Version of 30.9.02

In this introductory volume I set out, at a level which I hope will be suitable for students with no prior knowledge of the Lebesgue (or even Riemann) integral and with only a basic (but thorough) preparation in the techniques of ϵ-δ analysis, the theory of measure and integration up to the convergence theorems (§123). I add a third chapter (Chapter 13) of miscellaneous additional results, mostly chosen as being relatively elementary material necessary for topics treated in Volume 2 which does not have a natural place there.

The title of this volume is a little more emphatic than I should care to try to justify *au pied de la lettre*. I would certainly characterize the construction of Lebesgue measure on $\mathbb{R}$ (§114), the definition of the integral on an abstract measure space (§122) and the convergence theorems (§123) as indispensable. But a teacher who wishes to press on to further topics will find that much of Chapter 13 can be set aside for a while. I say 'teacher' rather than 'student' here, because if you are studying on your own I think you should aim to go slower than the text requires rather than faster; in my view, these ideas are genuinely difficult, and I think you should take the time to get as much practice at relatively elementary levels as you can.

Perhaps this is a suitable moment at which to set down some general thoughts on the teaching of measure theory. I have been teaching analysis for over thirty years now, and one of the few constants over that period has been the feeling, almost universal among teachers of analysis, that we are not serving most of our students well. We have all encountered students who are not stupid – who are indeed quite good at mathematics – but who seem to have a disproportionate difficulty with rigorous analysis. They are so exhausted and demoralised by the technical problems that they cannot make sense or use even of the knowledge they achieve. The natural reaction to this is to try to make courses shorter and easier. But I think that this makes it even more likely that at the end of the semester your students will be stranded in thorn-bushes half way up the mountain. Specifically, with Lebesgue measure, you are in danger of spending twenty hours teaching them how to integrate the indicator function of the rationals. This is not what the subject is for. Lebesgue's own presentations of the subject (Lebesgue 1904, Lebesgue 1918) emphasize the convergence theorems and the Fundamental Theorem of Calculus. I have put the former in Volume 1 and the latter in Volume 2, but it does seem to me that unless your students themselves want to know when one can expect to be able to interchange a limit and an integral, or which functions are indefinite integrals, or what the completions of $C([0,1])$ under the norms $\| \|_1$, $\| \|_2$ look like, then it is going to be very difficult for them to make anything of this material; and if you really cannot reach the point of explaining at least a couple of these matters in terms which they can appreciate, then it may not be worth starting. I would myself choose rather to omit a good many proofs than to come to the theorems for which the subject was created so late in the course that two thirds of my class have already given up before they are covered.

Of course I and others have followed that road too, with no better results (though usually with happier students) than we obtain by dotting every i and crossing every t in the proofs. Nearly every time I am consulted by a non-specialist who wants to be told a theorem which will solve his problem, I am reminded that pure mathematics in general, and analysis in particular, does not lie in the *theorems* but in the *proofs*. In so far as I have been successful in answering such questions, it has usually been by making a trifling adjustment to a standard argument to produce a non-standard theorem. The ideas are in the details. You have not understood Carathéodory's construction (§113) until you can, at the very least, reliably reproduce the argument which shows that it works. In the end, there is no alternative to going over every step of the ground, and while I have occasionally been ruthless in cutting out topics which seem to me to be marginal, I have tried to make sure – at the expense, frequently, of pedantry – that every necessary idea is signalled.

Faced, therefore, with any particular class, I believe that a teacher must compromise between scope and completeness. Exactly which compromises are most appropriate will depend on factors which it would be a waste of time for me to guess at. This volume is supposed to be a possible text on which to base a course; but I hope that no lecturer will set her class to read it at so many pages a week. My primary aim is to provide a concise and coherent basis on which to erect the structure of the later volumes. This involves me in pursuing, at more than one point, approaches which take slightly more difficult paths for the sake of developing a more refined technique. (Perhaps the most salient of these is my insistence that an integrable function need not be defined everywhere on the underlying measure space; see §§121-122.) It is the responsibility of the individual teacher to decide for herself whether such refinements are appropriate to the needs of her students, and, if not, to show them what translations are needed.

The above paragraphs are directed at teachers who are, supposedly, competent in the subject – certainly past the level treated in this volume – and who have access to some of the many excellent books already available, so that if they take the trouble to think out their aims, they should be able to choose which elements of my presentation are suitable. But I must also consider the position of a student who is setting out to learn this material on his own. I trust that you have understood from what I have already written that you should not be afraid to look ahead. You could, indeed, do worse than go to Volume 2, and take one of the wonderful theorems there – the Fundamental Theorem of Calculus (§222), for instance, or, if you are very ambitious, the strong law of large numbers (§273) – and use the index and the cross-references to try to extract a proof from first principles. If you are successful you will have every right to congratulate yourself. In the periods in which success seems elusive, however, you should be working systematically through the 'basic exercises' in the sections which seem to be relevant; and if all else fails, start again at the beginning. Mathematics is a difficult subject, that is why it is worth doing, and almost every section here contains some essential idea which you could not expect to find alone.

Version of 4.1.04

Chapter 11

Measure spaces

In this chapter I set out the fundamental concept of 'measure space', that is, a set in which some (not, as a rule, all) subsets may be assigned a 'measure', which you may wish to interpret as area, or mass, or volume, or thermal capacity, or indeed almost anything which you would expect to be additive – I mean, that the measure of the union of two disjoint sets should be the sum of their measures. The actual definition (in 112A) is not obvious, and depends essentially on certain technical features which make a preparatory section (§111) advisable. Furthermore, even with the definition well in hand, the original and most important examples of measures, Lebesgue measure on Euclidean space, remain elusive. I therefore devote a section (§113) to a method of constructing measures, before turning to the details of the arguments needed for Lebesgue measure in §§114-115. Thus the structure of the chapter is three sections of general theory followed by two (which are closely similar) on particular examples. I should say that the general theory is essentially easier; but it does rely on facility with certain manipulations of families of sets which may be new to you.

At some point I ought to comment on my arrangement of the material, and it may be helpful if I do so before you start work on this chapter. One of the many fundamental questions which any author on the subject must decide, is whether to begin with 'general' measure theory or with 'Lebesgue' measure and integration. The point is that Lebesgue measure is rather more than just the most important example of a measure space. It is so close to the heart of the subject that the great majority of the ideas of the elementary theory can be fully realised in theorems about Lebesgue measure. Looking ahead to Volume 2, I find that, with the exception of Chapter 21 – which is specifically devoted to extending your ideas of what measure spaces can be – only Chapter 27 and the second half of Chapter 25 really need the general theory to make sense, while Chapters 22, 26 and 28 are specifically about Lebesgue measure. Volume 3 is another matter, but even there more than half the mathematical content can be expressed in terms of Lebesgue measure. If you take the view, as I certainly do when it suits my argument, that the business of pure mathematics is to express and extend the logical capacity of the human mind, and that the actual theorems we work through are merely vehicles for the ideas, then you can correctly point out that all the really important things in the present volume can be done without going to the trouble of formulating a general theory of abstract measure spaces; and that by studying the relatively concrete example of Lebesgue measure on r-dimensional Euclidean space you can avoid a variety of irrelevant distractions.

If you are quite sure, as a teacher, that none of your pupils will wish to go beyond the elementary theory, there is something to be said for this view. I believe, however, that it becomes untenable if you wish to prepare any of your students for more advanced ideas. The difficulty is that, with the best will in the world, anyone who has worked through the full theory of Lebesgue measure, and then comes to the theory of abstract measure spaces, is likely to go through it too fast, and at the end find himself uncertain about just which ninety per cent of the facts he knows are generally applicable. I believe it is safer to keep the special properties of Lebesgue measure clearly labelled as such from the beginning.

It is of course the besetting sin of mathematics teachers at this level, to teach a class of twenty in a manner appropriate to perhaps two of them. But in the present case my own judgement is that very few students who are ready for the course at all will have any difficulty with the extra level of abstraction involved in 'Let (X, Σ, μ) be a measure space, ...'. I do assume knowledge of elementary linear algebra, and the grammar, at least, of arbitrary measure spaces is no worse than the grammar of arbitrary linear spaces. Moreover, the Lebesgue theory already involves statements of the form 'if E is a Lebesgue measurable set, ...', and in my experience students who can cope with quantification over subsets of the reals are not deterred by quantification over sets of sets (which anyway is necessary for any elementary description of the σ-algebra of Borel sets). So I believe that here, at least, the extra generality of the 'professional' approach is not an obstacle to the amateur.

I have written all this here, rather than later in the chapter, because I do wish to give you the choice. And if your choice is to learn the Lebesgue theory first, and leave the general theory to later, this is how to do it. You should read

paragraphs 114A-114C
114D, with 113A-113B and 112Ba, 112Bc
114E, with 113C-113D, 111A, 112A, 112Bb

114F

114G, with 111G and 111C-111F,

and then continue with Chapter 12. At some point, of course, you should look at the exercises for §§112-113; but, as in Chapters 12-13, you will do so by translating 'Let (X, Σ, μ) be a measure space' into 'Let μ be Lebesgue measure on $\mathbb{R}$, and Σ the σ-algebra of Lebesgue measurable sets'. Similarly, when you look at 111X-111Y, you will take Σ to be *either* the σ-algebra of Lebesgue measurable sets *or* the σ-algebra of Borel subsets of $\mathbb{R}$.

Version of 26.1.05

111 σ-algebras

In the introduction to this chapter I remarked that a measure space is 'a set in which some (not, as a rule, all) subsets may be assigned a measure'. All ordinary concepts of 'length' or 'area' or 'volume' apply only to reasonably regular sets. Modern measure theory is remarkably powerful in that an extraordinary variety of sets are regular enough to be measured; but we must still expect some limitation, and when studying any measure a proper understanding of the class of sets which it measures will be central to our work. The basic definition here is that of 'σ-algebra of sets'; all measures in the standard theory are defined on such collections. I therefore begin with a statement of the definition, and a brief discussion of the properties, of these classes.

111A Definition Let X be a set. A **σ-algebra of subsets of** X (sometimes called a **σ-field**) is a family Σ of subsets of X such that

(i) $\emptyset \in \Sigma$;

(ii) for every $E \in \Sigma$, its complement $X \setminus E$ in X belongs to Σ;

(iii) for every sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in Σ, its union $\bigcup_{n\in\mathbb{N}} E_n$ belongs to Σ.

111B Remarks (a) Almost any new subject in pure mathematics is likely to begin with definitions. At this point there is no substitute for rote learning. These definitions encapsulate years, sometimes centuries, of thought by many people; you cannot expect that they will always correspond to familiar ideas.

(b) Nevertheless, you should always seek immediately to find ways of making new definitions more concrete by finding examples within your previous mathematical experience. In the case of 'σ-algebra', the really important examples, to be described below, are going to be essentially new – supposing, that is, that you need to read this chapter at all. However, two examples should be immediately accessible to you, and you should bear these in mind henceforth:

(i) for any X, $\Sigma = \{\emptyset, X\}$ is a σ-algebra of subsets of X;

(ii) for any X, $\mathcal{P}X$, the set of all subsets of X, is a σ-algebra of subsets of X.

These are of course the smallest and largest σ-algebras of subsets of X, and while we shall spend little time with them, both are in fact significant.

