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from D.H.Fremlin

<u>Families of random triples</u> <u>Note of 12 July 1985</u> (version of 30.11.88)

1. <u>Definition</u> Let κ , λ and θ be cardinals, with $1 \le \theta \le \lambda \le \kappa$. Let $u \in [0,1]$. Say that

$$(\kappa,u) \Rightarrow [\lambda]^{\theta}$$

if: whenever $\langle E_I \rangle_{I \in [\kappa]^{\theta}}$ is a family of Lebesgue measurable subsets of $[0,1]$ such that $\mu E_I \ge u$ for every $I \in [\kappa]^{\theta}$, then there is a $K \in [\kappa]^{\lambda}$ such that $\bigcap_{I \in [K]^{\theta}} E_I \ne \emptyset$.

<u>Remark</u> Observe that $[0,1]$ can be replaced by any isomorphic measure space; in particular, by any atomless Radon probability on a Polish space.

2. <u>Lemma</u> If $1 \le k \le r \in \mathbb{N}$ and $u \in [0,1]$ and $(\omega,u) \Rightarrow [r]^k$, then there are $u' < u$ and $m \in \mathbb{N}$ such that $(m,u') \Rightarrow [r]^k$.

<u>proof</u> ~~Suppose~~ ? If not, then for each $m \ge r$ there is a family $\langle E^m_I \rangle_{I \in [m]^k}$ of Lebesgue measurable sets such that $\mu E^m_I \ge u - \frac{1}{m}$ for each $I \in [m]^k$ and $\bigcap_{I \in [K]^k} E_I = \emptyset$ for every $K \in [m]^r$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and for each finite $\mathcal{J} \subseteq [\mathbb{N}]^k$ set $s_{\mathcal{J}} = \lim_{m \to \mathcal{F}} \mu(\bigcap_{I \in \mathcal{J}} E_I)$. Because $[\mathbb{N}]^k$ is countable, there is a family $\langle F_I \rangle_{I \in [\mathbb{N}]^k}$ of measurable sets in $[0,1]$ such that $\mu(\bigcap_{I \in \mathcal{J}} F_I) = s_{\mathcal{J}}$ for every finite $\mathcal{J} \subseteq [\mathbb{N}]^k$; the F_I can be ~~supposed~~ taken to be finite unions of open intervals. We see that

$$\mu F_I = s_{\{I\}} = \lim_{m \to \mathcal{F}} \mu E^m_I \ge u \quad \forall\ I \in [\mathbb{N}]^k ,$$

$$\mu(\bigcap_{I\in[K]^k} F_I) = \lim_{m\to\infty}\mu(\bigcap_{I\in[K]^k} E_I^m) = 0 \quad \forall\ K \in [\underset{\sim}{N}]^r .$$

Because each F_I is open, it follows that $\bigcap_{I\in[K]^k} F_I = \emptyset$ whenever $K \in [\underset{\sim}{N}]^r$, so that $(\kappa, u) \not\Rightarrow [r]^k$.

<u>remark</u> This result was a precursor of Theorem 3Da of [2].

3. <u>Proposition</u> If $1 \leq k \leq r < \omega \leq \kappa$ and $u \in [0,1]$ and $(\kappa^+, u) \Rightarrow [r+1]^{k+1}$ then $(\kappa, u) \Rightarrow [r]^k$.

<u>proof</u> I prove the contrapositive. Suppose that $(\kappa, u) \not\Rightarrow [r]^k$. Let $\langle E(I)\rangle_{I\in[\kappa]^k}$ be a family of sets of measure $\geq u$ such that $\bigcap_{I\in[K]^k} E(I) = \emptyset$ for every $K \in [\kappa]^r$. For $\alpha < \kappa^+$ let $f_\alpha : \alpha \to \kappa$ be an injection. For $J \in [\kappa^+]^{k+1}$ set $F_J = E(f_\alpha[J \cap \alpha])$ where $\alpha = \max J$. Then $\mu F_J \geq u$ for every $J \in [\kappa^+]^{k+1}$. Suppose $K \in [\kappa^+]^{r+1}$. Set $\alpha = \max K$. Then

$$\begin{aligned} \bigcap\{ F_J : J \in [K]^{k+1} \} &\subseteq \bigcap\{ F_J : J \in [K]^{k+1},\ \alpha = \max J \} \\ &= \bigcap\{ E(f_\alpha[J\cap\alpha]) : J \in [K]^{k+1},\ \alpha = \max J \} \\ &= \bigcap\{ E(J) : J = [f_\alpha[K\cap\alpha]]^k \} = \emptyset , \end{aligned}$$

because $f_\alpha[K\cap\alpha] \in [\kappa]^r$. So $\langle F_J\rangle_{J\in[\kappa^+]^{r+1}}$ witnesses that $(\kappa^+, u) \not\Rightarrow [r+1]^{k+1}$.

