared

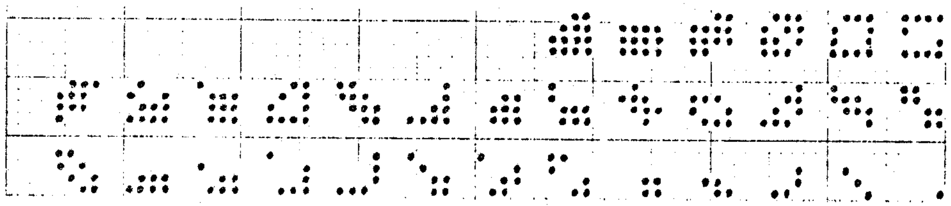

Table 2

P-positions for negative Nim-squared

P-positions for the positive game. (As is usual in games of this type, the N-positions greatly outnumber the P-positions). Unsurprisingly, many of these possess a rotational symmetry of some kind. The second list (Table 2) gives the P-positions for the negative game. Now I find it really surprising that the two lists overlap as much as they do. Has anyone any ideas?

References

(1) Bouton A. L., 'Nim, a game with a complete mathematical theory', Ann. Math. 3 (1902) 35-39.

(2) Sprague R. P., 'Über mathematische Kampfspiele', Tohoku Math. J. 41 (1935) 438-444.

(3) Grundy P. M. & Smith C. A. B., 'Disjunctive games with the last player losing', Proc. Cambridge Phil. Soc. 52 (1956) 527-533.

$y^2 + D = x^5$

by Bernard M. E. Wren

The most thoroughly investigated single Diophantine equation must be $y^2 + D = x^3$; see, for example, chapter 26 of Mordell [1].

The equation $y^2 + D = x^5$ however, is much less represented in the literature, but does yield a small amount to elementary twiddling.

For example, squares modulo 11 are $\{0, 1, 3, 4, 5, 9\}$ and fifth powers modulo 11 are $\{0, 1, 10\}$ whence the equation is insoluble in integers if $D \equiv 4 \mod 11$.

Theorem

Let D be a positive, squarefree integer, $D \neq 3$, with $D \not\equiv 7 \mod 8$. Then if 5 does not divide the class-number of $\mathbb{Q}(\sqrt{-D})$, the equation $y^2 + D = x^5$ has no solution in integers except when $D = 1, 19, 341$ when the only solutions are

$$(x, \pm y) = (1, 0), (55, 22434), (377, 2759646).$$

Proof:

The case $D = 1$ goes back to 1850, see Mordell [1], p. 301, so henceforth we assume $D \neq 1$.

x even implies $y^2 \equiv -D \mod 8$ which is impossible under the given hypotheses. So x is odd, $y^2 + D$ is odd, and thus $(y + \sqrt{-D}, 2) = 1$.

Then $(y + \sqrt{-D}, y - \sqrt{-D}) = (y + \sqrt{-D}, 2\sqrt{-D}) = \underbrace{(y + \sqrt{-D}, 2)}_{=1}$ since $(y, D) = 1$

Whence $(y + \sqrt{-D}) = \mathfrak{a}^5$ for some integral ideal $\mathfrak{a}$. Since 5 does not divide the class-number, $\mathfrak{a}$ is a principal ideal. Moreover, when $D \equiv 3 \mod 4$, the multiplicative index of $Z\left[\frac{1 + \sqrt{-D}}{2}\right]$ in $Z[\sqrt{-D}]$ is 3 so that if

$$\mathfrak{a} = \left(\frac{a + b\sqrt{-D}}{2}\right)$$ with a, b odd integers, then