***(c)** The phrase **measurable space** is often used to mean a pair (X, Σ), where X is a set and Σ is a σ-algebra of subsets of X; but I myself prefer to avoid this terminology, unless greatly pressed for time, as in fact many of the most interesting examples of such objects have no useful measures associated with them.

111C Infinite unions and intersections If you have not seen infinite unions before, it is worth pausing over the formula $\bigcup_{n\in\mathbb{N}} E_n$. This is the set of points belonging to one or more of the sets E_n; we may write it as

$$\bigcup_{n\in\mathbb{N}} E_n = \{x : \exists\ n \in \mathbb{N}, x \in E_n\}$$
$$= E_0 \cup E_1 \cup E_2 \cup \dots .$$

(I write $\mathbb{N}$ for the set of natural numbers $\{0, 1, 2, 3, \dots\}$.) In the same way,

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} E_n &= \{x : x \in E_n \ \forall\, n \in \mathbb{N}\}\\ &= E_0 \cap E_1 \cap E_2 \cap \dots .\end{aligned}$$

It is characteristic of the elementary theory of measure spaces that it demands greater facility with the set-operations $\cup$, $\cap$, $\setminus$ ('set difference': $E \setminus F = \{x : x \in E,\ x \notin F\}$), $\triangle$ ('symmetric difference': $E\triangle F = (E \setminus F) \cup (F \setminus E) = (E \cup F) \setminus (E \cap F)$) than you have probably needed before, with the added complication of infinite unions and intersections. I strongly advise spending at least a little time with Exercise 111Xa at some point.

111D Elementary properties of σ-algebras If Σ is a σ-algebra of subsets of X, then it has the following properties.

(a) $E \cup F \in \Sigma$ for all E, $F \in \Sigma$. **P** For if E, $F \in \Sigma$, set $E_0 = E$, $E_n = F$ for $n \ge 1$; then $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ and $E \cup F = \bigcup_{n\in\mathbb{N}} E_n \in \Sigma$. **Q**

(b) $E \cap F \in \Sigma$ for all E, $F \in \Sigma$. **P** By (ii) of the definition in 111A, $X \setminus E$ and $X \setminus F \in \Sigma$; by (a) of this paragraph, $(X \setminus E) \cup (X \setminus F) \in \Sigma$; by 111A(ii) again, $X \setminus ((X \setminus E) \cup (X \setminus F)) \in \Sigma$; but this is just $E \cap F$. **Q**

(c) $E \setminus F \in \Sigma$ for all E, $F \in \Sigma$. **P** $E \setminus F = E \cap (X \setminus F)$. **Q**

(d) Now suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and consider

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} E_n &= \{x : x \in E_n \ \forall\, n \in \mathbb{N}\}\\ &= E_0 \cap E_1 \cap E_2 \cap \dots\\ &= X \setminus \bigcup_{n\in\mathbb{N}} (X \setminus E_n);\end{aligned}$$

this also belongs to Σ.

111E More on infinite unions and intersections (a) So far I have considered infinite unions and intersections only in the context of sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ indexed by the set $\mathbb{N}$ of natural numbers itself. Many others will arise more or less naturally in the pages ahead. Consider, for instance, sets of the form

$$\textstyle\bigcup_{n\ge 4} E_n = E_4 \cup E_5 \cup E_6 \cup \dots,$$

$$\textstyle\bigcup_{n\in\mathbb{Z}} E_n = \{x : \exists\ n \in \mathbb{Z},\ x \in E_n\} = \dots \cup E_{-2} \cup E_{-1} \cup E_0 \cup E_1 \cup E_2 \cup \dots,$$

$$\textstyle\bigcup_{q\in\mathbb{Q}} E_q = \{x : \exists\ q \in \mathbb{Q},\ x \in E_q\},$$

where I write $\mathbb{Z}$ for the set of all integers and $\mathbb{Q}$ for the set of rational numbers. If every E_n, E_q belongs to a σ-algebra Σ, so will these unions. On the other hand,

$$\textstyle\bigcup_{t\in[0,1]} E_t = \{x : \exists\ t \in [0,1],\ x \in E_t\}$$

may fail to belong to a σ-algebra containing every E_t, and it is of the greatest importance to develop an intuition for those index sets, like $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$, which are 'safe' in this context, and those which are not.

(b) I rather hope that you have seen enough of Cantor's theory of infinite sets to make the following remarks a restatement of familiar material; but if not, I hope that they can stand as a first, and very partial, introduction to these ideas. The point about the first three examples is that we can re-index the families of sets involved as simple sequences of sets. For the first one, this is elementary; write $E'_n = E_{n+4}$ for $n \in \mathbb{N}$, and see that $\bigcup_{n\ge 4} E_n = \bigcup_{n\in\mathbb{N}} E'_n \in \Sigma$. For the other two, we need to know something about the sets $\mathbb{Z}$ and $\mathbb{Q}$. We can find sequences $\langle k_n\rangle_{n\in\mathbb{N}}$ of integers, and $\langle q_n\rangle_{n\in\mathbb{N}}$ of rational numbers, such that every integer appears (at least once) as a k_n, and every rational number appears (at least once) as a q_n; that is, the functions $n \mapsto k_n : \mathbb{N} \to \mathbb{Z}$ and $n \mapsto q_n : \mathbb{N} \to \mathbb{Q}$ are surjective. **P** There are many ways of doing this; one is to set

$$\begin{aligned} k_n &= \frac{n}{2} \text{ for even } n, \\ &= -\tfrac{n+1}{2} \text{ for odd } n, \\ q_n &= \tfrac{n-m^3-m^2}{m+1} \text{ if } m \in \mathbb{N} \text{ and } m^3 \le n < (m+1)^3. \end{aligned}$$

(You should check carefully that these formulae do indeed do what I claim they do.) **Q** Now, to deal with $\bigcup_{n\in\mathbb{Z}} E_n$, we can set

$$E'_n = E_{k_n} \in \Sigma$$

for $n \in \mathbb{N}$, so that

$$\bigcup_{n\in\mathbb{Z}} E_n = \bigcup_{n\in\mathbb{N}} E_{k_n} = \bigcup_{n\in\mathbb{N}} E'_n \in \Sigma,$$

while for the other case we have

$$\bigcup_{q\in\mathbb{Q}} E_q = \bigcup_{n\in\mathbb{N}} E_{q_n} \in \Sigma.$$

Note that the first case $\bigcup_{n\ge 4} E_n$ can be thought of as an application of the same principle; the map $n \mapsto n+4$ is a surjection from $\mathbb{N}$ onto $\{4,5,6,7,\dots\}$.

111F Countable sets (a) The common feature of the sets $\{n : n \ge 4\}$, $\mathbb{Z}$ and $\mathbb{Q}$ which makes this procedure possible is that they are 'countable'. For our purposes here, the most natural definition of countability is the following: a set K is **countable** if either it is empty or there is a surjection from $\mathbb{N}$ onto K. In this case, if Σ is a σ-algebra of sets and $\langle E_k\rangle_{k\in K}$ is a family in Σ indexed by K, then $\bigcup_{k\in K} E_k \in \Sigma$. **P** For if $n \mapsto k_n : \mathbb{N} \to K$ is a surjection, then $E'_n = E_{k_n} \in \Sigma$ for every $n \in \mathbb{N}$, and $\bigcup_{k\in K} E_k = \bigcup_{n\in\mathbb{N}} E'_n \in \Sigma$. This leaves out the case $K = \emptyset$; but in this case the natural interpretation of $\bigcup_{k\in K} E_k$ is

$$\{x : \exists\ k \in \emptyset,\, x \in E_k\}$$

which is itself $\emptyset$, and therefore belongs to Σ by clause (i) of 111A. **Q** (In a sense this treatment of $\emptyset$ is a conventional matter; but there are various contexts in which we shall wish to discuss $\bigcup_{k\in K} E_k$ without checking whether K actually has any members, and we need to be clear about what we will do in such cases.)

(b) There is an extensive, and enormously important, theory concerning countable sets. The only fragments which I think we must have explicit at this point are the following. (In §1A1 I add a few words to link this presentation to conventional approaches.)

(i) If K is countable and $L \subseteq K$, then L is countable. **P** If $L = \emptyset$, this is immediate. Otherwise, take any $l^* \in L$, and a surjection $n \mapsto k_n : \mathbb{N} \to K$ (of course K also is not empty, as $l^* \in K$); set $l_n = k_n$ if $k_n \in L$, l^* otherwise; then $n \mapsto l_n : \mathbb{N} \to L$ is a surjection. **Q**

(ii) The Cartesian product $\mathbb{N} \times \mathbb{N} = \{(m,n) : m,\, n \in \mathbb{N}\}$ is countable. **P** For each $n \in \mathbb{N}$, let k_n, $l_n \in \mathbb{N}$ be such that $n+1 = 2^{k_n}(2l_n+1)$; that is, k_n is the power of 2 in the prime factorisation of $n+1$, and $2l_n+1$ is the (necessarily odd) number $(n+1)/2^{k_n}$. Now $n \mapsto (k_n, l_n)$ is a surjection from $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$. **Q** It will be important to us later to know that $n \mapsto (k_n, l_n)$ is actually a bijection, as is readily checked.

(iii) It follows that if K and L are countable sets, so is $K \times L$. **P** If either K or L is empty, so is $K \times L$, so in this case $K \times L$ is certainly countable. Otherwise, let $\phi : \mathbb{N} \to K$ and $\psi : \mathbb{N} \to L$ be surjections; then we have a surjection $\theta : \mathbb{N}\times\mathbb{N} \to K \times L$ defined by setting $\theta(m,n) = (\phi(m), \psi(n))$ for all m, $n \in \mathbb{N}$. Now we know from (ii) just above that there is also a surjection $\chi : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$, so that $\theta\chi : \mathbb{N} \to K \times L$ is a surjection, and $K \times L$ must be countable. **Q**

(iv) An induction on r now shows us that if K_1, $K_2, \dots, K_r$ are countable sets, so is $K_1 \times \dots \times K_r$. In particular, such sets as $\mathbb{Q}^r \times \mathbb{Q}^r$ will be countable, for any integer $r \ge 1$.

(c) Putting 111Dd above together with these ideas, we see that if Σ is a σ-algebra of sets, K is a non-empty countable set, and $\langle E_k\rangle_{k\in K}$ is a family in Σ, then

$$\bigcap_{k\in K} E_k = \{x : x \in E_k \ \forall\, k \in K\}$$

belongs to Σ. **P** Let $n \mapsto k_n : \mathbb{N} \to K$ be a surjection; then $\bigcap_{k\in K} E_k = \bigcap_{n\in\mathbb{N}} E_{k_n} \in \Sigma$, as in 111Dd. **Q**

Note that there is a difficulty with the notion of $\bigcap_{k\in K} E_k$ if $K = \emptyset$; the natural interpretation of this formula is to read it as the universal class. So ordinarily, when there is any possibility that K might be empty, one needs some such formulation as $X \cap \bigcap_{k\in K} E_k$.

