In this introductory volume I set out, at a level which I hope will be suitable for students with no prior knowledge of the Lebesgue (or even Riemann) integral and with only a basic (but thorough) preparation in the techniques of ϵ-δ analysis, the theory of measure and integration up to the convergence theorems (§123). I add a third chapter (Chapter 13) of miscellaneous additional results, mostly chosen as being relatively elementary material necessary for topics treated in Volume 2 which does not have a natural place there.

The title of this volume is a little more emphatic than I should care to try to justify *au pied de la lettre*. I would certainly characterize the construction of Lebesgue measure on $\mathbb{R}$ (§114), the definition of the integral on an abstract measure space (§122) and the convergence theorems (§123) as indispensable. But a teacher who wishes to press on to further topics will find that much of Chapter 13 can be set aside for a while. I say 'teacher' rather than 'student' here, because if you are studying on your own I think you should aim to go slower than the text requires rather than faster; in my view, these ideas are genuinely difficult, and I think you should take the time to get as much practice at relatively elementary levels as you can.

Perhaps this is a suitable moment at which to set down some general thoughts on the teaching of measure theory. I have been teaching analysis for over thirty years now, and one of the few constants over that period has been the feeling, almost universal among teachers of analysis, that we are not serving most of our students well. We have all encountered students who are not stupid – who are indeed quite good at mathematics – but who seem to have a disproportionate difficulty with rigorous analysis. They are so exhausted and demoralised by the technical problems that they cannot make sense or use even of the knowledge they achieve. The natural reaction to this is to try to make courses shorter and easier. But I think that this makes it even more likely that at the end of the semester your students will be stranded in thorn-bushes half way up the mountain. Specifically, with Lebesgue measure, you are in danger of spending twenty hours teaching them how to integrate the indicator function of the rationals. This is not what the subject is for. Lebesgue's own presentations of the subject (Lebesgue 1904, Lebesgue 1918) emphasize the convergence theorems and the Fundamental Theorem of Calculus. I have put the former in Volume 1 and the latter in Volume 2, but it does seem to me that unless your students themselves want to know when one can expect to be able to interchange a limit and an integral, or which functions are indefinite integrals, or what the completions of $C([0,1])$ under the norms $\| \|_1$, $\| \|_2$ look like, then it is going to be very difficult for them to make anything of this material; and if you really cannot reach the point of explaining at least a couple of these matters in terms which they can appreciate, then it may not be worth starting. I would myself choose rather to omit a good many proofs than to come to the theorems for which the subject was created so late in the course that two thirds of my class have already given up before they are covered.

Of course I and others have followed that road too, with no better results (though usually with happier students) than we obtain by dotting every i and crossing every t in the proofs. Nearly every time I am consulted by a non-specialist who wants to be told a theorem which will solve his problem, I am reminded that pure mathematics in general, and analysis in particular, does not lie in the *theorems* but in the *proofs*. In so far as I have been successful in answering such questions, it has usually been by making a trifling adjustment to a standard argument to produce a non-standard theorem. The ideas are in the details. You have not understood Carathéodory's construction (§113) until you can, at the very least, reliably reproduce the argument which shows that it works. In the end, there is no alternative to going over every step of the ground, and while I have occasionally been ruthless in cutting out topics which seem to me to be marginal, I have tried to make sure – at the expense, frequently, of pedantry – that every necessary idea is signalled.

Faced, therefore, with any particular class, I believe that a teacher must compromise between scope and completeness. Exactly which compromises are most appropriate will depend on factors which it would be a waste of time for me to guess at. This volume is supposed to be a possible text on which to base a course; but I hope that no lecturer will set her class to read it at so many pages a week. My primary aim is to provide a concise and coherent basis on which to erect the structure of the later volumes. This involves me in pursuing, at more than one point, approaches which take slightly more difficult paths for the

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sake of developing a more refined technique. (Perhaps the most salient of these is my insistence that an integrable function need not be defined everywhere on the underlying measure space; see §§121-122.) It is the responsibility of the individual teacher to decide for herself whether such refinements are appropriate to the needs of her students, and, if not, to show them what translations are needed.