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4. Lemma (a) Let κ be an infinite cardinal. If $\varphi : [(2^\kappa)^+]^{<\omega} \to \kappa$ is any function, there is a set $A \subseteq (2^\kappa)^+$, of order type κ^+, and a function $\psi : [A]^{<\kappa} \to \kappa$ such that $\varphi(I \cup \{\xi\}) = \psi(I)$ whenever $I \in [A]^{<\omega}$ and $\max I < \xi \in A$.

(b) If $\varphi : [\omega_1]^{<\omega} \to \underset{\sim}{N}$ is any function, there is an infinite set $B \subseteq \omega_1$ and a function $\psi : [B]^{<\omega} \to \underset{\sim}{N}$ such that $\varphi(J \cup \{\xi\}) = \psi(J)$ whenever $J \in [B]^{<\omega}$ and $\max J < \xi \in B$.

(c) If $\varphi : [\mathfrak{c}^+]^{<\omega} \to \underset{\sim}{N}$ is any function, there is an infinite set $B \subseteq \mathfrak{c}^+$ and a function $\chi : [B]^{<\omega} \to \underset{\sim}{N}$ such that $\varphi(K \cup \{\xi, \eta\}) = \chi(K)$ whenever $K \in [B]^{<\omega}$, $\xi, \eta \in B$ and $\max K < \xi < \eta$.

proof (a) Let $\langle \zeta_\xi \rangle_{\xi < \kappa^+}$ be a strictly increasing family in $(2^\kappa)^+$ such that whenever $C \subseteq \zeta_\xi$, $\#(C) \le \kappa$, and $\zeta_\xi \le \beta < (2^\kappa)^+$ there is a γ such that $\zeta_\xi \le \gamma < \zeta_{\xi+1}$ and $\varphi(J \cup \{\beta\}) = \varphi(J \cup \{\gamma\})$ for every $J \in [C]^{<\omega}$; this is possible bcause there are at most 2^κ sets $C \in [\zeta_\xi]^{\le\kappa}$ and at most 2^κ functions from $[C]^{<\omega}$ to κ for each C. Set $\zeta = \sup_{\xi<\kappa^+} \zeta_\xi$. Now choose a ~~strictly increasing~~ family $\langle \alpha_\xi \rangle_{\xi<\kappa^+}$ such that $\zeta_\xi \le \alpha_\xi < \zeta_{\xi+1}$ and $\varphi(J \cup \{\zeta\}) = \varphi(J \cup \{\alpha_\xi\})$ for every $J \in [\{\alpha_\eta : \eta < \xi\}]^{<\omega}$, for each $\xi < \kappa^+$. Set $A = \{\alpha_\xi : \xi < \kappa^+\}$, $\psi(J) = \varphi(J \cup \{\zeta\})$ for $J \in [A]^{<\omega}$.

(b) Choose inductively $\beta_n < \omega_1$, $C_n \subseteq \omega_1$, for $n \in \underset{\sim}{N}$, as follows. $C_0 = \omega_1$. Given C_n, set $\beta_n = \min C_n$ and let $C_{n+1} \subseteq C_n \setminus \{\beta_n\}$ be an uncountable set such that $\varphi(J \cup \{\xi\}) = \varphi(J \cup \{\eta\})$ for every $J \subseteq \{\beta_i : i \le n\}$ and all $\xi, \eta \in C_{n+1}$. On completing the induction, set $B = \{\beta_n : n \in \underset{\sim}{N}\}$.

(c) Put (a) (with $\kappa = \omega$) and (b) together.

Note This lemma is due to P.Erdős & R.Rado.

5. Theorem Suppose that $1 \leq k \leq r \in \mathbb{N}$ and that $u \in [0,1]$.

(a) If $(\omega, u) \Rightarrow [r]^k$ then $(\omega_1, u) \Rightarrow [r+1]^{k+1}$ and $(\mathfrak{c}^+, u) \Rightarrow [r+2]^{k+2}$.