(d) As an example of the way in which these ideas will be used, consider the following. Suppose that X is a set, Σ is a σ-algebra of subsets of X, and $\langle E_{qn}\rangle_{q\in\mathbb{Q},n\in\mathbb{N}}$ is a family in Σ. Then

$$E = \bigcap_{q\in\mathbb{Q},q<\sqrt{2}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} E_{qn} = \bigcap_{q\in\mathbb{Q},q<\sqrt{2}} (\bigcup_{m\in\mathbb{N}} (\bigcap_{n\ge m} E_{qn})) \in \Sigma.$$

P Set $F_{qm} = \bigcap_{n\ge m} E_{qn} = \bigcap_{n\in\mathbb{N}} E_{q,m+n}$ for $q \in \mathbb{Q}$ and $m \in \mathbb{N}$; then every F_{qm} belongs to Σ, by 111Dd or (c) above. Set $G_q = \bigcup_{m\in\mathbb{N}} F_{qm}$ for $q \in \mathbb{Q}$; then every G_q belongs to Σ, by 111A(iii). Set $K = \{q : q \in \mathbb{Q},\ q < \sqrt{2}\}$; then K is countable, by 111E and (b-i) of this paragraph. So $\bigcap_{q\in K} G_q$ belongs to Σ, by (c). But $E = \bigcap_{q\in K} G_q$. **Q**

(e) And one final remark, which I give without proof here – though many proofs will be implicit in the work below, and I spell one out in 1A1Ha –

The set $\mathbb{R}$ of real numbers is not countable.

So you must resist any temptation to look for a list $a_0, a_1, \ldots$ running over the whole set of real numbers.

111G Borel sets I can describe here one type of non-trivial σ-algebra; the formulation is rather abstract, but the technique is important and the terminology is part of the basic vocabulary of measure theory.

(a) Let X be a set, and let $\mathfrak{S}$ be any non-empty family of σ-algebras of subsets of X. (Thus a *member* of $\mathfrak{S}$ is itself a *family* of sets; $\mathfrak{S} \subseteq \mathcal{P}(\mathcal{P}X)$.) Then

$$\bigcap \mathfrak{S} = \{E : E \in \Sigma \text{ for every } \Sigma \in \mathfrak{S}\},$$

the intersection of all the σ-algebras belonging to $\mathfrak{S}$, is a σ-algebra of subsets of X. **P** (i) By hypothesis, $\mathfrak{S}$ is not empty; take $\Sigma_0 \in \mathfrak{S}$; then $\bigcap\mathfrak{S} \subseteq \Sigma_0 \subseteq \mathcal{P}X$, so every member of $\bigcap\mathfrak{S}$ is a subset of X. (ii) $\emptyset \in \Sigma$ for every $\Sigma \in \mathfrak{S}$, so $\emptyset \in \bigcap\mathfrak{S}$. (iii) If $E \in \bigcap\mathfrak{S}$ then $E \in \Sigma$ for every $\Sigma \in \mathfrak{S}$, so $X \setminus E \in \Sigma$ for every $\Sigma \in \mathfrak{S}$ and $X \setminus E \in \bigcap\mathfrak{S}$. (iv) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\bigcap\mathfrak{S}$. Then for every $\Sigma \in \mathfrak{S}$, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, so $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$; as Σ is arbitrary, $\bigcup_{n\in\mathbb{N}} E_n \in \bigcap\mathfrak{S}$. **Q**

(b) Now let $\mathcal{A}$ be any family of subsets of X. Consider

$$\mathfrak{S} = \{\Sigma : \Sigma \text{ is a } \sigma\text{-algebra of subsets of } X,\ \mathcal{A} \subseteq \Sigma\}.$$

By definition, $\mathfrak{S}$ is a family of σ-algebras of subsets of X; also, it is not empty, because $\mathcal{P}X \in \mathfrak{S}$. So $\Sigma_{\mathcal{A}} = \bigcap\mathfrak{S}$ is a σ-algebra of subsets of X. Because $\mathcal{A} \subseteq \Sigma$ for every $\Sigma \in \mathfrak{S}$, $\mathcal{A} \subseteq \Sigma_{\mathcal{A}}$; thus $\Sigma_{\mathcal{A}}$ itself belongs to $\mathfrak{S}$; it is the smallest σ-algebra of subsets of X including $\mathcal{A}$.

We say that $\Sigma_{\mathcal{A}}$ is the σ-algebra of subsets of X **generated by** $\mathcal{A}$.

Examples (i) For any X, the σ-algebra of subsets of X generated by $\emptyset$ is $\{\emptyset, X\}$.

(ii) The σ-algebra of subsets of $\mathbb{N}$ generated by $\{\{n\} : n \in \mathbb{N}\}$ is $\mathcal{P}\mathbb{N}$.

(c)(i) We say that a set $G \subseteq \mathbb{R}$ is **open** if for every $x \in G$ there is a $\delta > 0$ such that the open interval $]x-\delta, x+\delta[$ is included in G.

(ii) Similarly, for any $r \ge 1$, we say that a set $G \subseteq \mathbb{R}^r$ is **open** in $\mathbb{R}^r$ if for every $x \in G$ there is a $\delta > 0$ such that $\{y : \|y-x\| < \delta\} \subseteq G$, where for $z = (\zeta_1, \ldots, \zeta_r) \in \mathbb{R}^r$ I write $\|z\| = \sqrt{\sum_{i=1}^r |\zeta_i|^2}$; thus $\|y-x\|$ is just the ordinary Euclidean distance from y to x.

(d) Now the **Borel sets** of $\mathbb{R}$, or of $\mathbb{R}^r$, are just the members of the σ-algebra of subsets of $\mathbb{R}$ or $\mathbb{R}^r$ generated by the family of open sets of $\mathbb{R}$ or $\mathbb{R}^r$; the σ-algebra itself is called the **Borel σ-algebra** in each case.

(e) Some readers will rightly feel that the development here gives very little idea of what a Borel set is 'really' like. (Open sets are much easier; see 111Ye.) In fact the importance of the concept derives largely from the fact that there are alternative, more explicit, and in a sense more concrete, ways of describing Borel sets. I shall return to this topic in Chapter 42 in Volume 4.

111X Basic exercises >(a) Practise the algebra of infinite unions and intersections until you can confidently interpret such formulae as

$$E \cap (\bigcup_{n\in\mathbb{N}} F_n), \qquad \bigcup_{n\in\mathbb{N}}(E_n \setminus F), \qquad E \cup (\bigcap_{n\in\mathbb{N}} F_n),$$

$$\bigcup_{n\in\mathbb{N}}(E \setminus F_n), \qquad E \setminus (\bigcup_{n\in\mathbb{N}} F_n), \qquad \bigcap_{n\in\mathbb{N}}(E_n \setminus F),$$

$$E \setminus (\bigcap_{n\in\mathbb{N}} F_n), \qquad \bigcap_{n\in\mathbb{N}}(E \cup F_n), \qquad (\bigcup_{n\in\mathbb{N}} E_n) \setminus F,$$

$$\bigcup_{n\in\mathbb{N}}(E \cap F_n), \qquad (\bigcap_{n\in\mathbb{N}} E_n) \setminus F, \qquad \bigcap_{n\in\mathbb{N}}(E \setminus F_n),$$

$$(\bigcup_{n\in\mathbb{N}} E_n) \cap (\bigcup_{n\in\mathbb{N}} F_n), \qquad \bigcap_{m,n\in\mathbb{N}}(E_m \setminus F_n), \qquad (\bigcap_{n\in\mathbb{N}} E_n) \cup (\bigcap_{n\in\mathbb{N}} F_n),$$

$$\bigcap_{m,n\in\mathbb{N}}(E_m \cup F_n), \qquad (\bigcap_{n\in\mathbb{N}} E_n) \setminus (\bigcup_{n\in\mathbb{N}} F_n), \qquad \bigcup_{m,n\in\mathbb{N}}(E_m \cap F_n),$$

and, in particular, can identify the nine pairs into which these formulae naturally fall.

>(b) In $\mathbb{R}$, show that all 'open intervals' $]a,b[$, $]-\infty,b[$, $]a,\infty[$ are open sets, and that all intervals (bounded or unbounded, open, closed or half-open) are Borel sets.

>(c) Let X and Y be sets and Σ a σ-algebra of subsets of X. Let $\phi : X \to Y$ be a function. Show that $\{F : F \subseteq Y,\ \phi^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y. (See 1A1B for the notation here.)

>(d) Let X and Y be sets and T a σ-algebra of subsets of Y. Let $\phi : X \to Y$ be a function. Show that $\{\phi^{-1}[F] : F \in \mathrm{T}\}$ is a σ-algebra of subsets of X.

(e) Let X be a set, $\mathcal{A}$ a family of subsets of X, and Σ the σ-algebra of subsets of X generated by $\mathcal{A}$. Suppose that Y is another set and $\phi : Y \to X$ a function. Show that $\{\phi^{-1}[E] : E \in \Sigma\}$ is the σ-algebra of subsets of Y generated by $\{\phi^{-1}[A] : A \in \mathcal{A}\}$.

(f) Let X be a set, $\mathcal{A}$ a family of subsets of X, and Σ the σ-algebra of subsets of X generated by $\mathcal{A}$. Suppose that $Y \subseteq X$. Show that $\{E \cap Y : E \in \Sigma\}$ is the σ-algebra of subsets of Y generated by $\{A \cap Y : A \in \mathcal{A}\}$.

111Y Further exercises (a) In $\mathbb{R}^r$, where $r \geq 1$, show that $G + a = \{x + a : x \in G\}$ is open whenever $G \subseteq \mathbb{R}^r$ is open and $a \in \mathbb{R}^r$. Hence show that $E + a$ is a Borel set whenever $E \subseteq \mathbb{R}^r$ is a Borel set and $a \in \mathbb{R}^r$. (*Hint*: show that $\{E : E + a$ is a Borel set$\}$ is a σ-algebra containing all open sets.)

(b) Let X be a set, Σ a σ-algebra of subsets of X and A any subset of X. Show that $\{(E \cap A) \cup (F \setminus A) : E,\ F \in \Sigma\}$ is a σ-algebra of subsets of X, the σ-algebra generated by $\Sigma \cup \{A\}$.

(c) Let $G \subseteq \mathbb{R}^2$ be an open set. Show that all the horizontal and vertical sections

$$\{\xi : (\xi,\eta) \in G\}, \quad \{\xi : (\eta,\xi) \in G\}$$

of G are open subsets of $\mathbb{R}$.

(d) Let $E \subseteq \mathbb{R}^2$ be a Borel set. Show that all the horizontal and vertical sections

$$\{\xi : (\xi,\eta) \in E\}, \quad \{\xi : (\eta,\xi) \in E\}$$

of E are Borel subsets of $\mathbb{R}$. (*Hint*: show that the family of subsets of $\mathbb{R}^2$ whose sections are all Borel sets is a σ-algebra of subsets of $\mathbb{R}^2$ containing all the open sets.)

(e) Let $G \subseteq \mathbb{R}$ be an open set. Show that G is uniquely expressible as the union of a countable (possibly empty) family $\mathcal{I}$ of open intervals (the 'components' of G) no two of which have any point in common. (*Hint*: for x, $y \in G$ say that $x \sim y$ if every point between x and y belongs to G. Show that $\sim$ is an equivalence relation. Let $\mathcal{I}$ be the set of equivalence classes.)

111 Notes and comments I suppose that the most important concept in this section is the one introduced tangentially in 111E-111F, the idea of 'countable' set. While it is possible to avoid much of the formal theory of infinite sets for the time being, I do not think it is possible to make sense of this chapter without a firm notion of the difference between 'finite' and 'infinite', and some intuitions concerning 'countability'. In particular, you must remember that infinite sets are not, in general, countable, and that σ-algebras are not, in general, closed under arbitrary unions.

