I introduce a class of completely regular Hausdorff spaces which form a common generalization of the (completely regular) K-analytic spaces and the "absolutely analytic" metric spaces, in much the same way that the "Čech-complete" spaces form a common generalization of the locally compact Hausdorff spaces and the complete metric spaces. This was originally designed to provide a formulation of Theorem 12 below.

I have not attempted to discuss spaces which are not completely regular and Hausdorff. Possibly Proposition 7 offers a definition of a more general type of Čech-analytic space.

1. Notation Write $\mathbb{N}^{(\mathbb{N})}$ for $\bigcup_{k\in\mathbb{N}}\mathbb{N}^k$, the set of all finite sequences in $\mathbb{N}$ (including the empty sequence $\emptyset$). For $\sigma\in\mathbb{N}^{(\mathbb{N})}$ write $\#(\sigma)$ for the number of terms in σ (which is also the domain of σ, if we identify $\mathbb{N}$ with the set of finite ordinals). If $\sigma, \tau \in \mathbb{N}^{(\mathbb{N})}$ and $\alpha\in\mathbb{N}^{\mathbb{N}}$, write $\sigma\subseteq\tau\subseteq\alpha$ to mean that α is an extension of τ and that τ is an extension of σ. If $\alpha\in\mathbb{N}^{\mathbb{N}}$, $\sigma\in\mathbb{N}^{(\mathbb{N})}$, $n\in\mathbb{N}$ write $\alpha|n = \langle\alpha(i)\rangle_{i<n}\in\mathbb{N}^{(\mathbb{N})}$, $\sigma|n = \langle\sigma(i)\rangle_{i<\min(n,\#(\sigma))}\in\mathbb{N}^{(\mathbb{N})}$, $\sigma^\frown n = \sigma\cup(\#(\sigma),n)\in\mathbb{N}^{(\mathbb{N})}$.

2. Definition A completely regular Hausdorff space A is Čech-analytic if there is a compact Hausdorff space Z and a set $R\subseteq\mathbb{N}^{\mathbb{N}}\times Z$ such that R is expressible as the intersection of a closed set with a G_δ set and A is homeomorphic to $\pi_2[R]\subseteq Z$.

3. Proposition If X is a compact Hausdorff space and $A\subseteq X$ is a Čech-analytic set, then there is a set $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ such that R is expressible as the intersection of a closed set with a G_δ set and $A=\pi_2[R]$.

proof We know that there is a compact Hausdorff space Z, a set $S\subseteq\mathbb{N}^{\mathbb{N}}\times Z$ which is expressible as the intersection of a closed set with a G_δ set, and a homeomorphism $f: A\to\pi_2[S]$. Suppose that $S = V\cap\bigcap_{n\in\mathbb{N}}W_n$ where V is closed and each W_n is open. Set

$$T = \overline{\{(\alpha,t) : \alpha \in \mathbb{N}^{\mathbb{N}},\ t \in A,\ (\alpha,f(t)) \in S\}} \subseteq \mathbb{N}^{\mathbb{N}} \times X$$

$$U_n = \bigcup\{U : U \subseteq \mathbb{N}^{\mathbb{N}} \times X \text{ open},\ \exists \text{ open } W \subseteq \mathbb{N}^{\mathbb{N}} \times Z,\ \overline{W} \subseteq W_n,\ f_1^{-1}[W] = U \cap (\mathbb{N}^{\mathbb{N}} \times A)\}$$

where $f_1(\alpha,t) = (\alpha,f(t))$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, $t \in A$. Set $R = T \cap \bigcap_{n\in\mathbb{N}} U_n$. I claim that $R = f_1^{-1}[S]$.

P (<u>i</u>) If $(\alpha,t) \in f_1^{-1}[S]$ then $(\alpha,t) \in \overline{f_1^{-1}[S]} = T$. Also, for each $n \in \mathbb{N}$, $(\alpha,f(t)) \in W_n$; as $\mathbb{N}^{\mathbb{N}} \times Z$ is regular, there is an open W such that $(\alpha,f(t)) \in W$ and $\overline{W} \subseteq W_n$. Now $(\alpha,t) \in f_1^{-1}[W]$ and f_1 is continuous for the induced topology on $\mathbb{N}^{\mathbb{N}} \times A$, so there is an open $U \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $f_1^{-1}[W] = U \cap (\mathbb{N}^{\mathbb{N}} \times A)$, and $(\alpha,t) \in U \subseteq U_n$. As n is arbitrary, $(\alpha,t) \in R$; as (α,t) is arbitrary, $f_1^{-1}[S] \subseteq R$.

(<u>ii</u>) If $(\alpha,t) \in R$, then for each $n \in \mathbb{N}$ we have $(\alpha,t) \in U_n$, so there are open $W'_n \subseteq \mathbb{N}^{\mathbb{N}} \times Z$, $U'_n \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $(\alpha,t) \in U'_n$, $f_1^{-1}[W'_n] = U'_n \cap (\mathbb{N}^{\mathbb{N}} \times A)$, and $\overline{W'_n} \subseteq W_n$. Now for each $n \in \mathbb{N}$, neighbourhood G of t in X, set

$$C^n_G = \{(\beta,f(u)) : \beta|n = \alpha|n,\ u \in A \cap G\} \cap \bigcap_{i\le n} W'_i \cap S,$$

$$D^n_G = \{(\beta,u) : \beta|n = \alpha|n,\ u \in G\} \cap \bigcap_{i\le n} U'_i \cap f_1^{-1}[S].$$