The above paragraphs are directed at teachers who are, supposedly, competent in the subject – certainly past the level treated in this volume – and who have access to some of the many excellent books already available, so that if they take the trouble to think out their aims, they should be able to choose which elements of my presentation are suitable. But I must also consider the position of a student who is setting out to learn this material on his own. I trust that you have understood from what I have already written that you should not be afraid to look ahead. You could, indeed, do worse than go to Volume 2, and take one of the wonderful theorems there – the Fundamental Theorem of Calculus (§222), for instance, or, if you are very ambitious, the strong law of large numbers (§273) – and use the index and the cross-references to try to extract a proof from first principles. If you are successful you will have every right to congratulate yourself. In the periods in which success seems elusive, however, you should be working systematically through the 'basic exercises' in the sections which seem to be relevant; and if all else fails, start again at the beginning. Mathematics is a difficult subject, that is why it is worth doing, and almost every section here contains some essential idea which you could not expect to find alone.

Version of 4.1.04

Chapter 11

Measure spaces

In this chapter I set out the fundamental concept of 'measure space', that is, a set in which some (not, as a rule, all) subsets may be assigned a 'measure', which you may wish to interpret as area, or mass, or volume, or thermal capacity, or indeed almost anything which you would expect to be additive – I mean, that the measure of the union of two disjoint sets should be the sum of their measures. The actual definition (in 112A) is not obvious, and depends essentially on certain technical features which make a preparatory section (§111) advisable. Furthermore, even with the definition well in hand, the original and most important examples of measures, Lebesgue measure on Euclidean space, remain elusive. I therefore devote a section (§113) to a method of constructing measures, before turning to the details of the arguments needed for Lebesgue measure in §§114-115. Thus the structure of the chapter is three sections of general theory followed by two (which are closely similar) on particular examples. I should say that the general theory is essentially easier; but it does rely on facility with certain manipulations of families of sets which may be new to you.

At some point I ought to comment on my arrangement of the material, and it may be helpful if I do so before you start work on this chapter. One of the many fundamental questions which any author on the subject must decide, is whether to begin with 'general' measure theory or with 'Lebesgue' measure and integration. The point is that Lebesgue measure is rather more than just the most important example of a measure space. It is so close to the heart of the subject that the great majority of the ideas of the elementary theory can be fully realised in theorems about Lebesgue measure. Looking ahead to Volume 2, I find that, with the exception of Chapter 21 – which is specifically devoted to extending your ideas of what measure spaces can be – only Chapter 27 and the second half of Chapter 25 really need the general theory to make sense, while Chapters 22, 26 and 28 are specifically about Lebesgue measure. Volume 3 is another matter, but even there more than half the mathematical content can be expressed in terms of Lebesgue measure. If you take the view, as I certainly do when it suits my argument, that the business of pure mathematics is to express and extend the logical capacity of the human mind, and that the actual theorems we work through are merely vehicles for the ideas, then you can correctly point out that all the really important things in the present volume can be done without going to the trouble of formulating a general theory of abstract measure spaces; and that by studying the relatively concrete example of Lebesgue measure on r-dimensional Euclidean space you can avoid a variety of irrelevant distractions.

If you are quite sure, as a teacher, that none of your pupils will wish to go beyond the elementary theory, there is something to be said for this view. I believe, however, that it becomes untenable if you wish to prepare any of your students for more advanced ideas. The difficulty is that, with the best will in the world, anyone who has worked through the full theory of Lebesgue measure, and then comes to the theory of abstract measure spaces, is likely to go through it too fast, and at the end find himself uncertain about just which ninety per cent of the facts he knows are generally applicable. I believe it is safer to keep the special properties of Lebesgue measure clearly labelled as such from the beginning.

