University of Essex

Department of Mathematics
Wivenhoe Park
Colchester CO4 3SQ

Tel: Colchester 44144 (STD Code 020 6)
Telegraphic address: University Colchester
Telex: 98440 (UNILIB COLCHSTR)

from D. H. Fremlin

Comparing σ-ideals — Note of 3.3.86

1. Definitions (a) Let P be a partially ordered set. Write

$$\mathrm{add}(P) = \inf\{\#(A) : A \subseteq P,\ A \text{ has no upper bound in } P\},$$

$$\mathrm{cf}(P) = \inf\{\#(Q) : Q \subseteq P \text{ is cofinal with } P\},$$

$$\mathrm{ci}(P) = \inf\{\#(Q) : Q \subseteq P \text{ is coinitial with } P\}$$

(taking $\inf\emptyset = \infty$ if need be).

(b) If X, Y are sets and $\mathcal{I}$, $\mathcal{J}$ are ideals of $\mathcal{P}X$ and $\mathcal{P}Y$ respectively, say that $(X,\mathcal{I}) \le (Y,\mathcal{J})$ if there is a set $S \subseteq X \times Y$ such that

$$\mathcal{I} = \{I : I \subseteq X,\ S[I] \in \mathcal{J}\},$$

where $S[I] = \{y : \exists\ x \in I,\ (x,y) \in S\}$.

2. Proposition $\le$ is transitive & reflexive.

3. Proposition If $(X,\mathcal{I}) \le (Y,\mathcal{J})$ then

$$\mathrm{add}(\mathcal{J}) \le \mathrm{add}(\mathcal{I}), \quad \mathrm{cf}(\mathcal{I}) \le \mathrm{cf}(\mathcal{J}).$$

proof If $S \subseteq X \times Y$ witnesses that $(X,\mathcal{I}) \le (Y,\mathcal{J})$ and $\mathcal{A} \subseteq \mathcal{I}$ witnesses that $\mathrm{add}(\mathcal{I}) \le \kappa$, then $\{S[E] : E \in \mathcal{A}\}$ witnesses that $\mathrm{add}(\mathcal{J}) \le \kappa$; if $\mathcal{K} \subseteq \mathcal{J}$ witnesses that $\mathrm{cf}(\mathcal{J}) \le \kappa$ then $\{I_K : K \in \mathcal{K}\}$ witnesses that $\mathrm{cf}(\mathcal{I}) \le \kappa$, where

$$I_K = \{x : x \in X,\ S[\{x\}] \subseteq K\} = \bigcup\{I : I \subseteq X,\ S[I] \subseteq K\}$$
$$= X \setminus S^{-1}[Y \setminus K].$$

4. <u>Proposition</u> If $\operatorname{add}(\mathcal{I}) = \operatorname{cf}(\mathcal{I}) = \operatorname{add}(\mathcal{J})$, then $(X,\mathcal{I}) \le (Y,\mathcal{J})$.

<u>proof</u> Set $\kappa = \operatorname{cf}(\mathcal{I})$ and let $\langle E_\xi\rangle_{\xi<\kappa}$ enumerate a cofinal subset of $\mathcal{I}$. Set $F_\xi = \bigcup_{\eta\le\xi} E_\eta$ for $\xi<\kappa$, so that $\langle F_\xi\rangle_{\xi<\kappa}$ is an increasing family in $\mathcal{I}$ (as $\operatorname{add}(\mathcal{I}) = \kappa \ge \omega$). Because $\operatorname{add}(\mathcal{J}) = \kappa$ there is a family $\langle G_\xi\rangle_{\xi<\kappa}$ in $\mathcal{J}$ such that $\bigcup_{\xi<\kappa} G_\xi \notin \mathcal{J}$. Set $H_\xi = \bigcup_{\eta<\xi} G_\eta \in \mathcal{J}$ for $\eta < \xi$. Now try

$$S = \bigcup_{\xi<\kappa} (X\setminus F_\xi)\times H_\xi \subseteq X \times Y.$$

Then we see easily that

$$S[E_\xi] \subseteq H_\xi \in \mathcal{J}\ \forall \xi<\kappa,$$

so that $S[E] \in \mathcal{J}$ for every $E \in \mathcal{I}$, while if $E \subseteq X$ and $E \notin \mathcal{I}$ then $E \not\subseteq F_\xi$ for every $\xi<\kappa$ and $S[E] = \bigcup_{\xi<\kappa} H_\xi \notin \mathcal{J}$.