Then $C^n_G = f_1[D^n_G]$. Because D^n_G is the intersection of a neighbourhood of (α,t) with $f_1^{-1}[S]$, and $(\alpha,t) \in T$, no D^n_G is empty and no C^n_G is empty. Consequently there is a $v \in \bigcap_{n,G} \overline{\pi_2[C^n_G]} \subseteq Z$, and $(\alpha,v) \in \bigcap_{n,G} \overline{C^n_G}$. But now $(\alpha,v) \in \overline{S} \cap \bigcap_{n\in\mathbb{N}} \overline{W'_n} \subseteq V \cap \bigcap_{n\in\mathbb{N}} W_n = S$. So $v \in \pi_2[S] = f[A]$. As $v \in \overline{f[A \cap G]}$ for every neighbourhood G of t in X, and f is a homeomorphism between A and $f[A]$, we must have $v = f(t)$. So $(\alpha,f(t)) \in S$. As (α,t) is arbitrary, $R \subseteq f_1^{-1}[S]$. **Q**

Npw $A = f^{-1}[\pi_2[S]] = \pi_2[f_1^{-1}[S]] = \pi_2[R]$, and R is expressible as the intersection of a closed set with a G_δ set.

4. <u>Theorem</u> (<u>a</u>) The product of a countable family of Čech-analytic spaces is Čech-analytic.

(b) If X is a completely regular Hausdorff space, the set of Čech-analytic subspaces of X is closed under the Souslin operation.

(c) If A is a Čech-analytic space, every Borel subset of A is Čech-analytic.

(d) If X and Y are completely regular Hausdorff spaces of which Y is Souslin, and $A \subseteq X \times Y$ is Čech-analytic, then $\pi_1[A] \subseteq X$ is Čech-analytic.

<u>proof</u> (<u>a</u>) ~~Let $\langle A_n \rangle_{n \in N}$ be a sequence of Čech-analytic spaces (the case of a finite product~~ Let $\langle A_i \rangle_{i \in I}$ be a countable family of Cech-analytic spaces. For each $i \in I$ express A_i as $\pi_2[R_i]$ where R_i is a subset of $\underset{\sim}{N}^{\underset{\sim}{N}} \times Z_i$ expressible as the intersection of a closed set with a G_δ set, Z_i being a compact Hausdorff space. Then $\prod_{i \in I} A_i$ may be identified with $\pi_2[R]$ in $\prod_{i \in I} Z_i = Z$, where $R = \prod_{i \in I} R_i$ is regarded as a subset of $(\underset{\sim}{N}^{\underset{\sim}{N}})^I \times Z$. As R is also the intersection of a closed set with a G_δ set, $\prod_{i \in I} A_i$ is Čech-analytic.

(<u>b</u>) Let $\langle A_\sigma \rangle_{\sigma \in \underset{\sim}{N}^{(\underset{\sim}{N})}}$ be a family of Čech-analytic subsets of X. Embed X in its Stone-Čech compactification Z. Express each A_σ as $\pi_2[R_\sigma]$ where $R_\sigma \subseteq \underset{\sim}{N}^{\underset{\sim}{N}} \times Z$ is expressible as the intersection of a closed set and a G_δ set (this is possible by Proposition 3). In $\underset{\sim}{N}^{\underset{\sim}{N}} \times (\underset{\sim}{N}^{\underset{\sim}{N}})^{\underset{\sim}{N}^{(\underset{\sim}{N})}} \times Z$ set

$$R = \{ (\alpha, \langle \beta_\sigma \rangle_{\sigma \in \underset{\sim}{N}^{(\underset{\sim}{N})}}, t) : (\beta_{\alpha|n}, t) \in R_{\alpha|n} \ \forall \ n \geq 1 \}$$

$$= \bigcap_{n \geq 1} \bigcup_{\tau \in \underset{\sim}{N}^n} S_\tau$$

where $S_\tau = \{ (\alpha, \langle \beta_\sigma \rangle_{\sigma \in \underset{\sim}{N}^{(\underset{\sim}{N})}}, t) : \alpha|n = \tau \ , \ (\beta_\tau, t) \in R_\tau \}$. For each $\tau \in \underset{\sim}{N}^n$, S_τ is expressible as the intersection of a closed set with a G_δ set; as the S_τ all lie in distinct sets of a partition of $\underset{\sim}{N}^{\underset{\sim}{N}} \times (\underset{\sim}{N}^{\underset{\sim}{N}})^{\underset{\sim}{N}^{(\underset{\sim}{N})}} \times Z$ into open sets, $\bigcup_{\tau \in \underset{\sim}{N}^n} S_\tau$ is also expressible as the intersection of a closed set with a G_δ set, and R is expressible as the intersection of a closed set with a G_δ set. Also

$\pi_2[R] = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \geq 1} A_{\alpha|n} \subseteq X$ and $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{(\mathbb{N})}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, so $\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \geq 1} A_{\alpha|n}$ is Čech-analytic. As $\langle A_\sigma \rangle_{\sigma \in \mathbb{N}^{(\mathbb{N})}}$ is arbitrary, (b) is proved.

(c) By (b), $\{ B : B \subseteq A$, B and $A \setminus B$ are both Čech-analytic $\}$ is a σ-algebra of subsets of A . Clearly it contains all closed sets so must contain all Borel sets.