It is of course the besetting sin of mathematics teachers at this level, to teach a class of twenty in a manner appropriate to perhaps two of them. But in the present case my own judgement is that very few students who are ready for the course at all will have any difficulty with the extra level of abstraction involved in 'Let (X,Σ,μ) be a measure space, $\ldots$'. I do assume knowledge of elementary linear algebra, and the grammar, at least, of arbitrary measure spaces is no worse than the grammar of arbitrary linear spaces. Moreover, the Lebesgue theory already involves statements of the form 'if E is a Lebesgue measurable set, $\ldots$', and in my experience students who can cope with quantification over subsets of the reals are not deterred by quantification over sets of sets (which anyway is necessary for any elementary description of the σ-algebra of Borel sets). So I believe that here, at least, the extra generality of the 'professional' approach is not an obstacle to the amateur.

I have written all this here, rather than later in the chapter, because I do wish to give you the choice. And if your choice is to learn the Lebesgue theory first, and leave the general theory to later, this is how to do it. You should read

paragraphs 114A-114C
114D, with 113A-113B and 112Ba, 112Bc
114E, with 113C-113D, 111A, 112A, 112Bb

114F

114G, with 111G and 111C-111F,

and then continue with Chapter 12. At some point, of course, you should look at the exercises for §§112-113; but, as in Chapters 12-13, you will do so by translating 'Let (X, Σ, μ) be a measure space' into 'Let μ be Lebesgue measure on $\mathbb{R}$, and Σ the σ-algebra of Lebesgue measurable sets'. Similarly, when you look at 111X-111Y, you will take Σ to be *either* the σ-algebra of Lebesgue measurable sets *or* the σ-algebra of Borel subsets of $\mathbb{R}$.

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111 σ-algebras

In the introduction to this chapter I remarked that a measure space is 'a set in which some (not, as a rule, all) subsets may be assigned a measure'. All ordinary concepts of 'length' or 'area' or 'volume' apply only to reasonably regular sets. Modern measure theory is remarkably powerful in that an extraordinary variety of sets are regular enough to be measured; but we must still expect some limitation, and when studying any measure a proper understanding of the class of sets which it measures will be central to our work. The basic definition here is that of 'σ-algebra of sets'; all measures in the standard theory are defined on such collections. I therefore begin with a statement of the definition, and a brief discussion of the properties, of these classes.

111A Definition Let X be a set. A **σ-algebra of subsets of** X is a family Σ of subsets of X such that

(i) $\emptyset \in \Sigma$;

(ii) for every $E \in \Sigma$, its complement $X \setminus E$ in X belongs to Σ;

(iii) for every sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in Σ, its union $\bigcup_{n\in\mathbb{N}} E_n$ belongs to Σ.

111D Elementary properties of σ-algebras If Σ is a σ-algebra of subsets of X, then it has the following properties.

(a) $E \cup F \in \Sigma$ for all E, $F \in \Sigma$.

(b) $E \cap F \in \Sigma$ for all E, $F \in \Sigma$.

(c) $E \setminus F \in \Sigma$ for all E, $F \in \Sigma$.

(d) Now suppose that $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and consider

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} E_n &= \{x : x \in E_n \ \forall\, n \in \mathbb{N}\} \\ &= E_0 \cap E_1 \cap E_2 \cap \ldots \\ &= X \setminus \bigcup_{n\in\mathbb{N}} (X \setminus E_n);\end{aligned}$$

this also belongs to Σ.

111F Countable sets (a) A set K is **countable** if either it is empty or there is a surjection from $\mathbb{N}$ onto K. In this case, if Σ is a σ-algebra of sets and $\langle E_k \rangle_{k\in K}$ is a family in Σ indexed by K, then $\bigcup_{k\in K} E_k \in \Sigma$.