5. <u>Examples</u> (<u>a</u>) If κ is a regular uncountable cardinal and NS_κ is the non-stationary ideal of κ, then

$$\operatorname{add}([\kappa]^{<\kappa}) = \operatorname{cf}([\kappa]^{<\kappa}) = \operatorname{add}(NS_\kappa) = \kappa < \operatorname{cf}(NS_\kappa),$$

so that $(\kappa,[\kappa]^{<\kappa}) \le (\kappa, NS_\kappa)$ but $(\kappa, NS_\kappa) \not\le (\kappa,[\kappa]^{<\kappa})$.

(<u>b</u>) $[\mathfrak{m} = \mathfrak{c}]$ If $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meagre and Lebesgue negligible subsets of $\mathbb{R}$, respectively, then $(\mathbb{R},\mathcal{M})$ and $(\mathbb{R},\mathcal{N})$ are isomorphic and $\operatorname{add}(\mathcal{M}) = \operatorname{add}(\mathcal{N}) = \operatorname{cf}(\mathcal{M}) = \operatorname{cf}(\mathcal{N}) = \mathfrak{c}$, so that $(\mathbb{R},\mathcal{M})$ and $(\mathbb{R},\mathcal{N})$ are equivalent to $(\mathfrak{c},[\mathfrak{c}]^{<\mathfrak{c}})$.

6. Lemma If $\langle A_\xi \rangle_{\xi < \omega_1}$ is a family of uncountable subsets of ω_1, there is a non-stationary set $A \subseteq \omega_1$ such that $A \cap A_\xi \neq \emptyset$ for every $\xi < \omega_1$.

proof Set $F = \{ \alpha : \alpha \in A'_\xi \ \forall\, \xi < \alpha \}$, where A'_ξ is the set of cluster points of A in the order topology of ω_1. Set $A = \omega_1 \setminus F'$.

7. Theorem If you add ω_2 random reals to a model of ZFC + GCH, you get a model in which

(a) $(X, \mathcal{I}) \le (\mathbb{R}, \mathcal{M})$ whenever $\mathcal{I}$ is a σ-ideal in $\mathcal{P}X$ and $\mathrm{cf}(\mathcal{I})$, $\mathrm{ci}(\mathcal{P}X \setminus \mathcal{I})$ are both $\le \mathfrak{c}$;

(b) $(\omega_1, NS_{\omega_1}) \le (\mathbb{R}, \mathcal{M})$;

(c) $(\mathbb{R}, \mathcal{N}) \le (\mathbb{R}, \mathcal{M})$;

(d) $(\mathbb{R}, \mathcal{M}) \not\le (\omega_1, \mathcal{I})$ for any ideal $\mathcal{I}$ of $\mathcal{P}\omega_1$ (in particular, $(\mathbb{R}, \mathcal{M}) \not\le (\omega_1, NS_{\omega_1})$);

(e) $(\omega_1, NS_{\omega_1}) \not\le (\mathbb{R}, \mathcal{N})$ (so that $(\mathbb{R}, \mathcal{M}) \not\le (\mathbb{R}, \mathcal{N})$).

proof We need to know the following facts about this kind of random-real model.

(A) $\mathfrak{c} = 2^{\omega_1}$.

(B) $\exists\ A \subseteq \mathbb{R}$, $\#(A) = \mathfrak{c}$, such that no uncountable subset of A belongs to $\mathcal{N}$. (Kunen 84, Theorem 3.18.)