(b) If $\kappa \geq \mathfrak{c}$ and $(\kappa^+, u) \Rightarrow [r]^k$ then $((2^\kappa)^+, u) \Rightarrow [r+1]^{k+1}$.

proof (a) Let $m \geq r$ and $\delta > 0$ be such that

$$(m, u - m^2\delta) \Rightarrow [r]^k$$

(Lemma 1). Let $\mathcal{H}$ be a countable family of measurable sets such that for every measurable $E \subseteq [0,1]$ there is an $H \in \mathcal{H}$ such that $\mu(E \Delta H) \leq \delta$.

(i) If $\langle E_I \rangle_{I \in [\omega_1]^{k+1}}$ is a family of measurable sets of measure $\geq u$, choose for each $I \in [\omega_1]^{k+1}$ an $H_I \in \mathcal{H}$ such that $\mu(E_I \Delta H_I) \leq \delta$. By 4b there is an infinite $B \subseteq \omega_1$ and a function $\chi : [B]^k \to \mathcal{H}$ such that $H_{J \cup \{\xi\}} = \chi(J)$ whenever $J \in [B]^k$ and $\max J < \xi \in B$. Let $C \subseteq B$ be a set of size m and take any $\gamma \in B$ with $\max C < \gamma$. Set

$$F_J = \bigcap \{ E_{J \cup \{\xi\}} : \xi \in C \cup \{\zeta\},\ \max J < \xi \}\ \forall \quad J \in [C]^k .$$

Then

$$\mu(F_J \Delta E_{J \cup \{\zeta\}}) \leq \sum_{\max J < \xi \in C \cup \{\zeta\}} \mu(E_{J \cup \{\xi\}} \Delta \chi(J))$$
$$\leq (m+1-k)\delta ,$$

so $\mu F_J \geq u - m\delta$ for each $J \in [C]^k$. Because $(m, u - m^2\delta) \Rightarrow [r]^k$, there is a $K \in [C]^r$ such that $\bigcap_{J \in [K]^k} F_J \neq \emptyset$. Now $\bigcap_{I \in [L]^{k+1}} E_I \neq \emptyset$, where $L = K \cup \{\zeta\} \in [\omega_1]^{r+1}$. As $\langle E_I \rangle_{I \in [\omega_1]^{k+1}}$

is arbitrary, $(\omega_1, u) \Rightarrow [r+1]^{k+1}$.

(ii) If $\langle E_I \rangle_{I \in [\mathfrak{c}^+]^{k+2}}$ is a family of measurable sets of measure $\geq u$, then for each $I \in [\mathfrak{c}^+]^{k+2}$ choose $H_I \in \mathcal{H}$ such that $\mu(E_I \Delta H_I) \leq \delta$. By 4c, there is an infinite $B \subseteq \mathfrak{c}^+$ and a function $\chi : [B]^k \to \mathcal{H}$ such that $H_{K \cup \{\xi, \eta\}} = \chi(K)$ whenever $K \in [B]^k$, $\xi, \eta \in B$ and $\max K < \xi < \eta$. Take $C \in [B]^m$, $\zeta, \vartheta \in B$ such that $\max C < \zeta < \vartheta$. For $K \in [C]^k$ set

$$F_K = \bigcap \{ E_{K \cup \{\xi, \eta\}} : \xi, \eta \in C \cup \{\zeta, \vartheta\} , \max K < \xi < \eta \} .$$

Then $\mu F_K \geq u - m^2 \delta$ for each $K \in [C]^k$ so there is an $L \in [C]^r$ such that $\bigcap_{K \in [L]^k} F_K \neq \emptyset$. In this case $\bigcap_{I \in [M]^{k+2}} E_I \neq \emptyset$ where $M = L \cup \{\zeta, \vartheta\} \in [\mathfrak{c}^+]^{r+2}$. As $\langle E_I \rangle_{I \in [\mathfrak{c}^+]^{k+2}}$ is arbitrary, $(\mathfrak{c}^+, u) \Rightarrow [r+2]^{k+2}$.