(b)(i) If K is countable and $L \subseteq K$, then L is countable.

(ii) The Cartesian product $\mathbb{N} \times \mathbb{N} = \{(m, n) : m,\, n \in \mathbb{N}\}$ is countable.

(iii) It follows that if K and L are countable sets, so is $K \times L$.

(iv) If $K_1, K_2, \ldots, K_r$ are countable sets, so is $K_1 \times \ldots \times K_r$. In particular, $\mathbb{Q}^r \times \mathbb{Q}^r$ will be countable, for any integer $r \geq 1$.

(c) If Σ is a σ-algebra of sets, K is a non-empty countable set, and $\langle E_k\rangle_{k\in K}$ is a family in Σ, then

$$\bigcap_{k\in K} E_k = \{x : x \in E_k \ \forall\ k \in K\}$$

belongs to Σ.

111G Borel sets (a) Let X be a set, and let $\mathfrak{S}$ be any non-empty family of σ-algebras of subsets of X. Then

$$\bigcap \mathfrak{S} = \{E : E \in \Sigma \text{ for every } \Sigma \in \mathfrak{S}\},$$

the intersection of all the σ-algebras belonging to $\mathfrak{S}$, is a σ-algebra of subsets of X.

(b) Now let $\mathcal{A}$ be any family of subsets of X. Consider

$$\mathfrak{S} = \{\Sigma : \Sigma \text{ is a } \sigma\text{-algebra of subsets of } X,\ \mathcal{A} \subseteq \Sigma\}.$$

$\Sigma_{\mathcal{A}} = \bigcap \mathfrak{S}$ is a σ-algebra of subsets of X; it is the smallest σ-algebra of subsets of X including $\mathcal{A}$.

We say that $\Sigma_{\mathcal{A}}$ is the σ-algebra of subsets of X **generated by** $\mathcal{A}$.

Examples (i) For any X, the σ-algebra of subsets of X generated by $\emptyset$ is $\{\emptyset, X\}$.

(ii) The σ-algebra of subsets of $\mathbb{N}$ generated by $\{\{n\} : n \in \mathbb{N}\}$ is $\mathcal{P}\mathbb{N}$.

(c)(i) We say that a set $G \subseteq \mathbb{R}$ is **open** if for every $x \in G$ there is a $\delta > 0$ such that the open interval $]x-\delta, x+\delta[$ is included in G.

(ii) Similarly, for any $r \geq 1$, we say that a set $G \subseteq \mathbb{R}^r$ is **open** in $\mathbb{R}^r$ if for every $x \in G$ there is a $\delta > 0$ such that $\{y : \|y - x\| < \delta\} \subseteq G$, where for $z = (\zeta_1, \dots, \zeta_r) \in \mathbb{R}^r$ I write $\|z\| = \sqrt{\sum_{i=1}^{r} |\zeta_i|^2}$.

(d) Now the **Borel sets** of $\mathbb{R}$, or of $\mathbb{R}^r$, are just the members of the σ-algebra of subsets of $\mathbb{R}$ or $\mathbb{R}^r$ generated by the family of open sets of $\mathbb{R}$ or $\mathbb{R}^r$; the σ-algebra itself is called the **Borel σ-algebra** in each case.

Version of 20.2.05/20.8.08

112 Measure spaces

We are now, I hope, ready for the second major definition, the definition on which all the work of this treatise is based.

112A Definition A **measure space** is a triple (X, Σ, μ) where

(i) X is a set;

(ii) Σ is a σ-algebra of subsets of X;

(iii) $\mu : \Sigma \to [0, \infty]$ is a function such that

(α) $\mu\emptyset = 0$;

(β) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, then $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$.

In this context, members of Σ are called **measurable** sets, and μ is called a **measure on** X.