(C) If $A \subseteq \mathbb{R}$ and $\#(A) \le \omega_1$ then $A \in \mathcal{M}$. (Kunen 84, 3.19.)

(D) If $A \subseteq \mathbb{R}$ and $A \notin \mathcal{N}$, then there is an $A' \subseteq A$ such that $\#(A') = \omega_1$ and $A' \notin \mathcal{N}$.

(E) If $A \subseteq \mathbb{R}$, $\#(A) = \omega_1$ and $A \notin \mathcal{N}$, then $\mathrm{cf}(\mathcal{N} \cap \mathcal{P}A) = \omega_1$.

Now for the main argument.

(a) Let $A \subseteq \mathbb{R}$ be a set of cardinal $\mathfrak{c}$ such that every uncountable subset of A is non-negligible (fact B). Fix on a meagre conegligible set $H \subseteq \mathbb{R}$ and set $E_a = a + H$ for $a \in A$. Then every E_a is meagre, while if $B \subseteq A$ is uncountable then $\bigcup_{a \in B} E_a = B + H = \mathbb{R}$, because B must meet the conegligible set $x - H$ for every $x \in \mathbb{R}$. Next, let $\mathcal{K} \subseteq \mathcal{J}$ and $\mathcal{V} \subseteq \mathcal{P}X \setminus \mathcal{J}$ be cofinal and coinitial, respectively, and of cardinal $\leq \mathfrak{c}$. Let $\varphi : \mathcal{K} \to A$ be any injection, and $\langle G_V \rangle_{V \in \mathcal{V}}$ a partition of $\mathbb{R}$ into non-meagre sets. (Recall that ZFC implies that there is a partition of $\mathbb{R}$ into $\mathfrak{c}$ non-meagre sets.) Choose any function $f : \mathbb{R} \to X$ such that

$$f(x) \in V \setminus \bigcup \{ K : K \in \mathcal{K},\ x \notin E_{\varphi(K)} \} \quad \text{if } V \in \mathcal{V} \text{ and } x \in G_V ;$$

there is such a function because $\{ K : x \notin E_{\varphi(K)} \}$ is always countable, so cannot cover V. Set $S = \{ (f(x), x) : x \in \mathbb{R} \} \subseteq X \times \mathbb{R}$. Then

$$S[K] = f^{-1}[K] \subseteq E_{\varphi(K)} \in \mathcal{M} \quad \forall\ K \in \mathcal{K},$$

$$S[V] = f^{-1}[V] \subseteq G_V \notin \mathcal{M} \quad \forall\ V \in \mathcal{V},$$

so S witnesses that $(X, \mathcal{J}) \leq (\mathbb{R}, \mathcal{M})$.

(b) Follows from (a), because $2^{\omega_1} = \mathfrak{c}$ (fact A).

(c) Also follows from (a), because $\mathrm{cf}(\mathcal{N}) \leq \mathfrak{c}$ (in any model of ZFC), while in our present model, $\mathrm{ci}(\mathcal{P}\mathbb{R} \setminus \mathcal{N}) \leq \#([\mathbb{R}]^{\omega_1}) = 2^{\omega_1} = \mathfrak{c}$, by fact D.

(d) Now if $S \subseteq \mathbb{R} \times \omega_1$ ~~and $S[\mathbb{R}]$ is stationary~~ there is a $B \in [\mathbb{R}]^{\omega_1}$ such that $S[B] = S[\mathbb{R}]$; since $B \in \mathcal{M}$ (fact C) and $\mathbb{R} \notin \mathcal{M}$, S cannot witness that $(\mathbb{R}, \mathcal{M}) \leq (\omega_1, \mathcal{J})$.