(d) Let Z , W be compact Hausdorff spaces with $X \subseteq Z$, $Y \subseteq W$. By Proposition 3, there is a set $R \subseteq \mathbb{N}^{\mathbb{N}} \times Z \times W$ such that R is expressible as the intersection of a closed set with a G_δ set and A is the projection of R onto $Z \times W$. Since Y is Souslin, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to Y$ (the case $Y = \emptyset$ is trivial). Now $S = \{ (\alpha,\beta,t) : \alpha , \beta \in \mathbb{N}^{\mathbb{N}} , t \in Z , (\alpha,t,f(\beta)) \in R \}$ is expressible as the intersection of a closed set with a G_δ set, so its projection in X is Čech-analytic; but this is just $\pi_1[A]$.

~~5. Corollary (a) If X is a compact Hausdorff space, a set $A \subseteq X$ is Cech-analytic iff it is the projection of a Borel set in $N^N \times X$.~~

5. Theorem Let X be a compact Hausdorff space, A a subset of X . Then the following are equivalent:

(i) A is Čech-analytic;

(ii) A is obtainable by the Souslin operation from $\{ E : E \subseteq X$ is either open or closed $\}$;

(iii) A is the projection of a Borel set in $\mathbb{N}^{\mathbb{N}} \times X$.

proof (a)(i)$\Rightarrow$(ii) Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be of the form $V \cap \bigcap_{n \in \mathbb{N}} W_n$, where V is closed and each W_n is open, with $\pi_2[R] = A$. For each $\sigma \in \mathbb{N}^{(\mathbb{N})}$ set $U =$ $\{ \alpha : \alpha \supseteq \sigma \} \subseteq \mathbb{N}^{\mathbb{N}}$. Set

$$G^n_\sigma = \bigcup\{ G : G \subseteq X \text{ open}, \ U_\sigma \times G \subseteq W_n \},$$

so that $W_n = \bigcup_{\sigma \in \underset{\sim}{N}^{(\underset{\sim}{N})}} U_\sigma \times G^n_\sigma$. Set $F_\sigma = \overline{\pi_2[V \cap (U_\sigma \times X)]}$. In $\underset{\sim}{N}^{\underset{\sim}{N}} \times \underset{\sim}{N}^{\underset{\sim}{N}} \times X$ consider

$$S = \{ (\alpha,\beta,t) : t \in F_{\alpha|n} \cap G^n_{\alpha|\beta(n)} \ \forall \ n \in \underset{\sim}{N} \} .$$

Let B be the projection of S in X ; it is easy to see that B is obtainable by the Souslin operation from $\{X,\emptyset\} \cup \{ F_\sigma : \sigma \in \underset{\sim}{N}^{(\underset{\sim}{N})} \} \cup \{ G^n_\sigma : n \in \underset{\sim}{N} , \sigma \in \underset{\sim}{N}^{(\underset{\sim}{N})} \}$. But also $A = B$. (i) If $t \in A$ then there is an $\alpha \in \underset{\sim}{N}^{\underset{\sim}{N}}$ such that $(\alpha,t) \in R$. We have $(\alpha,t) \in V \cap (U_{\alpha|n} \times X)$ so $t \in F_{\alpha|n}$ for every $n \in \underset{\sim}{N}$. Also, for each $n \in \underset{\sim}{N}$, $(\alpha,t) \in W_n$ so there is a $\sigma_n \subseteq \alpha$ such that $t \in G^n_{\sigma_n}$; set $\beta(n) = \#(\sigma_n)$; then $(\alpha,\beta,t) \in S$ and $t \in B$. (ii) If $t \in B$ then there are α , $\beta \in \underset{\sim}{N}^{\underset{\sim}{N}}$ such that $(\alpha,\beta,t) \in S$. Now $t \in F_{\alpha|n}$ for every $n \in \underset{\sim}{N}$ so that $(\alpha,t) \in V$ (because V is closed). Also, for each $n \in \underset{\sim}{N}$, $(\alpha,t) \in U_{\alpha|\beta(n)} \times G^n_{\alpha|\beta(n)} \subseteq W_n$. So $(\alpha,t) \in R$ and $t \in A$.

(b)(ii)$\Rightarrow$(i) Immediate from Theorem 4.

(c)(i)$\Rightarrow$(iii) by Proposition 3.

(d)(iii)$\Rightarrow$(i) by Theorem 4, parts (a), (c) and (d). (The point is that $\underset{\sim}{N}^{\underset{\sim}{N}}$ is a G_δ set in ~~its~~ some ~~compactification e.g. [0,1]~~ compactification, therefore Čech-analytic, so that $\underset{\sim}{N}^{\underset{\sim}{N}} \times X$ is Čech-analytic and any Borel set in $\underset{\sim}{N}^{\underset{\sim}{N}} \times X$ is Čech-analytic.)

5. Corollary (a) Every completely regular K-analytic space is Čech-analytic. (For if A is a completely regular K-analytic space, there is an usco-compact relation $R \subseteq \underset{\sim}{N}^{\underset{\sim}{N}} \times A$ with $\pi_2[R] = A$; now R is closed in $\underset{\sim}{N}^{\underset{\sim}{N}} \times \beta A$.)

(b) ~~A complete metric space is Čech-analytic~~ A Čech-complete space is Čech-analytic (being a G_δ subset of a compact Hausdorff space); in particular, a complete metric space is Čech-analytic.