The next thing to be sure of is that you can cope with the set-theoretic manipulations here, so that such formulae as $\bigcap_{n\in\mathbb{N}} E_n = X \setminus \bigcup_{n\in\mathbb{N}}(X \setminus E_n)$ (111Dd) are, if not yet transparent, at least not alarming. A large proportion of the volume will be expressed in this language, and reasonable fluency is essential.

Finally, for those who are looking for an actual idea to work on straight away, I offer the concept of σ-algebra 'generated' by a collection $\mathcal{A}$ (111G). The point of the definition here is that it involves consideration of a family $\mathfrak{S} \in \mathcal{P}(\mathcal{P}(\mathcal{P}X))$, even though both $\mathcal{A}$ and $\Sigma_{\mathcal{A}}$ are subsets of $\mathcal{P}X$; we need to work a layer or two up in the hierarchy of power sets. You may have seen, for instance, the concept of 'linear subspace U generated by vectors $u_1, \ldots , u_n$'. This can be defined as the intersection of all the linear subspaces containing the vectors $u_1, \ldots , u_n$, which is the method corresponding to that of 111Ga-b; but it also has an 'internal' definition, as the set of vectors expressible as $\alpha_1 u_1 + \ldots + \alpha_n u_n$ for scalars α_i. For σ-algebras, however, there is no such simple 'internal' definition available (though there is a great deal to be said in this direction which I think we are not yet ready for; some ideas may be found in §136). This is primarily because of (iii) in the definition 111A; a σ-algebra must be closed under an infinitary operation, that is, the operation of union applied to infinite sequences of sets. By contrast, a linear subspace of a vector space need be closed only under the finitary operations of scalar multiplication and addition, each involving at most two vectors at a time.

Version of 20.2.05/20.8.08

112 Measure spaces

We are now, I hope, ready for the second major definition, the definition on which all the work of this treatise is based.

112A Definition A **measure space** is a triple (X, Σ, μ) where
(i) X is a set;
(ii) Σ is a σ-algebra of subsets of X;
(iii) $\mu : \Sigma \to [0, \infty]$ is a function such that
(α) $\mu\emptyset = 0$;
(β) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, then $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$.
In this context, members of Σ are called **measurable** sets, and μ is called a **measure on** X.

112B Remarks (a) The use of ∞ In (iii) of the definition above, I declare that μ is to be a function taking values in '$[0, \infty]$', that is, the set comprising the non-negative real numbers with '∞' adjoined. I expect that you have already encountered various uses of the symbol ∞ in analysis; I hope you have realised that it means rather different things in different contexts, and that it is necessary to establish clear conventions for its use each time. The '∞ of measure' corresponds to the notion of infinite length or area or volume. The basic operation we need to perform on it is addition: $\infty + a = a + \infty = \infty$ for every $a \in [0, \infty[$ (that is, every real number $a \geq 0$), and $\infty + \infty = \infty$. This renders $[0, \infty]$ a semigroup under addition. It will be reasonably safe to declare $\infty - a = \infty$ for every $a \in \mathbb{R}$; but we must absolutely decline to interpret the formula $\infty - \infty$. As for multiplication, it turns out that it is usually right to interpret $\infty \cdot \infty$, $a \cdot \infty$ and $\infty \cdot a$ as ∞ for $a > 0$, while $0 \cdot \infty = \infty \cdot 0$ can generally be taken as 0.

We also have a natural total ordering of $[0, \infty]$, writing $a < \infty$ for every $a \in [0, \infty[$. This gives an idea of supremum and infimum of an arbitrary (non-empty) subset of $[0, \infty]$; and it will often be right to interpret

$\inf \emptyset$ as ∞, but I will try to signal this particular convention each time it is relevant. We also have a notion of limit; if $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in $[0,\infty]$, then it converges to $u \in [0,\infty]$ if

for every $v < u$ there is an $n_0 \in \mathbb{N}$ such that $v \le u_n$ for every $n \ge n_0$,

for every $v > u$ there is an $n_0 \in \mathbb{N}$ such that $v \ge u_n$ for every $n \ge n_0$.

Of course if $u = 0$ or $u = \infty$ then one of these clauses will be vacuously satisfied.

(See also §135.)

(b) I should say plainly what I mean by a 'disjoint' sequence: a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ is **disjoint** if no point belongs to more than one E_n, that is, if $E_m \cap E_n = \emptyset$ for all distinct m, $n \in \mathbb{N}$. Note that there is no bar here on one, or many, of the E_n being the empty set.

Similarly, if $\langle E_i \rangle_{i\in I}$ is a family of sets indexed by an arbitrary set I, it is **disjoint** if $E_i \cap E_j = \emptyset$ for all distinct i, $j \in I$.

(c) In interpreting clause (iii-β) of the definition above, we need to assign values to sums $\sum_{n=0}^{\infty} u_n$ for arbitrary sequences $\langle u_n \rangle_{n\in\mathbb{N}}$ in $[0,\infty]$. The natural way to do this is to say that $\sum_{n=0}^{\infty} u_n = \lim_{n\to\infty} \sum_{m=0}^{n} u_m$, using the definitions sketched in (a). If one of the u_m is itself infinite, say $u_k = \infty$, then $\sum_{m=0}^{n} u_m = \infty$ for every $n \ge k$, so of course $\sum_{n=0}^{\infty} u_n = \infty$. If all the u_m are finite, then, because they are all non-negative, the sequence $\langle \sum_{m=0}^{n} u_m \rangle_{n\in\mathbb{N}}$ of partial sums is monotonic non-decreasing, and either has a finite limit $\sum_{n=0}^{\infty} u_n \in \mathbb{R}$, or diverges to ∞; in which case we again interpret $\sum_{n=0}^{\infty} u_n$ as ∞.

(d) Once again, the important examples of measure spaces will have to wait until §§114 and 115 below. However, I can describe immediately one particular class of measure space, which should always be borne in mind, though it does not give a good picture of the most important and interesting parts of the subject. Let X be any set, and let $h : X \to [0,\infty]$ be any function. For every $E \subseteq X$ write $\mu E = \sum_{x\in E} h(x)$. To interpret this sum, note that there is no difficulty for finite sets E (taking $\sum_{x\in\emptyset} h(x) = 0$), while for infinite sets E we can take $\sum_{x\in E} h(x) = \sup\{\sum_{x\in I} h(x) : I \subseteq E \text{ is finite}\}$, because every $h(x)$ is non-negative. (You may well prefer to think about this at first with $X = \mathbb{N}$, so that $\sum_{n\in E} h(n) = \lim_{n\to\infty} \sum_{m\in E, m\le n} h(m)$; but I hope that a little thought will show you that the general case, in which X may even be uncountable, is not really more difficult.) Now $(X, \mathcal{P}X, \mu)$ is a measure space.

We are very far from being ready for the specialized vocabulary needed to describe different kinds of measure space, but when the time comes I will call measures of this kind **point-supported**.

Two particular cases recur often enough to be worth giving names to. If $h(x) = 1$ for every x, then μE is just the number of points of E if E is finite, and is ∞ if E is infinite. I will call this **counting measure** on X. If $x_0 \in X$, we can set $h(x_0) = 1$ and $h(x) = 0$ for $x \in X \setminus \{x_0\}$; then μE is 1 if $x_0 \in E$, and 0 for other E. I will call this the **Dirac measure on** X **concentrated at** x_0. Another simple example is with $X = \mathbb{N}$, $h(n) = 2^{-n-1}$ for every n; then $\mu X = \frac{1}{2} + \frac{1}{4} + \ldots = 1$.

(e) If (X, Σ, μ) is a measure space, then Σ is the domain of the function μ, and X is the largest member of Σ. We can therefore recover the whole triplet (X, Σ, μ) from its final component μ. This is not a game which is worth playing at this stage. However, it is convenient on occasion to introduce a measure without immediately giving a name to its domain, and when I do this I may say that 'μ **measures** E' or 'E is **measured by** μ' to mean that μE is defined, that is, that E belongs to the σ-algebra $\operatorname{dom}\mu$. **Warning!** Many authors use the phrase 'μ-measurable set' to mean something a little different from what I am discussing here.

112C Elementary properties of measure spaces Let (X, Σ, μ) be a measure space.

(a) If E, $F \in \Sigma$ and $E \cap F = \emptyset$ then $\mu(E \cup F) = \mu E + \mu F$.

(b) If E, $F \in \Sigma$ and $E \subseteq F$ then $\mu E \le \mu F$.

(c) $\mu(E \cup F) \le \mu E + \mu F$ for any E, $F \in \Sigma$.

(d) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is any sequence in Σ, then $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n$.

(e) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ (that is, $E_n \subseteq E_{n+1}$ for every $n \in \mathbb{N}$) then

$$\mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n = \sup_{n\in\mathbb{N}} \mu E_n.$$

(f) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ (that is, $E_{n+1} \subseteq E_n$ for every $n \in \mathbb{N}$), and if some μE_n is finite, then

$$\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n = \inf_{n\in\mathbb{N}} \mu E_n.$$

proof (a) Set $E_0 = E$, $E_1 = F$, $E_n = \emptyset$ for $n \ge 2$; then $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ and $\bigcup_{n\in\mathbb{N}} E_n = E\cup F$, so

$$\mu(E\cup F) = \sum_{n=0}^{\infty} \mu E_n = \mu E + \mu F$$

(because $\mu\emptyset = 0$).

(b) $F\setminus E \in \Sigma$ (111Dc) and $\mu(F\setminus E) \ge 0$ (because all values of μ are in $[0,\infty]$); so (using (a))

$$\mu F = \mu E + \mu(F\setminus E) \ge \mu E.$$

(c) $\mu(E\cup F) = \mu E + \mu(F\setminus E)$, by (a), and $\mu(F\setminus E) \le \mu F$, by (b).

(d) Set $F_0 = E_0$, $F_n = E_n \setminus \bigcup_{i<n} E_i$ for $n \ge 1$; then $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, $\bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} E_n$ and $F_n \subseteq E_n$ for every n. By (b) just above, $\mu F_n \le \mu E_n$ for each n; so

$$\mu(\bigcup_{n\in\mathbb{N}} E_n) = \mu(\bigcup_{n\in\mathbb{N}} F_n) = \sum_{n=0}^{\infty} \mu F_n \le \sum_{n=0}^{\infty} \mu E_n.$$

(e) Set $F_0 = E_0$, $F_n = E_n \setminus E_{n-1}$ for $n \ge 1$; then $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ and $\bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} E_n$. Consequently $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu F_n$. But an easy induction on n, using (a) for the inductive step, shows that $\mu E_n = \sum_{m=0}^{n} \mu F_m$ for every n. So

$$\sum_{n=0}^{\infty} \mu F_n = \lim_{n\to\infty} \sum_{m=0}^{n} \mu F_m = \lim_{n\to\infty} \mu E_n.$$

Finally, $\lim_{n\to\infty} \mu E_n = \sup_{n\in\mathbb{N}} \mu E_n$ because (by (b)) $\langle \mu E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing.