(e) Finally, ? suppose, if possible, that $S \subseteq \omega_1 \times \underset{\sim}{R}$ is such that, for $I \subseteq \omega_1$, $S[I] \in \mathcal{N}$ iff $I \in NS_{\omega_1}$. Then $S[\omega_1] \notin \mathcal{N}$ so (by fact D) there is an $A \subseteq S[\omega_1]$ such that $\#(A) = \omega_1$ and $A \notin \mathcal{N}$. Now $\mathrm{cf}(\mathcal{N} \cap \mathcal{P}A) = \omega_1$ (fact E); let $\langle A_\xi \rangle_{\xi < \omega_1}$ enumerated a cofinal subset of $\mathcal{N} \cap \mathcal{P}A$. Set $D_\xi = S^{-1}[A \setminus A_\xi]$ for each $\xi < \omega_1$. Then $S[D_\xi] \supseteq A \setminus A_\xi \notin \mathcal{N}$ so $D_\xi \notin NS_{\omega_1}$ and D_ξ is uncountable, for each $\xi < \omega_1$. Let $D \subseteq \omega_1$ be a non-stationary set such that $D \cap D_\xi \neq \emptyset$ for every ξ (Lemma 6). Then $S[D]$ meets $A \setminus A_\xi$ for every $\xi < \omega_1$, so $S[D] \cap A \notin \mathcal{N}$ and $S[D] \notin \mathcal{N}$.

So $(\omega_1, NS_{\omega_1}) \not\leq (\underset{\sim}{R}, \mathcal{N})$.

~~$\Vdash_{\mathbb{P}}$ " $A \in NS_{\omega_1}$ & $S[A] \notin \mathcal{N}$ ".~~

~~So $(\omega_1, NS_{\omega_1}) \not\leq (\mathbb{R}, \mathcal{N})$~~ .

8. <u>Remarks</u> (<u>a</u>) Theorem 7a is a version of a result of J.Cichoń, building on results of myself & M.Burke.

(<u>b</u>) I learnt fact D from H.Woodin.

(<u>c</u>) Using Cohen reals instead of random reals, we get the corresponding results with $\mathcal{M}$ and $\mathcal{N}$ interchanged.

(<u>d</u>) What the proof of 7e really shows is that (in this model) if $S \subseteq \omega_1 \times \mathbb{R}$ and $S[\omega_1] \notin \mathcal{N}$, there is an $A \in NS_{\omega_1}$ such that $S[A] \notin \mathcal{N}$.

9. <u>Problem</u> From 7b,7d and §5 we ~~real result, we~~ see that the following are both consistent with ZFC:

$$(\mathbb{R}, \mathcal{M}) \leq (\omega_1, NS_{\omega_1}) \not\leq (\mathbb{R}, \mathcal{M}) ,$$

$$(\omega_1, NS_{\omega_1}) \leq (\mathbb{R}, \mathcal{M}) \not\leq (\omega_1, NS_{\omega_1}) .$$

The question arises: is it consistent to suppose that $(\mathbb{R}, \mathcal{M})$ and $(\omega_1, NS_{\omega_1})$ are equivalent? or $(\mathbb{R}, \mathcal{N})$ and $(\omega_1, NS_{\omega_1})$?

I give a proposition related to this.

10. <u>Lemma</u> If $S \subseteq \omega_1^2$ is such that, for $A \subseteq \omega_1$,

$$S[A] \in NS_{\omega_1} \quad \text{iff} \quad A \in NS_{\omega_1} ,$$

then there is a closed unbounded $C \subseteq \omega_1$ such that

$$S^{-1}[\{\xi\}] = \{\xi\} \ \forall \ \xi \in C .$$

<u>proof</u> (<u>a</u>) Set

$$A = \{ \xi : \exists \eta < \xi , (\eta,\xi) \in S \} .$$

? If $A \notin NS_{\omega_1}$ then by the pressing-down lemma there is an $\eta < \omega_1$ such that

$$\{ \zeta : \zeta \in A , (\eta,\zeta) \in S \} \notin NS_{\omega_1} ;$$

but now $S[\{\eta\}] \notin NS_{\omega_1}$. X

(<u>b</u>) Set

$$B = \{ \xi : \exists \eta > \xi , (\eta,\xi) \in S \} .$$

For each $\xi \in B$ choose $f(\xi) > \xi$ such that $(f(\xi),\xi) \in S$. Set

$$F = \{ \zeta : \zeta < \omega_1 , f(\xi) < \zeta \ \forall \xi \in \zeta \cap B \} ;$$

then F is a closed unbounded sets and $S[\omega_1 \setminus F] \supseteq B$, so $B \in NS_{\omega_1}$.