(c) An absolutely analytic metric space is Čech-analytic (being an F-analytic subset of its completion). In fact we have the following:

6. Proposition A metric space X is Čech-analytic iff X is an F-analytic subset of its completion X .

proof If X is F-analytic in $\hat{X}$, it is Čech-analytic, by Theorem 4, since $\hat{X}$ is a G_δ set in its Stone-Čech compactification. If X is Čech-analytic, then embed $\hat{X}$ in its Stone-Čech compactification Z . We have $X = \pi_2[R]$ where $R \subseteq \mathbb{N}^{\mathbb{N}} \times Z$ is the intersection of a closed set and a G_δ set. Now R is a subset of $\mathbb{N}^{\mathbb{N}} \times \hat{X}$ and in $\mathbb{N}^{\mathbb{N}} \times X$ is an $F_{\sigma\delta}$ set, because $\mathbb{N}^{\mathbb{N}} \times X$ is metrizable. So $\pi_2[R]$ is F-analytic in X .

7. Proposition Let X be a completely regular Hausdorff space. Then X is Čech-analytic iff there is a set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ and a sequence $\langle \mathcal{G}_n \rangle_{n \in \mathbb{N}}$ of families of open sets in $\mathbb{N}^{\mathbb{N}} \times X$ such that

(i) $R \subseteq \bigcup \mathcal{G}_n \quad \forall \; n \in \mathbb{N}$

(ii) whenever $\mathcal{F}$ is a filter on $\mathbb{N}^{\mathbb{N}} \times X$, containing R , meeting every $\mathcal{G}_n$, and such that $\pi_1[[\mathcal{F}]]$ converges in $\mathbb{N}^{\mathbb{N}}$, then $\mathcal{F}$ has a cluster point in R ;

(iii) $\pi_2[R] = X$.

proof (a) If X is Čech-analytic, let $R \subseteq \mathbb{N}^{\mathbb{N}} \times \beta X$ be such that $\pi_2[R] = X$ and R is expressible as $V \cap \bigcap_{n \in \mathbb{N}} W_n$ where each W_n is open in $\mathbb{N}^{\mathbb{N}} \times \beta X$ and V is closed. Let $\mathcal{G}_n = \{ G \cap (\mathbb{N}^{\mathbb{N}} \times X) : G \subseteq \mathbb{N}^{\mathbb{N}} \times \beta X \text{ is open, } \overline{G} \subseteq W_n \}$. Then $R \subseteq \bigcup \mathcal{G}_n$ for each $n \in \mathbb{N}$. If $\mathcal{F}$ is a filter on $\mathbb{N}^{\mathbb{N}} \times X$, containing R , meeting every $\mathcal{G}_n$, and $\pi_1[[\mathcal{F}]]$ converges to α , then $\pi_2[[\mathcal{F}]]$ has a cluster point $t \in Z$ and (α,t) is a cluster point of $\mathcal{F}$. For each $n \in \mathbb{N}$, there is a $G \subseteq \mathbb{N}^{\mathbb{N}} \times \beta X$ such that $\overline{G} \subseteq W_n$ and $G \cap (\mathbb{N}^{\mathbb{N}} \times X) \in \mathcal{F}$, so $(\alpha,t) \in \overline{G} \subseteq W_n$. Also $R \in \mathcal{F}$ so $(\alpha,t) \in \overline{R} \subseteq V$; thus $(\alpha,t) \in R$. So R , $\langle \mathcal{G}_n \rangle_{n \in \mathbb{N}}$ satisfy (i)-(iii).

(b) Now suppose that there exist ~~fam~~ R, $\langle \mathcal{G}_n \rangle_{n \in \mathbb{N}}$ satisfying (i)-(iii). Embed X in its Stone-Čech compactification βX, and take $V = \overline{R}$ in $\mathbb{N}^{\mathbb{N}} \times \beta X$. Set $W_n = \bigcup \{ W : W \subseteq \mathbb{N}^{\mathbb{N}} \ \beta X$ open, $W \cap (\mathbb{N}^{\mathbb{N}} \times X) \in \mathcal{G}_n \}$. I claim that $R = V \cap \bigcap_{n \in \mathbb{N}} W_n$. **P** ($\alpha$) As $R \subseteq \bigcup \mathcal{G}_n = W_n \cap (\mathbb{N}^{\mathbb{N}} \times X)$ for each $n \in \mathbb{N}$, $R \subseteq V \cap \bigcap_{n \in \mathbb{N}} W_n$.
(β) Suppose that $(\alpha, t) \in V \cap \bigcap_{n \in \mathbb{N}} W_n$. Let be the filter on $\mathbb{N}^{\mathbb{N}}$ X

~~$\{ (\mathbb{N}^{\mathbb{N}} \times X) R \cup : U$ is a neighbourhood of $(\alpha, t) \}$~~ ;

~~as (α,t) V R̄, is a filter on R~~

generated by $\{ R \cap U : U$ is a neighbourhood of $(\alpha, t) \}$. (There is such a filter because $(\alpha, t) \in V = \overline{R}$.) Then $\pi_1[[\mathcal{F}]] \to \alpha$. Also, for each $n \in \mathbb{N}$, ~~there is a neighbourhood~~ $(\alpha, t) \in W_n$, so there is an open $W \subseteq \mathbb{N}^{\mathbb{N}} \times \beta X$ such that $(\alpha, t) \in W$ and $W \cap (\mathbb{N}^{\mathbb{N}} \times X) \in \mathcal{G}_n$; now $W \cap (\mathbb{N}^{\mathbb{N}} \times X) \in \mathcal{F} \cap \mathcal{G}_n$. By hypothesis (ii), $\mathcal{F}$ has a cluster point in R; but because $\mathbb{N}^{\mathbb{N}} \times \beta X$ is Hausdorff, this must be (α, t), and $(\alpha, t) \in R$, as required. **Q** So R is expressible as the intersection of a closed set with a G_δ set and X is Čech-analytic.