(f) Suppose that $\mu E_k < \infty$. Set $F_n = E_k \setminus E_{k+n}$ for $n \in \mathbb{N}$, $F = \bigcup_{n\in\mathbb{N}} F_n$; then $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ, so $\mu F = \lim_{n\to\infty} \mu F_n$, by (e) just above. Also, $\mu F_n + \mu E_{k+n} = \mu E_k$; because $\mu E_k < \infty$, we may safely write $\mu F_n = \mu E_k - \mu E_{k+n}$, so that

$$\mu F = \lim_{n\to\infty}(\mu E_k - \mu E_{k+n}) = \mu E_k - \lim_{n\to\infty} \mu E_n.$$

Next, $F \subseteq E_k$, so $\mu F + \mu(E_k\setminus F) = \mu E_k$, and (again because μE_k is finite) $\mu F = \mu E_k - \mu(E_k\setminus F)$. Thus we must have $\mu(E_k\setminus F) = \lim_{n\to\infty} \mu E_n$. But $E_k \setminus F$ is just $\bigcap_{n\in\mathbb{N}} E_n$.

Finally, $\lim_{n\to\infty} \mu E_n = \inf_{n\in\mathbb{N}} \mu E_n$ because $\langle \mu E_n\rangle_{n\in\mathbb{N}}$ is non-increasing.

Remark Observe that in (f) above it is essential to have $\inf_{n\in\mathbb{N}} \mu E_n < \infty$. The construction in 112Bd is already enough to show this. Take $X = \mathbb{N}$ and let μ be counting measure on X. Set $E_n = \{i : i \in \mathbb{N},\ i \ge n\}$ for each n. Then $E_{n+1} \subseteq E_n$ for each n, but

$$\mu(\bigcap_{n\in\mathbb{N}} E_n) = \mu\emptyset = 0 < \infty = \lim_{n\to\infty} \mu E_n.$$

112D Negligible sets Let (X, Σ, μ) be any measure space.

(a) A set $A \subseteq X$ is **negligible** (or **null**) if there is a set $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = 0$. (If there seems to be a possibility of doubt about which measure is involved, I will write **μ-negligible**.)

(b) Let $\mathcal{N}$ be the family of negligible subsets of X. Then (i) $\emptyset \in \mathcal{N}$ (ii) if $A \subseteq B \in \mathcal{N}$ then $A \in \mathcal{N}$ (iii) if $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{N}$, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{N}$. **P** (i) $\mu(\emptyset) = 0$. (ii) There is an $E \in \Sigma$ such that $\mu E = 0$ and $B \subseteq E$; now $A \subseteq E$. (iii) For each $n \in \mathbb{N}$ choose an $E_n \in \Sigma$ such that $A_n \subseteq E_n$ and $\mu E_n = 0$. Now $E = \bigcup_{n\in\mathbb{N}} E_n \in \Sigma$ and $\bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}} E_n$, and $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n$, by 112Cd, so $\mu(\bigcup_{n\in\mathbb{N}} E_n) = 0$. **Q**

I will call $\mathcal{N}$ the **null ideal** of the measure μ. (A family of sets satisfying the conditions (i)-(iii) here is called a **σ-ideal** of sets.)

(c) A set $A \subseteq X$ is **conegligible** if $X \setminus A$ is negligible; that is, there is a measurable set $E \subseteq A$ such that $\mu(X\setminus E) = 0$. Note that (i) X is conegligible (ii) if $A \subseteq B \subseteq X$ and A is conegligible then B is conegligible (iii) if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of conegligible sets, then $\bigcap_{n\in\mathbb{N}} A_n$ is conegligible.

(d) It is convenient, and customary, to use some relatively informal language concerning negligible sets. If $P(x)$ is some assertion applicable to members x of the set X, we say that

$$\text{‘}P(x) \text{ for almost every } x \in X\text{’}$$

or

$$\text{‘}P(x) \text{ a.e. } (x)\text{’}$$

or

$$\text{‘}P \text{ almost everywhere’}, \quad \text{‘}P \text{ a.e.’}$$

or, if it seems necessary to specify the measure involved,

$$\text{‘}P(x) \text{ for } \mu\text{-almost every } x\text{’}, \quad \text{‘}P(x)\,\mu\text{-a.e.}(x)\text{’}, \quad \text{‘}P\,\mu\text{-a.e.’},$$

to mean that

$$\{x : x \in X,\ P(x)\}$$

is conegligible in X, that is, that

$$\{x : x \in X,\ P(x) \text{ is false}\}$$

is negligible. Thus, for instance, if $f : X \to \mathbb{R}$ is a function, ‘$f > 0$ a.e.’ means that $\{x : f(x) \le 0\}$ is negligible.

(e) The phrases ‘**almost surely**’ (**a.s.**), ‘**presque partout**’ (**p.p.**) are also used for ‘almost everywhere’.

(f) I should call your attention to the fact that, on my definitions, a negligible set need not itself be measurable, though it must be included in some negligible measurable set. (Measure spaces in which all negligible sets are measurable are called **complete**. I will return to this question in §211.)

(g) When f and g are real-valued functions defined on conegligible subsets of a measure space, I will write $f =_{\text{a.e.}} g$, $f \le_{\text{a.e.}} g$ or $f \ge_{\text{a.e.}} g$ to mean, respectively,

$$f = g \text{ a.e., that is, } \{x : x \in \operatorname{dom}(f) \cap \operatorname{dom}(g),\ f(x) = g(x)\} \text{ is conegligible,}$$

$$f \le g \text{ a.e., that is, } \{x : x \in \operatorname{dom}(f) \cap \operatorname{dom}(g),\ f(x) \le g(x)\} \text{ is conegligible,}$$

$$f \ge g \text{ a.e., that is, } \{x : x \in \operatorname{dom}(f) \cap \operatorname{dom}(g),\ f(x) \ge g(x)\} \text{ is conegligible.}$$

112X Basic exercises >(a) Let (X, Σ, μ) be a measure space. Show that (i) $\mu(E \cup F) + \mu(E \cap F) = \mu E + \mu F$ (ii) $\mu(E \cup F \cup G) + \mu(E \cap F) + \mu(E \cap G) + \mu(F \cap G) = \mu E + \mu F + \mu G + \mu(E \cap F \cap G)$ for all E, F, $G \in \Sigma$. Generalize these results to longer sequences of sets. (You may prefer to begin with the case in which μE, μF and μG are all finite. But I hope you will be able to find arguments which deal with the general case.)

>(b) Let (X, Σ, μ) be a measure space and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence in Σ. Show that

$$\mu(\bigcup_{n \in \mathbb{N}} \bigcap_{m \ge n} E_m) \le \liminf_{n \to \infty} \mu E_n.$$

(c) Let (X, Σ, μ) be a measure space, and E, $F \in \Sigma$; suppose that $\mu E < \infty$. Show that $\mu(F \triangle E) = \mu F - \mu E + 2\mu(E \setminus F)$.

(d) Let (X, Σ, μ) be a measure space and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence of measurable sets such that $\mu(\bigcup_{n \in \mathbb{N}} E_n) < \infty$. (i) Show that $\limsup_{n \to \infty} \mu E_n \le \mu(\bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} E_m)$. (ii) Show that if $\bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} E_m = E = \bigcup_{n \in \mathbb{N}} \bigcap_{m \ge n} E_m$ then $\lim_{n \to \infty} \mu E_n$ exists and is equal to μE.

>(e) Let (X, Σ, μ) be a measure space, and $\mathcal{F}$ the set of real-valued functions whose domains are conegligible subsets of X. (i) Show that $\{(f, g) : f,\ g \in \mathcal{F},\ f \le_{\text{a.e.}} g\}$ and $\{(f, g) : f,\ g \in \mathcal{F},\ f \ge_{\text{a.e.}} g\}$ are reflexive transitive relations on $\mathcal{F}$, each the inverse of the other. (ii) Show that $\{(f, g) : f,\ g \in \mathcal{F},\ f =_{\text{a.e.}} g\}$ is their intersection, and is an equivalence relation on $\mathcal{F}$.

(f) Let (X, Σ, μ) be a measure space, Y a set, and $\phi : X \to Y$ a function. Set $\mathrm{T} = \{F : F \subseteq Y, \phi^{-1}[F] \in \Sigma\}$ and $\nu F = \mu\phi^{-1}[F]$ for $F \in \mathrm{T}$. Show that ν is a measure on Y. (ν is called the **image measure** on Y, and I will generally denote it $\mu\phi^{-1}$.)

112Y Further exercises (a) Let X be a set and Σ a σ-algebra of subsets of X. Let μ_1 and μ_2 be two measures on X, both with domain Σ. Set

$$\mu E = \inf\{\mu_1(E \cap F) + \mu_2(E \setminus F) : F \in \Sigma\}$$

for each $E \in \Sigma$. Show that μ is a measure on X, and that it is the greatest measure, with domain Σ, such that $\mu E \le \min(\mu_1 E, \mu_2 E)$ for every $E \in \Sigma$.

(b) Let X be a set and Σ a σ-algebra of subsets of X. Let μ_1 and μ_2 be two measures on X, both with domain Σ. Set

$$\mu E = \sup\{\mu_1(E \cap F) + \mu_2(E \setminus F) : F \in \Sigma\}$$

for each $E \in \Sigma$. Show that μ is a measure on X, and that it is the least measure, with domain Σ, such that $\mu E \ge \max(\mu_1 E, \mu_2 E)$ for every $E \in \Sigma$.

(c) Let X be a set and Σ a σ-algebra of subsets of X.

(i) Suppose that $\nu_0, \dots, \nu_n$ are measures on X, all with domain Σ. Set

$$\mu E = \inf\{\textstyle\sum_{i=0}^n \nu_i F_i : F_0, \dots, F_n \in \Sigma,\ E \subseteq \bigcup_{i \le n} F_i\}$$

for $E \in \Sigma$. Show that μ is a measure on X.

(ii) Let N be a non-empty family of measures on X, all with domain Σ. Set

$$\mu E = \inf\{\sum_{n=0}^\infty \nu_n F_n : \langle \nu_n \rangle_{n\in\mathbb{N}} \text{ is a sequence in } \mathrm{N},\ \langle F_n \rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma,\ E \subseteq \bigcup_{n\in\mathbb{N}} F_n\}$$

for $E \in \Sigma$. Show that μ is a measure on X.

(iii) Let N be a non-empty family of measures on X, all with domain Σ, and suppose that there is some $\nu' \in \mathrm{N}$ such that $\nu' X < \infty$. Set

$$\mu E = \inf\{\textstyle\sum_{i=0}^n \nu_i F_i : n \in \mathbb{N},\ \nu_0, \dots, \nu_n \in \mathrm{N},\ F_0, \dots, F_n \in \Sigma, E \subseteq \bigcup_{i \le n} F_i\}$$

for $E \in \Sigma$. Show that μ is a measure on X.

(iv) Suppose, in (iii), that N is downwards-directed, that is, for any ν_1, $\nu_2 \in \mathrm{N}$ there is a $\nu \in \mathrm{N}$ such that $\nu E \le \min(\nu_1 E, \nu_2 E)$ for every $E \in \Sigma$. Show that $\mu E = \inf_{\nu \in \mathrm{N}} \nu E$ for every $E \in \Sigma$.

(v) Show that in all the cases (i)-(iii) the measure constructed is the greatest measure μ with domain Σ such that $\mu E \le \inf_{\nu \in \mathrm{N}} \nu E$ for every $E \in \Sigma$.