(b) If $\langle E_I \rangle_{I \in [(2^\kappa)^+]^{k+1}}$ is any family of measurable sets, all ~~then for~~ of measure $\geq u$, then for each $I \in [(2^\kappa)^+]^{k+1}$ choose a Borel set $F_I \subseteq E_I$ of the same measure. As there are only $\mathfrak{c} \leq \kappa$ Borel sets, there is an $A \subseteq (2^\kappa)^+$, of order type κ^+ , ~~such that~~ and a ~~[illegible]~~ family $\langle G_J \rangle_{J \in [A]^k}$ such that $F_{J \cup \{\xi\}} = G_J$ whenever $J \in [A]^k$ and $\max J < \xi \in A$. Because $(\kappa^+, u) \Rightarrow [r]^k$, there is a set $C \in [A]^r$ such that $\bigcap_{J \in [C]^k} G_J \neq \emptyset$. Take any $\gamma \in A$ such that $\gamma > \max C$ and set $K = C \cup \{\gamma\}$; then $\bigcap_{I \in [K]^{k+1}} E_I \neq \emptyset$.

6. <u>Proposition</u> (<u>a</u>) $(\omega, \frac{5}{9}) \not\Rightarrow [4]^3$.

(<u>b</u>) $(\omega_1, u) \Rightarrow [4]^3$ iff $u > \frac{1}{2}$.

(<u>c</u>) $(\mathfrak{c}, \frac{1}{4}) \not\Rightarrow [4]^3$.

(<u>d</u>) $(\mathfrak{c}^+, u) \Rightarrow [4]^3$ iff $u > 0$.

<u>proof</u> (<u>a</u>) Let $X = \{0,1,2\}^{\underset{\sim}{N}}$ and let ν be the natural Radon measure on X , the product of the uniform measure on each factor. For $I = \{i,j,k\} \in [\underset{\sim}{N}]^3$ set

$E_I = \{ x :$ <u>either</u> $x(i)$, $x(j)$, $x(k)$ are all different

<u>or</u> $x(i) + x(j) + x(k) \equiv 1 \bmod 3 \}$.

Then $\nu E_I = \frac{5}{9}$ for every $I \in [\underset{\sim}{N}]^3$ but $\bigcap_{I \in [K]^3} E_I = \emptyset$ for every $K \in [\underset{\sim}{N}]^4$.

(<u>b</u>) By Prop. 3 and Theorem 5a, $(\omega_1, u) \Rightarrow [4]^3$ iff $(\omega, u) \Rightarrow [3]^2$. But this happens iff $u > \frac{1}{2}$ ([1], Theorem 1, or [2], 3G).

(<u>c</u>) Let $X = \{0,1\}^{\underset{\sim}{N}}$ and let ν be the usual Radon measure on X . Let $\leq$ be the lexigographic ordering of X and let $+$ be the natural group operation (identifying each factor $\{0,1\}$ with the cyclic group $\underset{\sim}{Z}_2$). Let $\varphi : \mathfrak{c} \to X$ be any injection. If $I \in [\mathfrak{c}]^3$, express I as $\{\xi, \eta, \zeta\}$ where $\xi < \eta < \zeta$, and set

$$E_I = \{ x : x \in X , x + \varphi(\xi) < x + \varphi(\eta) , x + \varphi(\zeta) < x + \varphi(\eta) \} .$$

Then $\nu E_I \geq \frac{1}{4}$ for each $I \in [\mathfrak{c}]^3$ but $\bigcap_{I \in [K]^3} E_I = \emptyset$ if $K \in [\mathfrak{c}]^4$.

(<u>d</u>) By Theorem 5a, $(\mathfrak{c}^+, u) \Rightarrow [4]^3$ whenever $(\omega, u) \Rightarrow [2]^1$; which is true whenever $u > 0$. And of course $(\mathfrak{c}^+, 0) \not\Rightarrow [4]^3$.

7. <u>Problems</u> (<u>a</u>) Find $\inf\{ u : (\omega,u) \Rightarrow [4]^3 \}$. (The methods of [2] are presumably relevant.)

(<u>b</u>) Is it consistent to suppose that $(\mathfrak{c},\frac{1}{2}) \Rightarrow [4]^3$? or that $(\omega_2,\frac{1}{2}) \not\Rightarrow [4]^3$?

<u>References</u> [1] P.ERdös & A.Hajnal, "Some remarks on set theory, IX. Combinatorial problems in measure theory and set theory", Mich. Math. J. 11 (1964) 107-127.

[2] D.H.Fremlin & M.Talagrand, "Subgraphs of random graphs", Trans. Amer. Math. Soc. 291 (1985) 551-582.