8. Proposition A hereditarily Lindelöf Čech-analytic space is K-analytic.

proof Let A be a hereditarily Lindelöf Čech-analytic space embedded in a compact Hausdorff space X. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be such that $\pi_2[R] = A$ and $R = V \cap \bigcap_{n \in \mathbb{N}} W_n$ where V is closed and every W_n is open. For each $n \in \mathbb{N}$, $\sigma \in \mathbb{N}^{(\mathbb{N})}$ let $\mathcal{G}_\sigma^n = \{ G : G \subseteq X$ a cozero set, $U_\sigma \times G \subseteq W_n \}$ where $U_\sigma = \{ \alpha : \alpha \supseteq \sigma \}$. Then there is a countable $\mathcal{H}_\sigma^n \subseteq \mathcal{G}_\sigma^n$ such that $A \cap \bigcup \mathcal{G}_\sigma^n \subseteq \bigcup \mathcal{H}_\sigma^n$. Set $W_n' = \bigcup_{\sigma \in \mathbb{N}^{(\mathbb{N})}} (U_\sigma \times \bigcup \mathcal{H}_\sigma^n)$; then W_n' is K_σ, so $R' = V \cap \bigcap_{n \in \mathbb{N}} W_n'$ is K-analytic.
As $W_n' \subseteq W_n$ for every $n \in \mathbb{N}$, $R' \subseteq R$. But in fact $R = R'$. **P** Take any $(\alpha, t) \in R$. Then for each $n \in \mathbb{N}$ there is a cozero neighbourhood G of t and a $\sigma \subseteq \alpha$ such that $U_\sigma \times G \subseteq W_n$. So $G \in \mathcal{G}_\sigma^n$ and $t \in A \cap \bigcup \mathcal{G}_\sigma^n$ and there must be an $H \in \mathcal{H}_\sigma^n$ such that $t \in H$; now $(\alpha, t) \in U_\sigma \times H \subseteq W_n'$. As n is arbitrary, $(\alpha, t) \in R'$. **Q** So $A = \pi_2[R] = \pi_2[R']$ is K-analytic.

9. Example Theorem 4 covers many of the properties that we expect a class of "analytic" sets to have; Theorem 5 shows that (for instance) a Čech-analytic subset of a completely regular space is universally measurable and has the strong Baire property. Since any discrete space is Čech-complete, therefore Čech-analytic, we do not always have the continuous image of a Čech-analytic space Čech-analytic. Nor do we have a set which is both Čech-analytic and co-Čech-analytic necessarily Borel. Give $[0,1]$ its usual topology, ω_1 its discrete topology; let $X = [0,1] \times \omega_1$, so that X is a complete metric space and is G_δ in its Stone-Cech compactification βX. For each $\xi < \omega_1$ let E_ξ be a Borel set in $[0,1]$ of Borel class $\geq \xi$. Then there is a closed set $F_\xi \subseteq \underset{\sim}{N}^{N} \times [0,1]$ such that $E_\xi = \pi_2[F_\xi]$. In $\underset{\sim}{N}^{N} \times X$ let

$$F = \{ (\alpha, t, \xi) : (\alpha, t) \in F_\xi \} ;$$

then F is closed, so that $\pi_2[F]$ is Čech-analytic. Similarly, $X \setminus \pi_2[F]$ is Čech-analytic (and $\beta X \setminus \pi_2[F]$ is Čech-analytic also). So $E = \pi_2[F]$ is both Čech-analytic and co-Čech-analytic (in both X and βX). But E cannot be Borel because the E_ξ are not of bounded Borel class.

10. Example I do not know of a useful necessary and sufficient condition for a Čech-analytic space to be K-analytic. "Hereditarily Lindelöf" is sufficient (proposition 8) but not necessary; "Lindelöf" is necessary but not sufficient. Let X be an uncountable set; adjoin one point ∞, and say that $G \subseteq X \cup \{\infty\}$ is open if either $X \setminus G$ is countable or $\infty \notin G$. Now $X \cup \{\infty\}$ is the union of two locally compact subspaces so is Čech-analytic. Clearly it is Lindelöf. But it is not K-analytic because any uncountable K-analytic space has a non-eventually-constant sequence with a cluster point, which $X \cup \{\infty\}$ does not have.

11. <u>Example</u> Give ω_1 its order topology. Then for a set $A \subseteq \omega_1$ the following are equivalent:

(i) A is Čech-analytic;

(ii) A is Borel;

(iii) A is expressible as $\bigcup_{n \in \underset{\sim}{N}}(F_n \cap G_n)$, where each F_n is closed and each G_n open;

(iv) there is an uncountable closed set $F \subseteq \omega_1$ such that either $A \supseteq F$ or $A \cap F = \emptyset$.

<u>proof</u> (i)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) .

12. <u>Theorem</u> (see [1], Theorem 2.1) Let X be a first-countable completely regular Hausdorff space, $A \subseteq X$ a Čech-analytic set which is not F_σ . Suppose <u>either</u> that X is ~~hereditarily~~ perfectly normal and paracompact <u>or</u> that A is hereditarily Lindelöf. Then there is a compact set $K \subseteq X$ such that $K \setminus A$ is countable, dense in K , and has no isolated points.