(d) Let X be a set and Σ a σ-algebra of subsets of X. Let N be a non-empty family of measures on X, all with domain Σ. Set

$$\mu E = \sup\{\sum_{i=0}^n \nu_i F_i : n \in \mathbb{N}, \nu_0, \dots, \nu_n \in \mathrm{N},\ F_0, \dots, F_n \text{ are disjoint subsets of } E \text{ belonging to } \Sigma\}$$

for $E \in \Sigma$. (i) Show that

$$\mu E = \sup\{\sum_{n=0}^\infty \nu_n F_n : \langle \nu_n \rangle_{n\in\mathbb{N}} \text{ is a sequence in } \mathrm{N},\ \langle F_n \rangle_{n\in\mathbb{N}} \text{ is a disjoint sequence in } \Sigma,\ \bigcup_{n\in\mathbb{N}} F_n \subseteq E\}$$

for every $E \in \Sigma$. (ii) Show that μ is a measure on X, and that it is the least measure, with domain Σ, such that $\mu E \geq \sup_{\nu \in \mathrm{N}} \nu E$ for every $E \in \Sigma$. (iii) Now suppose that N is upwards-directed, that is, for any ν_1, $\nu_2 \in \mathrm{N}$ there is a $\nu \in \mathrm{N}$ such that $\nu E \geq \max(\nu_1 E, \nu_2 E)$ for every $E \in \Sigma$. Show that $\mu E = \sup_{\nu \in \mathrm{N}} \nu E$ for every $E \in \Sigma$.

(e) Let (X, Σ, μ) be a measure space and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence of measurable sets. For each $k \in \mathbb{N}$ set $H_k = \{x : x \in X, \#(\{n : x \in E_n\}) \geq k\}$, the set of points belonging to E_n for k or more values of n. (i) Show that each H_k is measurable. (ii) Show that $\sum_{k=1}^{\infty} \mu H_k = \sum_{n=0}^{\infty} \mu E_n$. (*Hint*: start with the case in which $E_n = \emptyset$ for $n \geq n_0$.) (iii) Show that if $\sum_{n=0}^{\infty} \mu E_n$ is finite, then almost every point of X belongs to only finitely many E_n, and $\sum_{n=0}^{\infty} \mu E_n = \sum_{k=0}^{\infty} k \mu G_k$, where

$$G_k = H_k \setminus H_{k+1} = \{x : \#(\{n : x \in E_n\}) = k\}.$$

(f) Let X be a set and μ, ν two measures on X, with domains Σ, T respectively. Set $\Lambda = \Sigma \cap \mathrm{T}$ and define $\lambda : \Lambda \to [0, \infty]$ by setting $\lambda E = \mu E + \nu E$ for every $E \in \Lambda$. Show that (X, Λ, λ) is a measure space.

112 Notes and comments The calculations in such results as 112Ca-112Cc, 112Xa and 112Xc, involving only finitely many sets, are common to any additive concept of measure; you may have encountered them in elementary probability theory, but of course I am now asking you to consider also the possibility that one or more of the sets has measure ∞. I hope you will find that these results are entirely natural and unsurprising. I recommend Venn diagrams in this context; a result of this kind involving only finitely many measurable sets and only *addition*, with no *subtraction*, will be valid in general if and only if it is valid for the area of simple geometric shapes in the plane. The requirement '$\mu E < \infty$' in 112Xc is necessary only because we are subtracting μE; the corresponding additive result $\mu(F \triangle E) + \mu E = \mu F + 2\mu(E \setminus F)$ is true for all measurable E and F. Of course when sequences of sets enter the picture, we need to take a bit more care; the results 112Cd-112Cf are the basic ones to learn. I think however that the only trap is in the condition 'some μE_n is finite' in 112Cf. As noted in the remark at the end of 112C, this is essential, and for a decreasing sequence of measurable sets it is possible for the measure of the limit to be strictly less than the limit of the measures, though only when the latter is infinite.

Version of 6.4.05

113 Outer measures and Carathéodory's construction

I introduce the most important method of constructing measures.

113A Outer measures I come now to the third basic definition of this chapter.

Definition Let X be a set. An **outer measure** on X is a function $\theta : \mathcal{P}X \to [0, \infty]$ such that

(i) $\theta\emptyset = 0$,

(ii) if $A \subseteq B \subseteq X$ then $\theta A \leq \theta B$,

(iii) for every sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of subsets of X, $\theta(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n=0}^{\infty} \theta A_n$.

113B Remarks (a) For comments on the use of '∞', see 112B.

(b) Yet again, the most important outer measures must wait until §§114-115. The idea of the 'outer' measure of a set A is that it should be some kind of upper bound for the possible measure of A. If we are lucky, it may actually be the measure of A; but this is likely to be true only for sets with adequately smooth boundaries.

(c) Putting (i) and (iii) of the definition together, we see that if θ is an outer measure on X, and A, B are two subsets of X, then $\theta(A \cup B) \leq \theta A + \theta B$; compare 112Ca and 112Cc.

113C Carathéodory's Method: Theorem Let X be a set and θ an outer measure on X. Set

$$\Sigma = \{E : E \subseteq X,\, \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then Σ is a σ-algebra of subsets of X. Define $\mu : \Sigma \to [0,\infty]$ by writing $\mu E = \theta E$ for $E \in \Sigma$; then (X, Σ, μ) is a measure space.

proof (a) The first step is to note that for any E, $A \subseteq X$ we have $\theta(A \cap E) + \theta(A \setminus E) \ge \theta A$, by 113Bc; so that

$$\Sigma = \{E : E \subseteq X,\, \theta A \ge \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}.$$

(b) Evidently $\emptyset \in \Sigma$, because

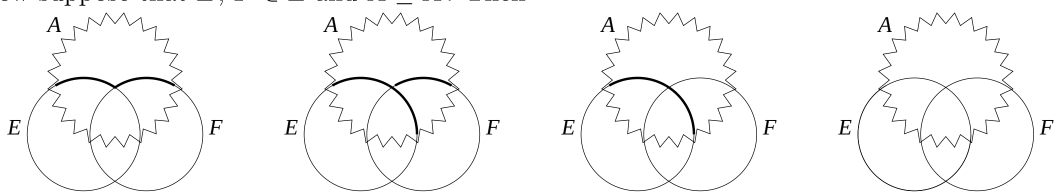
$$\theta(A \cap \emptyset) + \theta(A \setminus \emptyset) = \theta\emptyset + \theta A = \theta A$$

for every $A \subseteq X$. If $E \in \Sigma$, then $X \setminus E \in \Sigma$, because

$$\theta(A \cap (X \setminus E)) + \theta(A \setminus (X \setminus E)) = \theta(A \setminus E) + \theta(A \cap E) = \theta A$$

for every $A \subseteq X$.

(c) Now suppose that E, $F \in \Sigma$ and $A \subseteq X$. Then



(i) (ii) (iii) (iv)

$$\begin{aligned} &\theta(A \cap (E \cup F)) + \theta(A \setminus (E \cup F)) && \text{diagram (i)}\\ &\quad = \theta(A \cap (E \cup F) \cap E) + \theta(A \cap (E \cup F) \setminus E) + \theta(A \setminus (E \cup F)) && \text{diag. (ii)} \end{aligned}$$

(because $E \in \Sigma$ and $A \cap (E \cup F) \subseteq X$)

$$\begin{aligned} &\quad = \theta(A \cap E) + \theta((A \setminus E) \cap F) + \theta((A \setminus E) \setminus F)\\ &\quad = \theta(A \cap E) + \theta(A \setminus E) && \text{diag. (iii)} \end{aligned}$$

(because $F \in \Sigma$)

$$\quad = \theta A \qquad \text{diag. (iv)}$$

(again because $E \in \Sigma$). Because A is arbitrary, $E \cup F \in \Sigma$.

(d) Thus Σ is closed under simple unions and complements, and contains $\emptyset$. Now suppose that $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, with $E = \bigcup_{n\in\mathbb{N}} E_n$. Set

$$G_n = \bigcup_{m\le n} E_m;$$

then $G_n \in \Sigma$ for each n, by induction on n. Set

$$F_0 = G_0 = E_0, \quad F_n = G_n \setminus G_{n-1} = E_n \setminus G_{n-1} \text{ for } n \ge 1;$$

then $E = \bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} G_n$.

Take any $n \ge 1$ and any $A \subseteq X$. Then

$$\begin{aligned} \theta(A \cap G_n) &= \theta(A \cap G_n \cap G_{n-1}) + \theta(A \cap G_n \setminus G_{n-1})\\ &= \theta(A \cap G_{n-1}) + \theta(A \cap F_n). \end{aligned}$$

An induction on n shows that $\theta(A \cap G_n) = \sum_{m=0}^{n} \theta(A \cap F_m)$ for every $n \ge 0$.

Suppose that $A \subseteq X$. Then $A \cap E = \bigcup_{n\in\mathbb{N}} A \cap F_n$, so

$$\theta(A\cap E)\le\sum_{n=0}^{\infty}\theta(A\cap F_n)$$

$$=\lim_{n\to\infty}\sum_{m=0}^{n}\theta(A\cap F_m)=\lim_{n\to\infty}\theta(A\cap G_n).$$

On the other hand,

$$\theta(A\setminus E)=\theta(A\setminus\bigcup_{n\in\mathbb{N}}G_n)$$

$$\le\inf_{n\in\mathbb{N}}\theta(A\setminus G_n)=\lim_{n\to\infty}\theta(A\setminus G_n),$$

using 113A(ii) to see that $\langle\theta(A\setminus G_n)\rangle_{n\in\mathbb{N}}$ is non-increasing and that $\theta(A\setminus E)\le\theta(A\setminus G_n)$ for every n. Accordingly

$$\theta(A\cap E)+\theta(A\setminus E)\le\lim_{n\to\infty}\theta(A\cap G_n)+\lim_{n\to\infty}\theta(A\setminus G_n)$$

$$=\lim_{n\to\infty}(\theta(A\cap G_n)+\theta(A\setminus G_n))=\theta A$$

because every G_n belongs to Σ, so $\theta(A\cap G_n)+\theta(A\setminus G_n)=\theta A$ for every n. But A is arbitrary, so $E\in\Sigma$, by the remark in (a) above.

Because $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, condition (iii) of 111A is satisfied, and Σ is a σ-algebra of subsets of X.

(e) Now let us turn to μ, the restriction of θ to Σ, and Definition 112A. Of course $\mu\emptyset=\theta\emptyset=0$. So let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any disjoint sequence in Σ. Set $G_n=\bigcup_{m\le n}E_m$ for each n, as in (d), and

$$E=\bigcup_{n\in\mathbb{N}}E_n=\bigcup_{n\in\mathbb{N}}G_n.$$

As in (d),

$$\mu G_{n+1}=\theta G_{n+1}=\theta(G_{n+1}\cap E_{n+1})+\theta(G_{n+1}\setminus E_{n+1})$$

$$=\theta E_{n+1}+\theta G_n=\mu E_{n+1}+\mu G_n$$

for each n, so $\mu G_n=\sum_{m=0}^{n}\mu E_m$ for every n.

Now

$$\mu E=\theta E\le\sum_{n=0}^{\infty}\theta E_n=\sum_{n=0}^{\infty}\mu E_n.$$

But also

$$\mu E=\theta E\ge\theta G_n=\mu G_n=\sum_{m=0}^{n}\mu E_m$$

for each n, so $\mu E\ge\sum_{n=0}^{\infty}\mu E_n$.