8. <u>Note added 6.4.87</u> (in response to a question of J. Steprāns): For finite λ , the definition in §1 can be re-written: whenever $\langle E_I \rangle_{I\in[\kappa]^\theta}$ is a family of Lebesgue measurable subsets of $[0,1]$ such that $\mu E_I \geq u$ for every $I \in [\kappa]^\theta$, then there is a $K \in [\kappa]^\lambda$ such that $\mu(\bigcap_{I\in[K]^\theta} E_I) > 0$. (Because we can replace the E_I by $\varphi(E_I)$ where φ is a multiplicative lifting for Lebesgue measure.)

~~9. Lemma~~ (added 11.11.88) Let X be a set with $n \geq 2$ elements, $\mathcal{J} \subseteq [X]^3$ a set such that $[K]^3 \not\subseteq \mathcal{J}$ for every $K \in [X]^4$; then $\#(\mathcal{J}) \leq n(n-1)(2n-3)/18$.

<u>proof</u> For $J \in [X]^2$ set $a_J = \#(\{ I : J \subseteq I \in \mathcal{J} \})$. Then $\sum_{J\in[X]^2} a_J = 3m$ where $m = \#(\mathcal{J})$. Also, if $I \in \mathcal{J}$, then

$$\sum_{J\in[I]^2} a_J \leq 2n-3$$

because writing $A_J = \{ k : J \cup \{k\} \in \mathcal{J} \}$ then $\bigcap_{J\in[I]^2} A_J$ must be empty so $\sum_{J\in[I]^2} \#(A_J \setminus I) \leq 2(n-3)$ while $A_J \cap I = I \setminus J$ has just one member for each $J \in [I]^2$, and $2(n-3) + 3 = 2n-3$.

Supplement 30.11.88

I add a note to integrate the above with the work of [3].

9. Definition If $1 \le k \le r \le s \in \mathbb{N}$, write $T^*(k,r,s)$ for
$\max\{ \#(\mathcal{A}) : \mathcal{A} \subseteq [s]^k , [K]^k \not\subseteq \mathcal{A} \ \forall \ K \in [s]^r \}$.

10. Lemma If $1 \le k \le r \le s \in \mathbb{N}$, $u \in [0,1]$ then $(s,u) \Rightarrow [r]^k$ iff $u > T^*(k,r,s)/\binom{s}{k}$.

proof (a) If $u > T^*(k,r,s)/\binom{s}{k}$ and $\langle E_I \rangle_{I \in [s]^k}$ is a family of measurable subsets of $[0,1]$ all of measure $\ge u$, then there must be a $t \in [0,1]$ such that $\mathcal{A}_t = \{ I : t - E_I \}$ has cardinal at least $\binom{s}{k}u > T^*(k,r,s)$; in which case there is a $K \in [s]^r$ such that $[K]^k \subseteq \mathcal{A}_t$ i.e. $t \in \bigcap_{I \in [K]^k} E_I$.

(b) Fix $\mathcal{A} \subseteq [s]^k$, of cardinal $T^*(k,r,s)$, such that $[K]^k \not\subseteq \mathcal{A}$ for every $K \in [s]^r$. Let F be the set of all bijections from s to s and let $\varphi : [0,1] \to F$ be a function such that $\mu\varphi^{-1}[\{f\}] = \frac{1}{s!}$ for every $f \in F$. For $I \in [s]^k$ set

$$E_I = \{ t : \varphi(t)[I] \in \mathcal{A} \} ;$$

then $\mu E_I = \#(\mathcal{A})/\binom{s}{k}$ for each I . If $K \in [s]^r$, $t \in [0,1]$ then there is a $J \in [\varphi(t)\ [K]]^k \setminus \mathcal{A}$ so that $t \notin E_I$ where $I = \varphi(t)^{-1}[J] \in [K]^k$. Thus $\langle E_I \rangle_{I \in [s]^k}$ witnesses that $(s,u) \not\Rightarrow [r]^k$ where $u = T^*(k,r,s)/\binom{s}{k}$.