112B Remarks (c) In interpreting clause (iii-β) of the definition above, we need to assign values to sums $\sum_{n=0}^{\infty} u_n$ for arbitrary sequences $\langle u_n\rangle_{n\in\mathbb{N}}$ in $[0, \infty]$. If one of the u_m is itself infinite, $\sum_{n=0}^{\infty} u_n = \infty$. If all the u_m are finite, then the sequence $\langle \sum_{m=0}^{n} u_m\rangle_{n\in\mathbb{N}}$ of partial sums is monotonic non-decreasing, and either has a finite limit $\sum_{n=0}^{\infty} u_n \in \mathbb{R}$, or diverges to ∞; in which case we again interpret $\sum_{n=0}^{\infty} u_n$ as ∞.

(d) Let X be any set, and let $h : X \to [0, \infty]$ be any function. For every $E \subseteq X$ write $\mu E = \sum_{x\in E} h(x)$. To interpret this sum, note that there is no difficulty for finite sets E (taking $\sum_{x\in\emptyset} h(x) = 0$), while for infinite sets E we can take $\sum_{x\in E} h(x) = \sup\{\sum_{x\in I} h(x) : I \subseteq E \text{ is finite}\}$, because every $h(x)$ is non-negative. Now $(X, \mathcal{P}X, \mu)$ is a measure space.

I will call measures of this kind **point-supported**.

Two particular cases recur often enough to be worth giving names to. If $h(x)=1$ for every x, then μE is just the number of points of E if E is finite, and is ∞ if E is infinite. I will call this **counting measure** on X. If $x_0\in X$, we can set $h(x_0)=1$ and $h(x)=0$ for $x\in X\setminus\{x_0\}$; then μE is 1 if $x_0\in E$, and 0 for other E. I will call this the **Dirac measure on X concentrated at** x_0. Another simple example is with $X=\mathbb{N}$, $h(n)=2^{-n-1}$ for every n; then $\mu X=\frac{1}{2}+\frac{1}{4}+\ldots=1$.

(e) If (X,Σ,μ) is a measure space I may say that 'μ **measures** E' or 'E is **measured by** μ' to mean that μE is defined.

112C Elementary properties of measure spaces Let (X,Σ,μ) be a measure space.
(a) If E, $F\in\Sigma$ and $E\cap F=\emptyset$ then $\mu(E\cup F)=\mu E+\mu F$.
(b) If E, $F\in\Sigma$ and $E\subseteq F$ then $\mu E\le\mu F$.
(c) $\mu(E\cup F)\le\mu E+\mu F$ for any E, $F\in\Sigma$.
(d) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence in Σ, then $\mu(\bigcup_{n\in\mathbb{N}}E_n)\le\sum_{n=0}^{\infty}\mu E_n$.
(e) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ (that is, $E_n\subseteq E_{n+1}$ for every $n\in\mathbb{N}$) then

$$\mu(\textstyle\bigcup_{n\in\mathbb{N}}E_n)=\lim_{n\to\infty}\mu E_n=\sup_{n\in\mathbb{N}}\mu E_n.$$

(f) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ (that is, $E_{n+1}\subseteq E_n$ for every $n\in\mathbb{N}$), and if some μE_n is finite, then

$$\mu(\textstyle\bigcap_{n\in\mathbb{N}}E_n)=\lim_{n\to\infty}\mu E_n=\inf_{n\in\mathbb{N}}\mu E_n.$$

112D Negligible sets Let (X,Σ,μ) be any measure space.

(a) A set $A\subseteq X$ is **negligible** (or **null**) if there is a set $E\in\Sigma$ such that $A\subseteq E$ and $\mu E=0$. (If there seems to be a possibility of doubt about which measure is involved, I will write **μ-negligible**.)

(b) Let $\mathcal{N}$ be the family of negligible subsets of X. Then (i) $\emptyset\in\mathcal{N}$ (ii) if $A\subseteq B\in\mathcal{N}$ then $A\in\mathcal{N}$ (iii) if $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{N}$, $\bigcup_{n\in\mathbb{N}}A_n\in\mathcal{N}$.
I will call $\mathcal{N}$ the **null ideal** of the measure μ. (A family of sets satisfying the conditions (i)-(iii) here is called a **σ-ideal** of sets.)