(<u>c</u>) Set

$$D = \{ \xi : \xi < \omega_1 , (\xi,\xi) \notin S \} .$$

Consdier $S[D] \setminus (A \cup B)$. ? If $\xi \in S[D] \setminus (A \cup B)$, there is an $\eta \in D$ such that $(\eta,\xi) \in S$; as $\xi \notin A \cup B$, $\eta = \xi$; so $\xi \in D$ and $(\xi,\xi) \in S$. X So $S[D] \subseteq A \cup B \in NS_{\omega_1}$ and $D \in NS_{\omega_1}$.

(<u>d</u>) Now take any closed unbounded $C \subseteq \omega_1 \setminus (A \cup B \cup D)$.

11. <u>Proposition</u> If $(X,\mathcal{J}) \leq (\omega_1, NS_{\omega_1}) \leq (X,\mathcal{J})$, there is a $Y \subseteq X$ such that $(Y, \mathcal{J} \cap \mathcal{P}Y) \cong (\omega_1, NS_{\omega_1})$.

<u>proof</u> We have $S \subseteq X \times \omega_1$, $T \subseteq \omega_1 \times X$ such that, for $I \subseteq X$ and $J \subseteq \omega_1$,

$$S[I] \in NS_{\omega_1} \quad \text{iff} \quad I \in \mathcal{J} ,$$

$$T[J] \in \mathcal{J} \quad \text{iff} \quad J \in NS_{\omega_1} .$$

Let U be the composition $S \circ T \subseteq \omega_1^2$. Then $U[J] \in NS_{\omega_1}$ ~~xxxthen~~ iff $J \in NS_{\omega_1}$, so by Lemma 10 there is a closed unbounded set $C \subseteq \omega_1$ such that, for every $\xi \in C$,

$$T^{-1}[S^{-1}[\{\xi\}]] = U^{-1}[\{\xi\}] = \{\xi\} .$$

For $\xi \in C$ choose $h(\xi) \in X$ such that $(\xi, h(\xi)) \in T$ and $(h(\xi), \xi) \in S$; such exists because $(\xi,\xi) \in S \circ T$. If ξ, η are distinct members of C then

$$h(\eta) \in S^{-1}[\{\eta\}] , \quad \xi \notin T^{-1}[S^{-1}[\{\eta\}]]$$

so $(\xi, h(\eta)) \notin T$ and $h(\xi) \neq h(\eta)$. Thus $h : C \to X$ is injective.

If $J \subseteq C$ and $J \in NS_{\omega_1}$ then ~~$S[h[J]] \subseteq$~~ $h[J] \subseteq T[J] \in \mathcal{J}$. If $J \subseteq C$ and $J \notin NS_{\omega_1}$ then $S[h[J]] \supseteq J$ so $h[J] \notin \mathcal{J}$. Thus, taking $Y = h[C]$,

$$(Y, \mathcal{J} \cap \mathcal{P}Y) \cong (C, NS_{\omega_1} \cap \mathcal{P}C) .$$

But of course ~~any~~ the strictly increasing enumeartion of C witnesses that

$$(C, \mathcal{P}C \cap NS_{\omega_1}) \cong (\omega_1, NS_{\omega_1}) .$$

12. <u>Remarks</u> Thus Problem 9 is related to the question, of independent interest:

is it consistent to assume that there is a subset Y of $\underset{\sim}{R}$ such that $(Y, \mathcal{M} \cap \mathcal{P}Y) \cong (\omega_1, NS_{\omega_1})$?