11. Lemma ~~(a)~~ If $0 \le k \le s \le s+k \le n \in \mathbb{N}$ then

(a) $$\sum_{i=0}^{k} \binom{n-2k}{n-i-s}\binom{k}{i}\binom{n-i}{k-i} = \binom{n-k}{s-k}\binom{s}{k}$$

(b) $$\sum_{i=0}^{k} \binom{n-2k}{n-i-s}\binom{k}{i}^2 \Big/ \binom{n}{i} = \binom{n-k}{s-k}^2 \Big/ \binom{n}{s} .$$

Note $\binom{n}{i} = \#([n]^i)$ is to be taken as 0 if $i < 0$ or $i > n$.

proof (for completeness; these are "standard" facts)

(a) Set

$$p = \sum_{i=0}^{k} \binom{n-2k}{n-i-s}\binom{k}{i}\binom{n-i}{k-i}\binom{n-k}{k} = \sum_{i=0}^{k} \binom{n-k}{k}\binom{k}{i}\binom{n-2k}{s-2k+i}\binom{n-i}{k-i}$$

$$= \sum_{i=0}^{k} \#(\{ (J,K,L,M) : K \in [n \setminus k]^k ,\ M \in [K]^i ,\ k \subseteq L \in [n]^s ,\ K \cap L = K \setminus M ,\ M \subseteq J \in [n]^k \})$$

$$= \sum_{i=0}^{k} \#(\{ (J,K,L) : K \in [n \setminus k]^k ,\ k \subseteq L \in [n]^s ,\ K \setminus L \subseteq J \in [n]^k \})$$

$$= \binom{n-k}{s-k} p'$$

where

$$p' = \#(\{ (J,K) : K \in [n \setminus k]^k ,\ K \setminus s \subseteq J \in [n]^k \}) .$$

Similarly, if we write

$$q = \binom{n-k}{s-k}\binom{s}{k}\binom{n-k}{k} \ , \cancel{\binom{n-k}{k}\binom{n-k}{s-k}\binom{s}{k}}$$

then $q = \binom{n-k}{s-k} q'$ where

$$q' = \#(\{ (J,K) : K \in [n \setminus k]^k ,\ J \in [s]^k \}) .$$

I claim that $p' = q'$. To see this, it will be enough to check that for every $D \subseteq n \setminus s$,

$$\#(\{ (J,K) : J \in [n]^k ,\ K \in [n \setminus k]^k ,\ D \subseteq K \setminus J \})$$
$$= \#(\{ (J,K) : J \in [n]^k ,\ K \in [n \setminus k]^k ,\ D \subseteq J \}) .$$

But if $\#(D) = i$ then these quantities are, respectively,

$$\binom{n-i}{k}\binom{n-k-i}{k-i} = \frac{(n-i)!(n-k-i)!}{k!(n-k-i)!(k-i)!(n-2k)!}$$

and

$$\binom{n-i}{k-i}\binom{n-k}{k} = \frac{(n-i)!(n-k)!}{(n-k)!(k-i)!(n-2k)!k!}$$

which of course are equal. So $p' = q'$ and $p = q$ and (a) is proved.

(b) Now (b) follows at once on multiplying both sides of (a) by $\binom{n-k}{s-k}/\binom{n}{s}\binom{s}{k}$ because

$$\binom{n-i}{k-i}\binom{n-k}{s-k}/\binom{n}{s}\binom{s}{k} = \frac{(n-i)!(n-k)!s!(n-s)!k!(s-k)!}{(k-i)!(n-k)!(s-k)!(n-s)!n!s!} = \binom{k}{i}/\binom{n}{i} .$$

12. Theorem If $1 \le k \le r \le s \le s+k \le n \in \mathbb{N}$ then

$$T^*(k,r,m) \le \frac{\binom{n-k}{s-k}T^*(k,r,s) - \binom{n-2k}{s-k}}{\binom{n-k}{s-k}^2/\binom{n}{s} - \binom{n-2k}{s-k}/\binom{n}{k}} .$$

proof Take $\mathcal{S} \subseteq [n]^k$; set $m = \#(\mathcal{S})$ and for $H \in [n]^s$ set $X_H = \#\{ S : H \supseteq S \in \mathcal{S}\}$. For $i \le k$ set

$$a_i = \#(\{ (I,J) : I , J \in \mathcal{S} , \#(I \cap J) = i \}),$$
$$b_i = \#(\{ (L,I,J) : I , J \in \mathcal{S} , L \in [I \cap J]^i \}) .$$