(c) A set $A\subseteq X$ is **conegligible** if $X\setminus A$ is negligible. Note that (i) X is conegligible (ii) if $A\subseteq B\subseteq X$ and A is conegligible then B is conegligible (iii) if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of conegligible sets, then $\bigcap_{n\in\mathbb{N}}A_n$ is conegligible.

(d) If $P(x)$ is some assertion applicable to members x of the set X, we say that

'$P(x)$ for almost every $x\in X$'

or

'$P(x)$ a.e. (x)'

or

'P almost everywhere', 'P a.e.'

or

'$P(x)$ for μ-almost every x', '$P(x)$ μ-a.e.(x)', 'P μ-a.e.',

to mean that

$$\{x:x\in X,\ P(x)\}$$

is conegligible in X. Thus if $f:X\to\mathbb{R}$ is a function, '$f>0$ a.e.' means that $\{x:f(x)\le 0\}$ is negligible.

(g) When f and g are real-valued functions defined on conegligible subsets of a measure space, I will write $f=_{\text{a.e.}}g$, $f\le_{\text{a.e.}}g$ or $f\ge_{\text{a.e.}}g$ to mean, respectively,

$$f = g \text{ a.e., that is, } \{x : x \in \operatorname{dom}(f) \cap \operatorname{dom}(g),\ f(x) = g(x)\} \text{ is conegligible,}$$

$$f \le g \text{ a.e., that is, } \{x : x \in \operatorname{dom}(f) \cap \operatorname{dom}(g),\ f(x) \le g(x)\} \text{ is conegligible,}$$

$$f \ge g \text{ a.e., that is, } \{x : x \in \operatorname{dom}(f) \cap \operatorname{dom}(g),\ f(x) \ge g(x)\} \text{ is conegligible.}$$

Version of 6.4.05

113 Outer measures and Carathéodory's construction

I introduce the most important method of constructing measures.

113A Outer measures: Definition Let X be a set. An **outer measure** on X is a function $\theta : \mathcal{P}X \to [0, \infty]$ such that

(i) $\theta\emptyset = 0$,

(ii) if $A \subseteq B \subseteq X$ then $\theta A \le \theta B$,

(iii) for every sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of subsets of X, $\theta(\bigcup_{n\in\mathbb{N}} A_n) \le \sum_{n=0}^{\infty} \theta A_n$.

113C Carathéodory's Method: Theorem Let X be a set and θ an outer measure on X. Set

$$\Sigma = \{E : E \subseteq X,\ \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then Σ is a σ-algebra of subsets of X. Define $\mu : \Sigma \to [0, \infty]$ by writing $\mu E = \theta E$ for $E \in \Sigma$; then (X, Σ, μ) is a measure space.

113D Remark Note that in this construction

$$\Sigma = \{E : E \subseteq X,\ \theta(A \cap E) + \theta(A \setminus E) \le \theta A \text{ whenever } A \subseteq X \text{ and } \theta A < \infty\}.$$

Version of 14.6.05

114 Lebesgue measure on $\mathbb{R}$

Following the very abstract ideas of §§111-113, we have an urgent need for a non-trivial example of a measure space. By far the most important example is the real line with Lebesgue measure, and I now proceed to a description of this measure (114A-114E), with a few of its basic properties.

The principal ideas of this section are repeated in §115, and if you have encountered Lebesgue measure before, or feel confident in your ability to deal with two- and three-dimensional spaces at the same time as doing some difficult analysis, you could go directly to that section, turning back to this one only when a specific reference is given.