<u>proof</u> (<u>a</u>) Let Z be the Stone-Čech compactification of X ; henceforth all notions of closure, interior will be taken in Z . Let $R \subseteq \underset{\sim}{N}^{\underset{\sim}{N}} \times Z$ be a set such that $\pi_2[R] = A$ and R is expressible as $V \cap \bigcap_{n \in \underset{\sim}{N}} W_n$ where V is closed and every W_n is open (Proposition 3). Let $\mathcal{I}$ be the σ-ideal of subsets of A generated by $\{ F : F \subseteq A , \overline{F} \cap X \subseteq A \}$; our hypothesis that A is not F_σ in X becomes just $A \notin \mathcal{I}$. Let $\mathcal{A}$ be

$$\{ B : B \subseteq A , B \neq \emptyset , B \cap G \notin \mathcal{I} \text{ whenever } G \text{ open, } B \cap G \neq \emptyset \} .$$

(<u>b</u>) If $B \subseteq A$ and $\mathcal{G}$ is a collection of open sets such that $B \cap G \in \mathcal{I}$ for every $G \in \mathcal{G}$, then $B \cap \bigcup\mathcal{G} \in \mathcal{I}$. (<u>i</u>) If A is hereditarily Lindelöf, then $B \cap \bigcup\mathcal{G}$ is Lindelöf, so there is a countable $\mathcal{G}_o \subseteq \mathcal{G}$ such that $B \cap \bigcup\mathcal{G} = B \cap \bigcup\mathcal{G}_o \in \mathcal{I}$. ~~(ii) If X is hereditarily paracompact, there~~

(<u>ii</u>) If X is perfectly normal and paracompact, then $X \cap \bigcup\mathcal{G}$ is relatively F_σ in X ; let $\langle E_m \rangle_{m\in\mathbb{N}}$ be such that $\overline{E}_m \cap X = E_m$ for each $m \in \mathbb{N}$, $\bigcup_{m\in\mathbb{N}} E_m = X \cap \bigcup\mathcal{G}$. If $B \cap \bigcup\mathcal{G} \notin \mathcal{I}$, there is an $m \in \mathbb{N}$ such that $B \cap E_m \notin \mathcal{I}$. As X is paracompact, there is a refinement $\mathcal{H}$ of $\mathcal{G}$ such that $E_m \subseteq \bigcup\mathcal{H}$ and every point of X has a neighbourhood meeting only finitely many elements of $\mathcal{H}$. For each $H \in \mathcal{H}$, $B \cap H \in \mathcal{I}$ so that there is a sequence $\langle F_H^n \rangle_{n\in\mathbb{N}}$ such that $B \cap H = \bigcup_{n\in\mathbb{N}} F_H^n$ and $\overline{F_H^n} \cap X \subseteq A$ for every $n \in \mathbb{N}$. Set $F_n = \bigcup_{H\in\mathcal{H}} F_H^n$; then $\overline{F}_n \cap X = \bigcup_{H\in\mathcal{H}} \overline{F_H^n} \cap X \subseteq A$ because $\langle F_H^n \rangle_{H\in\mathcal{H}}$ is locally finite in X . Now $B \cap E_m \subseteq B \cap \bigcup\mathcal{H} = \bigcup_{n\in\mathbb{N}} F_n \in \mathcal{I}$. So $B \cap \bigcup\mathcal{G} \in \mathcal{I}$, as required.

~~covering $X \cap \bigcup \mathcal{G}$~~

~~is a refinement $\mathcal{H}$ of $\mathcal{G}$ such that every point of $X \cap \bigcup \mathcal{G}$ has a neighbourhood meeting only finitely many elements of $\mathcal{H}$. (This is because X is dense in Z.) For each $H \in \mathcal{H}$, $B \cap H \in \mathcal{J}$ so there is a sequence $\langle F^n_H \rangle_{n \in \mathbb{N}}$ of subsets of A such that $B \cap H \subseteq \bigcup_{n \in \mathbb{N}} F^n_H \subseteq H$, $\overline{F^n_H} \cap X \subseteq A$ for every $n \in \mathbb{N}$. Set $F_n = \bigcup_{H \in \mathcal{H}} F^n_H$; then $\overline{F}_n \cap X = \bigcup_{H \in \mathcal{H}} \overline{F^n_H} \cap X \subseteq A$ because $\langle F^n_H \rangle_{H \in \mathcal{H}}$ is locally finite in X. Now $B \cap \bigcup \mathcal{G} = B \cap \bigcup \mathcal{H} = B \cap \bigcup_{n \in \mathbb{N}, H \in \mathcal{H}} F^n_H \subseteq B \cap \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{J}$.~~ ~~**Q**~~

(c) So if $B \subseteq A$ and $B \notin \mathcal{J}$ there is a $C \subseteq B$ such that $C \in \mathcal{A}$.

P Set $C = B \setminus \bigcup \{ G : G \text{ open}, B \cap G \in \mathcal{J} \}$. **Q**

(d) If $S \subseteq R$ and $n \in \mathbb{N}$ and $\pi_2[S] \notin \mathcal{J}$, there is an $S_1 \subseteq S$ such that $\pi_2[S_1] \notin \mathcal{J}$ and $\overline{S_1} \subseteq W_n$. **P** For $\sigma \in \mathbb{N}^{(\mathbb{N})}$ set $U_\sigma = \{ \alpha : \alpha \supseteq \sigma \} \subseteq \mathbb{N}^{\mathbb{N}}$, $\mathcal{G}_\sigma = \{ G : G \subseteq Z \text{ open}, \overline{U_\sigma \times G} \subseteq W_n \}$, $G_\sigma = \bigcup \mathcal{G}_\sigma$. We have $S \subseteq W_n = \bigcup_{\sigma \in \mathbb{N}^{(\mathbb{N})}} U_\sigma \times G_\sigma$. So there is a $\sigma \in \mathbb{N}^{(\mathbb{N})}$ such that $\pi_2[S \cap (U_\sigma \times G_\sigma)] \notin \mathcal{J}$. Now $\pi_2[S \cap (U_\sigma \times G_\sigma)] = \pi_2[S \cap (U_\sigma \times Z)] \cap G_\sigma$ so by part (b) there is a $G \in \mathcal{G}_\sigma$ such that $\pi_2[S \cap (U_\sigma \times Z)] \cap G \notin \mathcal{J}$. Set $S_1 = S \cap (U_\sigma \times G)$. **Q**