Then

$$b_i = \sum_{j=i}^{k} \binom{j}{i} a_j = \sum_{j=i}^{k} \binom{j}{j-i} a_i ,$$

so

$$\sum_{i=0}^{k} \binom{n-2k}{i+s-2k} b_i = \sum_{i \le j \le k} \binom{n-2k}{i+s-2k}\binom{j}{i} a_j = \sum_{j \le r} \Big(\sum_{i \le j} \binom{n-2k}{n-s-i}\binom{j}{i} \Big) a_j$$
$$= \sum_{j \le r} \binom{n-2k+j}{n-s} a_j$$
$$= \#(\{(I,J,H) : I , J \in \mathcal{S} , I \cup J \subseteq H \in [n]^s \})$$
$$= \sum_{H \in [n]^s} X_H^2 .$$

Now $b_k = m$ and for $i < k$

$$b_i = \sum_{L \in [n]^i} (\#(\{ I : L \subseteq I \in \mathcal{S} \}))^2$$
$$\ge \Big(\sum_{L \in [n]^i} \#\{ I : L \subseteq I \in \mathcal{S} \}\Big)^2/\binom{n}{i} \quad \text{(by Cauchy's inequality)}$$
$$= (\#\{ (L,I) : I \in \mathcal{S} , L \in [I]^i \})^2/\binom{n}{i}$$
$$= m^2\binom{k}{i}^2/\binom{n}{i} .$$

So

$$\sum_{H\in[n]^s} X_H^2 \;\geq\; \sum_{i=0}^{k-1} \binom{n-2k}{i+s-2k} m^2 \binom{k}{i}^2 \Big/ \binom{n}{i} \;+\; m\binom{n-2k}{s-k} \;.$$

Now suppose that $[K]^k \not\subseteq \mathcal{S}$ for every $K \in [n]^r$. In this case $\#(\mathcal{S} \cap [L]^k) \leq T^*(k,r,s)$ for every $L \in [n]^s$ i.e. $X_H \leq T^*(k,r,s)$ for every $H \in [n]^s$. But in thais case

$$\begin{aligned} \sum_{H\in[n]^s} X_H^2 &\leq T^*(k,r,s) \sum_{H\in[n]^s} X_H \\ &= T^*(k,r,s)\ \#\{\ (I,H)\ :\ I \in \mathcal{S}\ ,\ I \subseteq H \in [n]^s\ \} \\ &= T^*(k,r,s)\, m \binom{n-k}{s-k} \;. \end{aligned}$$

So

$$\begin{aligned} T^*(k,r,s)\ \binom{n-k}{s-k} &\geq \binom{n-2k}{s-k} + \sum_{i=0}^{k-1} \binom{n-2k}{i+s-2k} m \binom{k}{i}^2 \Big/ \binom{n}{i} \\ &= \binom{n-2k}{s-k} + m\binom{n-k}{s-k}^2 \Big/ \binom{n}{s} - m\binom{n-2k}{s-k} \Big/ \binom{n}{k} \end{aligned}$$

by Lemma 11(b). Turning this into an inequality of the form $m \leq \ldots$ we have the required result.

<u>Remark</u> This is taken directly from [3] .

13. <u>Corollary</u> If $1 \leq k \leq r \leq \ \leq s \in \mathbb{N}$ and

$$u > \frac{T^*(k,r,s) - 1}{\binom{s}{k} - 1}$$

then $(\omega,u) \Rightarrow [r]^k$.

<u>Proof</u> An elementary computation shows that

$$\lim_{n\to\infty} \frac{\binom{n-k}{s-k} T^*(k,r,s) - \binom{n-2k}{s-k}}{\binom{n-k}{s-k}^2 \Big/ \binom{n}{s} - \binom{n-2k}{s-k} \Big/ \binom{n}{k}} \cdot \binom{n}{k} = \frac{T^*(k,r,s) - 1}{\binom{s}{k} - 1}$$

so there is some n such that $u > T^*(k,r,n)/\binom{n}{k}$ and $(n,u) \Rightarrow [r]^k$, so of course $(\omega,u) \Rightarrow [r]^k$.

14. <u>Corollary</u> If $u > \frac{7}{11}$ then $(\omega,u) \Rightarrow [4]^3$.

<u>proof</u> As reported in [3], $T^*(3,4,8) = 36$.

<u>Remark added 16.12.88</u> My own calculations confirm that $T^*(3,4,8) = 36$.

<u>Reference</u> [3] D. de Caen, "A note on the probabilistic approach to Turan's problem", J. Combinatorial Theory (b) 34 (1983) 340-349.