114A Definitions (a) For the purposes of this section, a **half-open interval** in $\mathbb{R}$ is a set of the form $[a, b[= \{x : a \le x < b\}$, where $a, b \in \mathbb{R}$.

(b) We define the **length** λI of a half-open interval I by setting

$$\lambda\emptyset = 0, \quad \lambda\, [a, b[= b - a \text{ if } a < b.$$

114B Lemma If $I \subseteq \mathbb{R}$ is a half-open interval and $\langle I_j \rangle_{j\in\mathbb{N}}$ is a sequence of half-open intervals covering I, then $\lambda I \le \sum_{j=0}^{\infty} \lambda I_j$.

114C Definition Now define $\theta : \mathcal{P}\mathbb{R} \to [0, \infty]$ by writing

$$\theta A = \inf\{\sum_{j=0}^{\infty} \lambda I_j : \langle I_j \rangle_{j\in\mathbb{N}} \text{ is a sequence of half-open intervals such that } A \subseteq \bigcup_{j\in\mathbb{N}} I_j\}.$$

θ is called **Lebesgue outer measure** on $\mathbb{R}$.

114D Proposition (a) θ is an outer measure on $\mathbb{R}$.
(b) $\theta I = \lambda I$ for every half-open interval $I \subseteq \mathbb{R}$.

114E Definition Because Lebesgue outer measure is an outer measure, we may use it to construct a measure μ, using Carathéodory's method. This measure is **Lebesgue measure on** $\mathbb{R}$. The sets E measured by μ are called **Lebesgue measurable**.
Sets which are negligible for μ are called **Lebesgue negligible**.

114F Lemma Let $x \in \mathbb{R}$. Then $H_x = \left]-\infty, x\right[$ is Lebesgue measurable for every $x \in \mathbb{R}$.

114G Proposition All Borel subsets of $\mathbb{R}$ are Lebesgue measurable; in particular, all open sets, and all sets of the following classes, together with countable unions of them:
(i) open intervals $]a, b[$, $]-\infty, b[$, $]a, \infty[$, $]-\infty, \infty[$, where $a < b \in \mathbb{R}$;
(ii) closed intervals $[a, b]$, where $a \le b \in \mathbb{R}$;
(iii) half-open intervals $[a, b[$, $]a, b]$, $]-\infty, b]$, $[a, \infty[$, where $a < b$ in $\mathbb{R}$.
We have the following formula for the measures of such sets, writing μ for Lebesgue measure:

$$\mu\,]a, b[= \mu[a, b] = \mu\,[a, b[= \mu\,]a, b] = b - a$$

whenever $a \le b$ in $\mathbb{R}$, while all the unbounded intervals have infinite measure. It follows that every countable subset of $\mathbb{R}$ is measurable and of zero measure.

Version of 21.7.05

115 Lebesgue measure on $\mathbb{R}^r$

Following the very abstract ideas of §§111-113, there is an urgent need for non-trivial examples of measure spaces. By far the most important examples are the Euclidean spaces $\mathbb{R}^r$ with Lebesgue measure, and I now proceed to a definition of these measures (115A-115E), with a few of their basic properties. Except at one point (in the proof of the fundamental lemma 115B) this section does not rely essentially on §114; but nevertheless most students encountering Lebesgue measure for the first time will find it easier to work through the one-dimensional case carefully before embarking on the multi-dimensional case.

115A Definitions (a) For practically the whole of this section (the exception is the proof of Lemma 115B) r will denote a fixed integer greater than or equal to 1. I will use Roman letters a, b, c, d, x, y to denote members of $\mathbb{R}^r$, and Greek letters for their coordinates, so that $a = (\alpha_1, \dots, \alpha_r)$, $b = (\beta_1, \dots, \beta_r)$, $x = (\xi_1, \dots, \xi_r)$.