(e) We can now choose inductively, for $\sigma \in \mathbb{N}^{(\mathbb{N})}$, $\tau(\sigma)$, S_σ, N_σ, t_σ such that, for every $\sigma \in \mathbb{N}^{(\mathbb{N})}$,

(i) $\tau(\sigma) \in \mathbb{N}^{(\mathbb{N})}$, $\#(\tau(\sigma)) = \#(\sigma) + 1$

(ii) if $\sigma' \subseteq \sigma$ then $\tau(\sigma') \subseteq \tau(\sigma)$

(iii) $S_\sigma \subseteq R \cap (U_{\tau(\sigma)} \times Z)$, $\overline{S_\sigma} \subseteq W_{\#(\sigma)}$

(iv) if $\sigma' \subseteq \sigma$ then $S_{\sigma'} \supseteq S_\sigma$

(v) $\pi_2[S_\sigma] \notin \mathcal{J}$, $\pi_2[S_\sigma] \subseteq N_\sigma$

(vi) $t_\sigma \in X \cap \overline{\pi_2[S_\sigma]} \setminus A$

(vii) $\overline{\pi_2[S_{\sigma^\frown i}]} \cap \overline{\pi_2[S_{\sigma^\frown j}]} = \emptyset$ if $i \neq j$

(viii) $\langle N_{\sigma^\frown i} \rangle_{i \in \mathbb{N}}$ is a decreasing sequence forming a neighbourhood base for t_σ.

<u>construction</u> Set $N_\emptyset = Z$. We have $A = \pi_2[R] = \bigcup_{\#(\tau)=1}\pi_2[R\cap(U_\tau \times Z)]$; take $\tau(\emptyset)$ such that $\#(\tau(\emptyset)) = 1$, $\pi_2[R\cap(U_{\tau(\emptyset)}\times Z)] \notin \mathcal{I}$. Let $S_\emptyset \subseteq R\cap(U_{\tau(\emptyset)}\times Z)$ be such that $\overline{S_\emptyset} \subseteq W_0$ and $\pi_2[S_\emptyset] \notin \mathcal{I}$ (using (d) above).

Having chosen $S_\sigma \subseteq R\cap(U_{\tau(\sigma)}\times Z)$ such that $\pi_2[S_\sigma] \notin \mathcal{I}$, proceed as follows. Let $C \in \mathcal{A}$ be such that $C \subseteq \pi_2[S_\sigma]$ (part (c) above). Since $C \notin \mathcal{I}$, while $C \subseteq A$, we must have $X\cap\overline{C} \not\subseteq A$; choose $t_\sigma \in X\cap\overline{C}\setminus A$, so that $t_\sigma \in X\cap\overline{\pi_2[S_\sigma]}\setminus A$. Because X is first-countable, t_σ must have a countable base of neighbourhoods in Z; take $\langle N_{\sigma^\frown i}\rangle_{i\in\mathbb{N}}$ to enumerate a decreasing sequence of open sets forming such a base. Choose $\langle s_i\rangle_{i\in\mathbb{N}}$, all distinct, such that $s_i \in C\cap N_{\sigma^\frown i}$. Choose neighbourhoods H_i of s_i such that all the $\overline{H_i}$ are disjoint and $H_i \subseteq N_{\sigma^\frown i}$ for each $i \in \mathbb{N}$. Because $C \in \mathcal{A}$ and $s_i \in C\cap H_i$, we have $C\cap H_i \notin \mathcal{I}$. Also, for each $i \in \mathbb{N}$, we have

$$C\cap H_i \subseteq \pi_2[S_\sigma] = \bigcup_{k\in\mathbb{N}}\pi_2[S_\sigma\cap(U_{\tau(\sigma)^\frown k}\times Z)]$$

so there must be a $k(i) \in \mathbb{N}$ such that

$$C\cap H_i\cap\pi_2[S_\sigma\cap(U_{\tau(\sigma)^\frown k(i)}\times Z)] \notin \mathcal{I}.$$

Set $\tau(\sigma^\frown i) = \tau(\sigma)^\frown k(i)$. Set

$$P_i = S_\sigma\cap(U_{\tau(\sigma^\frown i)}\times Z)\cap\pi_2^{-1}[C\cap H_i]$$

so that $\pi_2[P_i] \notin \mathcal{I}$; choose $S_{\sigma^\frown i} \subseteq P_i$ such that $\overline{S_{\sigma^\frown i}} \subseteq W_{\#(\sigma)+1}$ and $\pi_2[S_{\sigma^\frown i}] \notin \mathcal{I}$. It is easy to check that this procedure gives us (i)-(viii).

