Linguistic Phenomena in Mathematics

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A Language for Mathematics

- Joint work with Thomas Barnet-Lamb.
- Long-term project.
- Only discuss linguistic aspects here.

Mathematics as a Linguistic Domain

- Clean, total semantics.
- Every mathematical term is formally defined.
- Can extract all syntactic and semantic information from definitions.
- *Adaptivity* — language starts with a small core, expands via definitions.
- Benefits of both closed domains and open domains.

Example: Real Mathematics

Sylow's Theorems

Let G be a finite group whose order is divisible by the prime p. Suppose p^m is the highest power of p which is a factor of $|G|$ and set

$$k = \frac{|G|}{p^m}.$$

Then

1. the group G contains at least one subgroup of order p^m,
2. any two subgroups of G of order p^m are conjugate, and
3. the number of subgroups of G of order p^m is congruent to 1 modulo p and is a factor of k.

Example: MIZAR (truncated)

```
theorem :: GROUP_10:12
  for G being finite Group,
  p being prime (natural number)
  holds ex P being Subgroup of G st
    P is_Sylow_p-subgroup_of_prime p;

theorem :: GROUP_10:14
  for G being finite Group,
  p being prime (natural number) holds
    (for H being Subgroup of G st
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        ...
```

Comparison

Sylow's Theorems

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    (for P1,P2 being Subgroup of G
      st P1 is_Sylow_p-subgroup_of_prime p
      & P2 is_Sylow_p-subgroup_of_prime p
      holds P1,P2 are_conjugated);

theorem :: GROUP_10:15
  for G being finite Group,
    p being prime (natural number) holds
    card the_sylow_p-subgroups_of_prime(p,G)
       mod p = 1 &
    card the_sylow_p-subgroups_of_prime(p,G)
       divides ord G;
```

Example: A New Language

Theorem 72 *("Sylow's Theorems")*

Let G be a finite group whose order is divisible by a prime p. Let m be the integer s.t. p^m is the highest power of p which divides |G| and set

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Natural Formal Languages

- Fuse concepts from formal languages and natural languages.
- Requires solving substantive some problems, but...
- Find formal language features counteract weaknesses of natural language and vice versa.
- E.g. type (FL concept) defangs problems caused by ambiguity (NL concept).
- Major examples take too much space... .
- Below: Discuss a minor issue of particular interest to linguists.

Mathematics and Logic

- Then $V = U \cap H$ for some U in $\mathcal{T}$, by definition of $\mathcal{T}_H$, and $U \cap H = i^{-1}(U)$, so $g^{-1}(V) = g^{-1}(i^{-1}(U)) = (i \circ g)^{-1}(U)$.

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Mathematics and Logic

- Then $V = U \cap H$ for some U in $\mathcal{T}$ and $U \cap H = i^{-1}(U)$.

Mathematics and Logic

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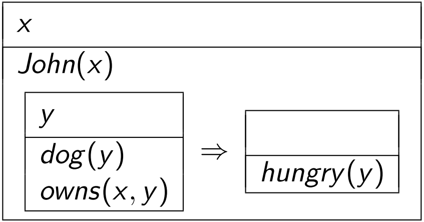
unbound variable

Varying the Quantifier

- $\alpha[U]$ for **some** U in $\mathcal{T}$ and $\beta[U]$.
- *$\alpha[U]$ for **every** U in $\mathcal{T}$ and $\beta[U]$.
- Asymmetric treatment of quantifiers required.
- All standard logics fail.

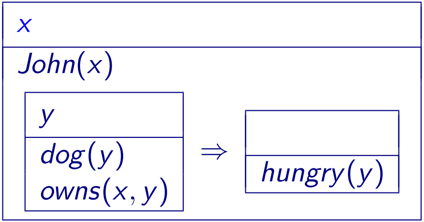
Discourse Representation Theory



All John's dogs are hungry.

(I.e. If y is a dog which John owns, then y is hungry.)

Discourse Representation Theory



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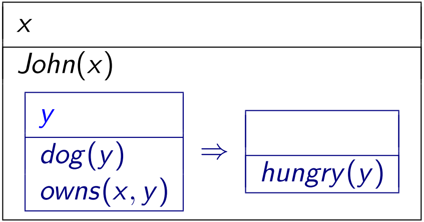

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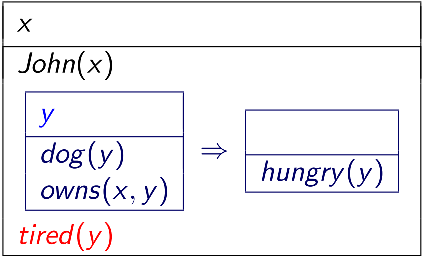

Donkey Sentences and DRT

Every farmer who owns a donkey beats it.

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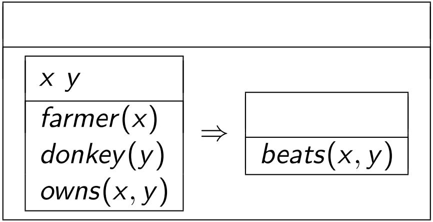

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Mathematical Variables as Referents

Every natural number which is greater than 1 has a prime divisor.

If $n > 1$ is a natural number, then there is a prime p such that $p|n$.

Mathematical Variables as Referents

x	$\Rightarrow$	y
$natural\_number(x)$		$is\_prime(y)$
$greater\_than(x, 1)$		$divides(y, x)$

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$\mathbf{n}$	$\Rightarrow$	$\mathbf{p}$
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If $n > 1$ is a natural number, then there is a prime p such that $p|n$.

DRT in Action

$$\begin{array}{|l|}\hline \mathbf{U} \\ \hline \mathit{in}(\mathbf{U},\mathcal{T}) \\ \hat{\alpha}[\mathbf{U}] \\ \hline \end{array}$$

$$\alpha[U] \text{ for some } U \text{ in } \mathcal{T}$$

$$\begin{array}{|l|}\hline \\ \hline \hat{\beta}[\mathbf{U}] \\ \hline \end{array}$$

$$\beta[U]$$

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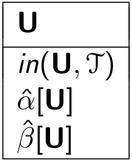

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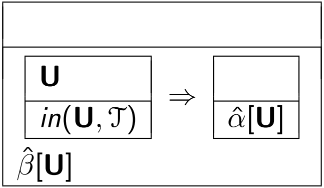

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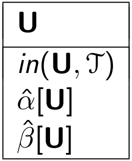

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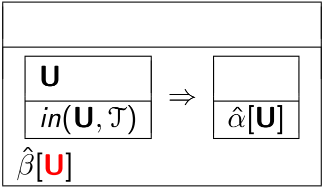

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Conclusion

- Mathematical language contains variables, normally associated with formal languages.
- But a semantic theory fitted to formal languages (Predicate Calculus) cannot describe their behaviour.
- Need to use a theory designed for natural languages (DRT).
- Such hybridised formal language/natural language concepts recur throughout mathematics.
- We need a ‘natural formal language’ to describe mathematics.