(b) For the purposes of this section, a **half-open interval** in $\mathbb{R}^r$ is a set of the form $[a, b[= \{x : \alpha_i \le \xi_i < \beta_i \ \forall\ i \le r\}$, where a, $b \in \mathbb{R}^r$. Observe that I allow $\beta_i \le \alpha_i$ in this formula; if this happens for any i, then $[a, b[= \emptyset$.

(c) If $I = [a, b[\subseteq \mathbb{R}^r$ is a half-open interval, then either $I = \emptyset$ or

$$\alpha_i = \inf\{\xi_i : x \in I\}, \quad \beta_i = \sup\{\xi_i : x \in I\}$$

for every $i \le r$; in the latter case, the expression of I as a half-open interval is unique. We may therefore define the r**-dimensional volume** λI of a half-open interval I by setting

$$\lambda\emptyset = 0, \quad \lambda\,[a, b[= \prod_{i=1}^{r} \beta_i - \alpha_i \text{ if } \alpha_i < \beta_i \text{ for every } i.$$

115B Lemma If $I \subseteq \mathbb{R}^r$ is a half-open interval and $\langle I_j \rangle_{j\in\mathbb{N}}$ is a sequence of half-open intervals covering I, then $\lambda I \le \sum_{j=0}^{\infty} \lambda I_j$.

115C Definition Now, and for the rest of this section, define $\theta : \mathcal{P}(\mathbb{R}^r) \to [0,\infty]$ by writing

$$\theta A = \inf\{\sum_{j=0}^{\infty} \lambda I_j : \langle I_j \rangle_{j\in\mathbb{N}} \text{ is a sequence of half-open intervals such that } A \subseteq \bigcup_{j\in\mathbb{N}} I_j\}.$$

This function θ is called **Lebesgue outer measure** on $\mathbb{R}^r$; the phrase is justified by (a) of the next proposition.

115D Proposition (a) θ is an outer measure on $\mathbb{R}^r$.
(b) $\theta I = \lambda I$ for every half-open interval $I \subseteq \mathbb{R}^r$.

115E Definition Because Lebesgue outer measure is an outer measure, we may use it to construct a measure μ, using Carathéodory's method. This measure is **Lebesgue measure on** $\mathbb{R}^r$. The sets E for which μE is defined are called **Lebesgue measurable**.

Sets which are negligible for μ are called **Lebesgue negligible**.

115F Lemma If $i \le r$ and $\xi \in \mathbb{R}$, then $H_{i\xi} = \{y : \eta_i < \xi\}$ is Lebesgue measurable.

115G Proposition All Borel subsets of $\mathbb{R}^r$ are Lebesgue measurable; in particular, all open sets, and all sets of the following classes, together with countable unions of them:

open intervals $]a,b[= \{x : x \in \mathbb{R}^r,\ \alpha_i < \xi_i < \beta_i\ \forall\ i \le r\}$, where $\alpha_i,\ \beta_i \in \mathbb{R} \cup \{-\infty,\infty\}$ for each $i \le r$;

closed intervals $[a,b] = \{x : x \in \mathbb{R}^r,\ \alpha_i \le \xi_i \le \beta_i\ \forall\ i \le r\}$, where $\alpha_i,\ \beta_i \in \mathbb{R} \cup \{-\infty,\infty\}$ for each $i \le r$.

We have the following formula for the measures of such sets, writing μ for Lebesgue measure:

$$\mu\,]a,b[= \mu[a,b] = \prod_{i=1}^{r} \beta_i - \alpha_i$$

whenever $a \le b$ in $\mathbb{R}^r$. Consequently every countable subset of $\mathbb{R}^r$ is measurable and of zero measure.

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Concordance for Chapter 11

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

112E-112F Image measures These paragraphs, referred to in the 2001 and 2003 editions of Volume 2, and the 2003 and 2006 editions of Volume 4, have been moved to 234C-234D in Volume 2.

112Ya Sums of measures This material, referred to in the 2001 and 2003 editions of Volume 2, has been moved to 234G in Volume 2.