(<u>f</u>) Take $K = \overline{\{t_\sigma : \sigma\in\mathbb{N}^{(\mathbb{N})}\}} \subseteq Z$. Then $K\setminus\{t_\sigma : \sigma\in\mathbb{N}^{(\mathbb{N})}\} \subseteq A$. ¶ Let $t \in K\setminus\{t_\sigma : \sigma\in\mathbb{N}^{(\mathbb{N})}\}$. Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}^{(\mathbb{N})}$ such that $t = \lim_{\sigma\to\mathcal{F}} t_\sigma$. ? Suppose, if possible, that there is an $n \in \mathbb{N}$ such that $\{\sigma : \sigma \supseteq \sigma_0\} \notin \mathcal{F}$ for every $\sigma_0 \in \mathbb{N}^n$. Then there is a least such n, and $n > 0$. Let $\sigma_0 \in \mathbb{N}^{n-1}$ be such that $\{\sigma : \sigma \supseteq \sigma_0\} \in \mathcal{F}$. Of course $\{\sigma_0\} \notin \mathcal{F}$, because $t \neq t_{\sigma_0}$. So $\{\sigma : \sigma\supseteq\sigma_0,\ \sigma(n) \geq m\} \in \mathcal{F}$ for every $m \in \mathbb{N}$. Now if $\sigma \supset \sigma_0$, $\sigma(n) = j \geq m$,

$$t_\sigma \in \overline{\pi_2[S_\sigma]} \subseteq \overline{\pi_2[S_{\sigma_0^\frown j}]} \subseteq \overline{N_{\sigma_0^\frown j}} \subseteq \overline{N_{\sigma_0^\frown m}}.$$

So $t \in \bigcap_{m\in\mathbb{N}} \overline{N_{\sigma_0{}^\frown m}}$. But $\{ N_{\sigma_0{}^\frown m} : m \in \mathbb{N} \}$ is a neighbourhood base for t_{σ_0}, so $t = t_{\sigma_0}$. $\mathbf{X}$

Accordingly there is for each $n \in \mathbb{N}$ a $\sigma_n \in \mathbb{N}^n$ such that $\{ \sigma : \sigma \supseteq \sigma_n \} \in \mathcal{F}$. There must now be an $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_n = \alpha|n$ for every $n \in \mathbb{N}$. As just above, we have $t_\sigma \in \overline{\pi_2[S_{\alpha|n}]}$ whenever $\sigma \supseteq \alpha|n$, so that $t \in \overline{\pi_2[S_{\alpha|n}]}$ for every $n \in \mathbb{N}$.

Now let $\mathcal{H}$ be the filter on $\mathbb{N}^{\mathbb{N}} \times Z$ generated by

$$\{ S_{\alpha|n} : n \in \mathbb{N} \} \cup \{ \mathbb{N}^{\mathbb{N}} \times G : G \text{ is a neighbourhood of } t \} .$$

We have $U_{\tau(\alpha|n)} \times Z \supseteq S_{\alpha|n} \in \mathcal{H}$ for each $n \in \mathbb{N}$. But the condition (ii) of ~~the inductive hypothesis in~~ (e) above shows that there is a $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\tau(\alpha|n) \subseteq \beta$ for every $n \in \mathbb{N}$. Now $\pi_1[[\mathcal{H}]] \to \beta$, so that $\mathcal{H} \to (\beta,t)$. As $R \in \mathcal{H}$, $(\beta,t) \in \overline{R} \subseteq V$. Also, for each $n \in \mathbb{N}$, $\overline{S_{\alpha|n}} \subseteq W_n$, so that $(\beta,t) \in W_n$. Thus $(\beta,t) \in V \cap \bigcap_{n\in\mathbb{N}} W_n$ and $(\beta,t) \in R$ and $t \in A$, as required. $\mathbf{Q}$

(g) It follows that $K \subseteq X$; as K is closed in Z , K is compact. We have $K \setminus A = \{ t_\sigma : \sigma \in \mathbb{N}^{(\mathbb{N})} \}$ countable and dense in K . Also, because $t_{\sigma^\frown i} \in \overline{\pi_2[S_{\sigma^\frown i}]} \subseteq \overline{N_{\sigma^\frown i}}$ for each σ , i we have $\lim_{i\to\infty} t_{\sigma^\frown i} = t_\sigma$ for each σ, and $K \setminus A$ is without isolated points. (Note that all the $t_{\sigma^\frown i}$ are distinct, by (vii) of (e).)

13. <u>Applications</u> Cases in which the hypotheses are satisfied include:

(a) X any metrizable space, $A \subseteq X$ an absolutely analytic set (because now X is ~~hereditarily~~ perfectly normal and paracompact).

(b) X a ~~first countable~~ perfectly normal ~~space~~ Hausdorff space, $A \subseteq X$ a K-analytic set. In this case I have to show that A is hereditarily Lindelöf. But $A \cap F$ is K-analytic, therefore Lindelöf, for every closed $F \subseteq X$, and therefore for every F_σ set $F \subseteq X$; and open sets in X are F_σ . (This is the case given in [1]. Observe that actually any Lindelöf Čech-analytic set A will do.)

14. Remarks The main value of Theorem 12 seems to be that $K \cap A$ cannot be an F_σ set (because K is not the union of countably many nowhere dense closed sets). In [1] some further remarks on the structure of K are given.

~~15. Problem~~

References 1. J.E.Jayne & C.A.Rogers, "Borel isomorphisms at the first level, I", Mathematika 26 (1979) 125-156.

2. Z.Frolik, "The concept of non-separable analytic set", pp. 449-461 of Topology (ed. A.Csaszar), North-Holland, 1980 (Colloq. Math. Soc. Janos Bolyai 23).