Accordingly $\mu E=\sum_{n=0}^{\infty}\mu E_n$. As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, 112A(iii-β) is satisfied and (X,Σ,μ) is a measure space.

113D Remark Note from (a) in the proof above that in this construction

$$\Sigma=\{E:E\subseteq X,\ \theta(A\cap E)+\theta(A\setminus E)\le\theta A\text{ for every }A\subseteq X\}.$$

Since $\theta(A\cap E)+\theta(A\setminus E)$ is necessarily less than or equal to θA when $\theta A=\infty$,

$$\Sigma=\{E:E\subseteq X,\ \theta(A\cap E)+\theta(A\setminus E)\le\theta A\text{ whenever }A\subseteq X\text{ and }\theta A<\infty\}.$$

113X Basic exercises >(a) Let X be a set and θ an outer measure on X, and let μ be the measure on X defined from θ by Carathéodory's method. Show that if $\theta A=0$, then μ measures A, so that a set $A\subseteq X$ is μ-negligible iff $\theta A=0$, and μ is 'complete' in the sense of 112Df.

(b) Let X be a set. (i) Show that if θ_1, θ_2 are outer measures on X, so is $\theta_1+\theta_2$, setting $(\theta_1+\theta_2)(A)=\theta_1A+\theta_2A$ for every $A\subseteq X$. (ii) Show that if $\langle\theta_i\rangle_{i\in I}$ is any non-empty family of outer measures on X, so is

$\theta = \sup_{i\in I} \theta_i$, setting $\theta A = \sup_{i\in I} \theta_i A$ for every $A \subseteq X$. (iii) Show that if θ_1, θ_2 are outer measures on X so is $\theta_1 \wedge \theta_2$, setting

$$(\theta_1 \wedge \theta_2)(A) = \inf\{\theta_1 B + \theta_2(A \setminus B) : B \subseteq A\}$$

for every $A \subseteq X$.

>(c) Let X and Y be sets, θ an outer measure on X, and $f : X \to Y$ a function. Show that the functional $B \mapsto \theta(f^{-1}[B]) : \mathcal{P}Y \to [0, \infty]$ is an outer measure on Y.

>(d) Let X be a set and θ an outer measure on X; let Y be any subset of X. (i) Show that $\theta\restriction\mathcal{P}Y$, the restriction of θ to subsets of Y, is an outer measure on Y. (ii) Show that if $E \subseteq X$ is measured by the measure on X defined from θ by Carathéodory's method, then $E \cap Y$ is measured by the measure on Y defined from $\theta\restriction\mathcal{P}Y$.

>(e) Let X and Y be sets, θ an outer measure on Y, and $f : X \to Y$ a function. Show that the functional $A \mapsto \theta(f[A]) : \mathcal{P}X \to [0, \infty]$ is an outer measure.

(f) Let X and Y be sets, θ an outer measure on X, and $R \subseteq X \times Y$ a relation. Show that the map $B \mapsto \theta(R^{-1}[B]) : \mathcal{P}Y \to [0, \infty]$ is an outer measure on Y, where $R^{-1}[B] = \{x : \exists y \in B, (x, y) \in R\}$ (1A1Bc). Explain how this is a common generalization of (c), (d-i) and (e) above, and how it can be proved by putting them together.

(g) Let X be a set and θ an outer measure on X. Suppose that $E \subseteq X$ is measured by the measure on X defined from θ by Carathéodory's method. Show that $\theta(E \cap A) + \theta(E \cup A) = \theta E + \theta A$ for every $A \subseteq X$.

(h) Let X be a set and $\theta : \mathcal{P}X \to [0, \infty]$ a functional such that $\theta\emptyset = 0$, $\theta A \le \theta B$ whenever $A \subseteq B \subseteq X$, and $\theta(A \cup B) \le \theta A + \theta B$ whenever A, $B \subseteq X$. Set

$$\Sigma = \{E : E \subseteq X,\ \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}.$$

Show that $\emptyset$, $X \setminus E$ and $E \cup F$ belong to Σ for all E, $F \in \Sigma$, so that $E \setminus F$, $E \cap F \in \Sigma$ for all E, $F \in \Sigma$. Show that $\theta(E \cup F) = \theta E + \theta F$ whenever E, $F \in \Sigma$ and $E \cap F = \emptyset$.

113Y Further exercises (a) Let (X, Σ, μ) be a measure space. For $A \subseteq X$ set $\mu^* A = \inf\{\mu E : E \in \Sigma,\ A \subseteq E\}$. Show that for every $A \subseteq X$ the infimum is attained, that is, there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = \mu^* A$. Show that μ^* is an outer measure on X.

(b) Let (X, Σ, μ) be a measure space and D any subset of X. Show that $\Sigma_D = \{E \cap D : E \in \Sigma\}$ is a σ-algebra of subsets of D. Set $\mu_D = \mu^*\restriction\Sigma_D$, the function with domain Σ_D such that $\mu_D B = \mu^* B$ for every $B \in \Sigma_D$, where μ^* is defined as in (a) above; show that (D, Σ_D, μ_D) is a measure space. (μ_D is the **subspace measure** on D.)

(c) Let (X, Σ, μ) be a measure space and let μ^* be the associated outer measure on X, as in 113Ya. Let $\check\mu$ be the measure on X constructed by Carathéodory's method from μ^*, and $\check\Sigma$ its domain. Show that $\Sigma \subseteq \check\Sigma$ and that $\check\mu$ extends μ.

(d) Let X be a set and $\tau : \mathcal{P}X \to [0, \infty]$ any function such that $\tau\emptyset = 0$. For $A \subseteq X$ set

$$\theta A = \inf\{\sum_{j=0}^{\infty} \tau C_j : \langle C_j\rangle_{j\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ such that } A \subseteq \bigcup_{j\in\mathbb{N}} C_j\}.$$

Show that θ is an outer measure on X. (*Hint*: you will need 111F(b-ii) or something equivalent.)

(e) Let X be a set and θ_1, θ_2 two outer measures on X. Show that $\theta_1 \wedge \theta_2$, as described in 113Xb(iii), is the outer measure derived by the process of 113Yd from the functional $\tau C = \min(\theta_1 C, \theta_2 C)$.

(f) Let X be a set and $\langle\theta_i\rangle_{i\in I}$ any non-empty family of outer measures on X. Set $\tau C = \inf_{i\in I}\theta_i C$ for each $C \subseteq X$. Show that the outer measure derived from τ by the process of 113Yd is the largest outer measure θ such that $\theta A \le \theta_i A$ whenever $A \subseteq X$ and $i \in I$.

(g) Let X be a set and $\phi : \mathcal{P}X \to [0,\infty]$ a functional such that

$\phi\emptyset = 0$;

$\phi(A \cup B) \ge \phi A + \phi B$ for all disjoint A, $B \subseteq X$;

if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X and $\phi A_0 < \infty$ then $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty}\phi A_n$;

if $\phi A = \infty$ and $a \in \mathbb{R}$ there is a $B \subseteq A$ such that $a \le \phi B < \infty$.

Set

$$\Sigma = \{E : E \subseteq X,\ \phi(A\cap E) + \phi(A\setminus E) = \phi A \text{ for every } A \subseteq X\}.$$

Show that $(X, \Sigma, \phi\restriction\Sigma)$ is a measure space.

(h) Let (X,Σ,μ) be a measure space and for $A \subseteq X$ set $\mu_* A = \sup\{\mu E : E \in \Sigma,\ E \subseteq A,\ \mu E < \infty\}$. Show that $\mu_* : \mathcal{P}X \to [0,\infty]$ satisfies the conditions of 113Yg, and that if $\mu X < \infty$ then the measure defined from μ_* by the method of 113Yg extends μ.

(i) Let X be a set and $\mathcal{A}$ an **algebra** of subsets of X, that is, a family of subsets of X such that

$\emptyset \in \mathcal{A}$,

$X \setminus E \in \mathcal{A}$ for every $E \in \mathcal{A}$,

$E \cup F \in \mathcal{A}$ whenever E, $F \in \mathcal{A}$.

Let $\phi : \mathcal{A} \to [0,\infty]$ be a function such that

$\phi\emptyset = 0$,

$\phi(E \cup F) = \phi E + \phi F$ whenever E, $F \in \mathcal{A}$ and $E \cap F = \emptyset$,

$\phi E = \lim_{n\to\infty}\phi E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$ with union E.

Show that there is a measure μ on X extending ϕ. (*Hint*: set $\phi A = \infty$ for $A \in \mathcal{P}X \setminus \mathcal{A}$; define θ from ϕ as in 113Yd, and μ from θ.)

(j) (T.de Pauw) Let X be a set, T a σ-algebra of subsets of X, and θ an outer measure on X. Set $\Sigma = \{E : E \in \mathrm{T},\ \theta A = \theta(A\cap E) + \theta(A\setminus E) \text{ for every } A \in \mathrm{T}\}$. Show that Σ is a σ-algebra of subsets of X and that $\theta\restriction\Sigma$ is a measure.

(k) Let X, $\tau : \mathcal{P}X \to [0,\infty]$ and θ be as in 113Yd; let μ be the measure defined by Carathéodory's method from θ, and Σ the domain of μ. Suppose that $E \subseteq X$ is such that $\theta(C\cap E) + \theta(C\setminus E) \le \tau C$ whenever $C \subseteq X$ is such that $0 < \tau C < \infty$. Show that $E \in \Sigma$.

113 Notes and comments We are proceeding by the easiest stages I can devise to the construction of a non-trivial measure space, that is, Lebesgue measure on $\mathbb{R}$. There are many constructions of Lebesgue measure, but in my view Carathéodory's method (113C) is the right one to begin with, because it is the most powerful and versatile single technique for constructing measures. It is, of course, abstract – it deals with arbitrary outer measures on arbitrary sets; but I really think that the Lebesgue theory, intertwined as it is with the rich structure of Euclidean space, is harder than the abstract theory of measure. We do at least have here a serious theorem for you to get your teeth into, mastery of which should be both satisfying and useful. I must say that I think it very remarkable that such a direct construction should be effective. Looking at the proof, it is perhaps worth while distinguishing between the 'algebraic' or 'finite' parts ((a)-(c)) and the parts involving sequences of sets ((d)-(e)); the former amount to a proof of 113Xh. Outer measures of various kinds appear throughout measure theory, and I sketch a few of the relevant constructions in 113X-113Y.

Version of 14.6.05

114 Lebesgue measure on $\mathbb{R}$

Following the very abstract ideas of §§111-113, we have an urgent need for a non-trivial example of a measure space. By far the most important example is the real line with Lebesgue measure, and I now proceed to a description of this measure (114A-114E), with a few of its basic properties.

The principal ideas of this section are repeated in §115, and if you have encountered Lebesgue measure before, or feel confident in your ability to deal with two- and three-dimensional spaces at the same time as doing some difficult analysis, you could go directly to that section, turning back to this one only when a specific reference is given.

114A Definitions (a) For the purposes of this section, a **half-open interval** in $\mathbb{R}$ is a set of the form $[a,b[\,=\{x : a \le x < b\}$, where a, $b \in \mathbb{R}$.

Observe that I allow $b \le a$ in this formula; in this case $[a,b[\,=\emptyset$ (see 1A1A).