(& similarly for~~m~~ $\mathcal{N}$). Bringing the problem back to ω_1, we can ask:

is it consistent to assume that there is a countably-generated σ-subalgebra ~~of~~ Σ of $\mathcal{P}\omega_1$ such that (i) $\mathcal{J} = \Sigma \cap NS_{\omega_1}$ is cofinal with NS_{ω_1} (ii) the algebra $\Sigma/\mathcal{J}$ is ccc?

13. <u>Yet another question</u> In Theorem 7, it is left open whether $(\underset{\sim}{R},\mathcal{N}) \leq (\omega_1, NS_{\omega_1})$. If instead of adding ω_2 random reals we add ω_3 random reals, then we can answer this as follows, because of the following. First, another fact for the list of Theorem 7:

(F) $cf(NS_{\omega_1}) = \omega_2$.

Now add another result to (a)-(e):

(f) $(\mathfrak{c},[\mathfrak{c}]^{\leq\omega}) \leq (\underset{\sim}{R},\mathcal{N})$ so $cf(\mathcal{N}) = \mathfrak{c}$ and (if $\mathfrak{c} > \omega_2$) $(\underset{\sim}{R},\mathcal{N}) \not\leq (\omega_1, NS_{\omega_1})$.

P Take the set A of fact B and enumerate it as $\{a_\xi : \xi < \mathfrak{c}\}$. Set $S = \{(\xi, a_\xi) : \xi < \mathfrak{c}\}$; then, for $I \subseteq \mathfrak{c}$, $S[I] \in \mathcal{N}$ iff I is countable. Consequently $cf(\mathcal{N}) \geq cf([\mathfrak{c}]^{\leq\omega})$ (Prop. 3) $= \mathfrak{c}$ (because $cf(\mathfrak{c}) > \omega$), and of course $cf(\mathcal{N}) \leq \mathfrak{c}$. On the other hand, $cf(NS_{\omega_1}) = \omega_2 < \mathfrak{c}$, so $(\underset{\sim}{R},\mathcal{N}) \not\leq (\omega_1, NS_{\omega_1})$ by Prop. 3 again.

Q

I do not know what happens when $\mathfrak{c} = \omega_2$. Indeed, I do not know whether (in ZFC) $(\omega_2,[\omega_2]^{\leq\omega}) \leq (\omega_1, NS_{\omega_1})$. There is a simple combinatorial characterization of this question, since for any $(X,\mathcal{J})$ such that $\mathcal{J}$ is a σ-ideal we can see that $(\kappa,[\kappa]^{\leq\omega}) \leq (X,\mathcal{J})$ iff there is a family $\langle A_\xi \rangle_{\xi<\kappa}$ in $\mathcal{J}$ such that $\bigcup_{\xi\in I} A_\xi \notin \mathcal{J}$ for every uncountable $I \subseteq \kappa$. So I ask: is there such a family when $\kappa = \omega_2$, $\mathcal{J} = NS_{\omega_1}$?

14. <u>Prime ideals</u> The arguments above are designed for the special ideals $\mathcal{M}$, $\mathcal{N}$ and NS_{ω_1}. However the basic relation $\leq$ is of interest in other contexts. In particular, there is a simplification in the context of prime ideals, as follows. If $\mathcal{J} \triangleleft \mathcal{P}X$ is a prime ideal and $(X,\mathcal{J}) \leq (Y,\mathcal{J})$ then there are a $Z \subseteq Y$ and an $f : Z \to X$ such that $\mathcal{J} =$

$\{ I : I \subseteq X , f^{-1}[I] \in \mathfrak{J} \}$. **P** Let $S \subseteq X \times Y$ be such that $\mathfrak{I} = \{ I : I \subseteq X , S[I] \in \mathfrak{J} \}$. Then $S[X] \notin \mathfrak{J}$. Set $Z = S[X]$ and let $f : Z \to X$ be any selector for S^{-1} . If $I \in \mathfrak{I}$ then $f^{-1}[I] \subseteq S[I] \in \mathfrak{J}$. If $I \subseteq X$ and $I \notin \mathfrak{I}$ then $X \setminus I \in \mathfrak{I}$ so $J = f^{-1}[X \setminus I] \in \mathfrak{J}$ and

$$f^{-1}[I] = f^{-1}[X] \setminus J = Z \setminus J \notin \mathfrak{J} .$$

So Z , f serve. **Q**

Accordingly, for prime ideals, $\leq$ corresponds to the Rudin-Keisler ordering of ultrafilters.

15. <u>Alternative relations</u> ~~The partial order ≤ is chosen to be~~ I fixed on $\leq$ as the largest straightforward relation for which it seemed natural to say that if $(X, \mathfrak{I}) \leq (Y, \mathfrak{J})$ then $\mathfrak{I}$ is derivable from $\mathfrak{J}$. There are several smaller relations of interest e.g.

$(X, \mathfrak{I}) \leq_1 (Y, \mathfrak{J})$ if $\exists$ $f : X \to Y$ such that
$$\mathfrak{I} = \{ I : I \subseteq X , f[I] \in \mathfrak{J} \} ,$$
$(X, \mathfrak{I}) \leq_2 (Y, \mathfrak{J})$ if $\exists$ $f : Y \to X$ such that
$$\mathfrak{I} = \{ I : I \subseteq X , f^{-1}[I] \in \mathfrak{J} \}$$

(cf. §14). There is also a larger relation given by

$(X, \mathfrak{I}) \leq^* (Y, \mathfrak{J})$ if there are functions $I \mapsto G_I : \mathfrak{I} \to \mathfrak{J}$, $J \mapsto H_J : \mathfrak{J} \to \mathfrak{I}$ such that $I \subseteq H_J$ whenever $G_I \subseteq J$.

(If $(X, \mathfrak{I}) \leq (Y, \mathfrak{J})$ then $(X, \mathfrak{I}) \leq^* (Y, \mathfrak{J})$ because we can take $G_I = S[I]$, $H_J = X \setminus S^{-1}[Y \setminus J]$.) The point of $\leq^*$ is the Bartoszyński-Raisonnier-Stern result that (in ZFC) $(\mathbb{R}, \mathcal{M}) \leq^* (\mathbb{R}, \mathcal{N})$ (see Fremlin 85a).

We can see that of the arguments above, Prop. 3 applies to $\leq^*$, while Theorem 7a-c applies to $\leq_2$.

16. <u>Note added 7.5.86</u> M. Burke has referred me to Abraham & Shelah 86, where the question "is $(\omega_2,[\omega_2]^{\le\omega}) \le (\omega_1,\mathrm{NS}_{\omega_1})$?" is treated. They remark that F.Galvin proved that if $\kappa > \omega$ and $2^\lambda \le \kappa$ for every $\lambda < \kappa$, then $(\kappa^+,[\kappa^+]^{\le\kappa}) \not\le (\kappa,\mathrm{NS}_\kappa)$; so that, in particular, the continuum hypothesis implies that $(\omega_2,[\omega_2]^{\le\omega}) \not\le (\omega_1,\mathrm{NS}_{\omega_1})$. Abraham and Shelah also describe models in which $(\omega_2,[\omega_2]^{\le\omega}) \le (\omega_1,\mathrm{NS}_{\omega_1})$; one of them allows $\mathfrak{m} = \mathfrak{c}$.

A proof of Galvin's result is given in Baumgartner Hajnal & Maté 75. It is easy to see that the theorem

$$(\omega_2,[\omega_2]^{\le\omega}) \not\le (\omega_1,\mathrm{NS}_{\omega_1})$$

is unaffected by ccc forcing; so that, in particular, it is true in the random-real models considered in Theorem 7 above. So in all these models we have $(\underset{\sim}{\mathrm{R}},\mathcal{N}) \not\le (\omega_1,\mathrm{NS}_{\omega_1})$.

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