(b) If $I \subseteq \mathbb{R}$ is a half-open interval, then either $I = \emptyset$ or $I = [\inf I, \sup I[$, so that its endpoints are well defined. We may therefore define the **length** λI of a half-open interval I by setting

$$\lambda\emptyset = 0, \quad \lambda\,[a,b[\,= b-a \text{ if } a<b.$$

114B Lemma If $I \subseteq \mathbb{R}$ is a half-open interval and $\langle I_j\rangle_{j\in\mathbb{N}}$ is a sequence of half-open intervals covering I, then $\lambda I \le \sum_{j=0}^{\infty} \lambda I_j$.

proof (a) If $I=\emptyset$ then of course $\lambda I = 0 \le \sum_{j=0}^{\infty}\lambda I_j$. Otherwise, take $I=[a,b[$, where $a<b$. For each $x\in\mathbb{R}$ let H_x be the half-line $]-\infty, x[$, and consider the set

$$A = \{x : a \le x \le b,\ x-a \le \textstyle\sum_{j=0}^{\infty}\lambda(I_j\cap H_x)\}.$$

(Note that if $I_j = [c_j, d_j[$ then $I_j \cap H_x = [c_j, \min(d_j,x)[$, so $\lambda(I_j\cap H_x)$ is always defined.) We have $a\in A$ (because $a-a=0\le\sum_{j=0}^{\infty}\lambda(I_j\cap H_a)$) and of course $A\subseteq[a,b]$, so $c=\sup A$ is defined, and belongs to $[a,b]$.

(b) We find now that $c\in A$.

$$\mathbf{P}\ c-a = \sup_{x\in A} x-a$$
$$\le \sup_{x\in A}\sum_{j=0}^{\infty}\lambda(I_j\cap H_x) \le \sum_{j=0}^{\infty}\lambda(I_j\cap H_c).\ \mathbf{Q}$$

(c) ? Suppose, if possible, that $c<b$. Then $c\in[a,b[$, so there is some $k\in\mathbb{N}$ such that $c\in I_k$. Express I_k as $[c_k,d_k[$; then $x=\min(d_k,b)>c$. For each j, $\lambda(I_j\cap H_x)\ge\lambda(I_j\cap H_c)$, while

$$\lambda(I_k\cap H_x) = \lambda(I_k\cap H_c)+x-c.$$

So

$$\sum_{j=0}^{\infty}\lambda(I_j\cap H_x) \ge \sum_{j=0}^{\infty}\lambda(I_j\cap H_c)+x-c$$
$$\ge c-a+x-c = x-a,$$

so $x\in A$; but $x>c$ and $c=\sup A$. **X**

(d) We conclude that $c=b$, so that $b\in A$ and

$$b-a \le \textstyle\sum_{j=0}^{\infty}\lambda(I_j\cap H_b) \le \sum_{j=0}^{\infty}\lambda I_j,$$

as claimed.

114C Definition Now, and for the rest of this section, define $\theta : \mathcal{P}\mathbb{R} \to [0, \infty]$ by writing

$$\theta A = \inf\{\sum_{j=0}^{\infty} \lambda I_j : \langle I_j \rangle_{j \in \mathbb{N}} \text{ is a sequence of half-open intervals such that } A \subseteq \bigcup_{j \in \mathbb{N}} I_j\}.$$

Observe that every A can be covered by some sequence of half-open intervals – e.g., $A \subseteq \bigcup_{n \in \mathbb{N}} [-n, n[$; so that if we interpret the sums in $[0, \infty]$, as in 112Bc above, we always have a non-empty set to take the infimum of, and θA is always defined in $[0, \infty]$. This function θ is called **Lebesgue outer measure** on $\mathbb{R}$; the phrase is justified by (a) of the next proposition.

114D Proposition (a) θ is an outer measure on $\mathbb{R}$.

(b) $\theta I = \lambda I$ for every half-open interval $I \subseteq \mathbb{R}$.

proof (a)(i) θ takes values in $[0, \infty]$ because every θA is the infimum of a non-empty subset of $[0, \infty]$.

(ii) $\theta\emptyset = 0$ because (for instance) if we set $I_j = \emptyset$ for every j, then every I_j is a half-open interval (on the convention I am using) and $\emptyset \subseteq \bigcup_{j \in \mathbb{N}} I_j$, $\sum_{j=0}^{\infty} \lambda I_j = 0$.

(iii) If $A \subseteq B$ then whenever $B \subseteq \bigcup_{j \in \mathbb{N}} I_j$ we have $A \subseteq \bigcup_{j \in \mathbb{N}} I_j$, so θA is the infimum of a set at least as large as that involved in the definition of θB, and $\theta A \le \theta B$.

(iv) Now suppose that $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of subsets of $\mathbb{R}$, with union A. For any $\epsilon > 0$, we can choose, for each $n \in \mathbb{N}$, a sequence $\langle I_{nj} \rangle_{j \in \mathbb{N}}$ of half-open intervals such that $A_n \subseteq \bigcup_{j \in \mathbb{N}} I_{nj}$ and $\sum_{j=0}^{\infty} \lambda I_{nj} \le \theta A_n + 2^{-n}\epsilon$. (You should perhaps check that this formulation is valid whether θA_n is finite or infinite.) Now by 111F(b-ii) there is a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$; express this in the form $m \mapsto (k_m, l_m)$. Then $\langle I_{k_m, l_m} \rangle_{m \in \mathbb{N}}$ is a sequence of half-open intervals, and

$$A \subseteq \bigcup_{m \in \mathbb{N}} I_{k_m, l_m}.$$

P If $x \in A = \bigcup_{n \in \mathbb{N}} A_n$ there must be an $n \in \mathbb{N}$ such that $x \in A_n \subseteq \bigcup_{j \in \mathbb{N}} I_{nj}$, so there is a $j \in \mathbb{N}$ such that $x \in I_{nj}$. Now $m \mapsto (k_m, l_m)$ is surjective, so there is an $m \in \mathbb{N}$ such that $k_m = n$ and $l_m = j$, in which case $x \in I_{k_m, l_m}$. **Q**

Next,

$$\sum_{m=0}^{\infty} \lambda I_{k_m, l_m} \le \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \lambda I_{nj}.$$

P If $M \in \mathbb{N}$, then $N = \max(k_0, k_1, \dots, k_M, l_0, l_1, \dots, l_M)$ is finite; because every λI_{nj} is greater than or equal to 0, and any pair (n, j) can appear at most once as a (k_m, l_m),

$$\sum_{m=0}^{M} \lambda I_{k_m, l_m} \le \sum_{n=0}^{N} \sum_{j=0}^{N} \lambda I_{nj} \le \sum_{n=0}^{N} \sum_{j=0}^{\infty} \lambda I_{nj} \le \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \lambda I_{nj}.$$

So

$$\sum_{m=0}^{\infty} \lambda I_{k_m, l_m} = \lim_{M \to \infty} \sum_{m=0}^{M} \lambda I_{k_m, l_m} \le \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \lambda I_{nj}. \quad \textbf{Q}$$

Accordingly

$$\begin{aligned}\theta A &\le \sum_{m=0}^{\infty}\lambda I_{k_m,l_m}\\ &\le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\lambda I_{nj}\\ &\le \sum_{n=0}^{\infty}(\theta A_n+2^{-n}\epsilon)\\ &= \sum_{n=0}^{\infty}\theta A_n+\sum_{n=0}^{\infty}2^{-n}\epsilon\\ &= \sum_{n=0}^{\infty}\theta A_n+2\epsilon.\end{aligned}$$

Because ϵ is arbitrary, $\theta A \le \sum_{n=0}^{\infty}\theta A_n$ (again, you should check that this is valid whether or not $\sum_{n=0}^{\infty}\theta A_n$ is finite). As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, θ is an outer measure.

(b) Because we can always take $I_0 = I$, $I_j = \emptyset$ for $j \ge 1$, to obtain a sequence of half-open intervals covering I with $\sum_{j=0}^{\infty}\lambda I_j = \lambda I$, we surely have $\theta I \le \lambda I$. For the reverse inequality, use 114B: if $I \subseteq \bigcup_{j\in\mathbb{N}} I_j$, then $\lambda I \le \sum_{j=0}^{\infty}\lambda I_j$; as $\langle I_j\rangle_{j\in\mathbb{N}}$ is arbitrary, $\theta I \ge \lambda I$ and $\theta I = \lambda I$, as required.

Remark There is an ungainly shift in the argument of (a-iv) above, in the stage

‘$\theta A \le \sum_{m=0}^{\infty}\lambda I_{k_m,l_m} \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\lambda I_{nj}$’.

I dare say you would have believed me if I had suppressed the k_m, l_m altogether and simply written ‘because $A \subseteq \bigcup_{n,j\in\mathbb{N}} I_{nj}$, $\theta A \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\lambda I_{nj}$’. I hope that you will not find it too demoralizing if I suggest that such a jump is not quite safe. My reasons for interpolating a name for a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$, and taking a couple of lines to say explicitly that $\sum_{m=0}^{\infty}\lambda I_{k_m,l_m} \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\lambda I_{nj}$, are the following. To start with, there is the formal point that the definition 114C demands a simple sequence, not a double sequence. Is it really obvious that it doesn't matter here? If so, why? There can be no way to justify the shift which does not rely on the facts that $\mathbb{N}\times\mathbb{N}$ is countable and every λI_{nj} is non-negative. If either of those were untrue, the method would be in grave danger of failing.

At some point we shall certainly need to discuss sums over infinite index sets other than $\mathbb{N}$, including uncountable index sets. I have already touched on these in 112Bd, and I will return to them in 226A in Volume 2. For the moment, I feel that we have quite enough new ideas to cope with, and that what we need here is a reasonably honest expedient to deal with the question immediately before us.

You may have noticed, or guessed, that some of the inequalities ‘$\le$’ here must actually be equalities; if so, check your guess in 114Ya.

114E Definition Because Lebesgue outer measure (114C) is indeed an outer measure (114Da), we may use it to construct a measure μ, using Carathéodory's method (113C). This measure is **Lebesgue measure on** $\mathbb{R}$. The sets E measured by μ (that is, for which $\theta(A\cap E)+\theta(A\setminus E)=\theta A$ for every $A\subseteq\mathbb{R}$) are called **Lebesgue measurable**.

Sets which are negligible for μ are called **Lebesgue negligible**; note that these are just the sets A for which $\theta A = 0$, and are all Lebesgue measurable (113Xa).

114F Lemma Let $x\in\mathbb{R}$. Then $H_x = \left]-\infty, x\right[$ is Lebesgue measurable for every $x\in\mathbb{R}$.

proof (a) The point is that $\lambda I = \lambda(I\cap H_x)+\lambda(I\setminus H_x)$ for every half-open interval $I\subseteq\mathbb{R}$. **P** If either $I\subseteq H_x$ or $I\cap H_x=\emptyset$, this is trivial. Otherwise, I must be of the form $[a,b[$, where $a<x<b$. Now $I\cap H_x=[a,x[$ and $I\setminus H_x=[x,b[$ are both half-open intervals, and

$$\lambda(I\cap H_x)+\lambda(I\setminus H_x)=(x-a)+(b-x)=b-a=\lambda I. \ \mathbf{Q}$$

(b) Now suppose that A is any subset of $\mathbb{R}$, and $\epsilon > 0$. Then we can find a sequence $\langle I_j\rangle_{j\in\mathbb{N}}$ of half-open intervals such that $A \subseteq \bigcup_{j\in\mathbb{N}} I_j$ and $\sum_{j=0}^{\infty} \lambda I_j \le \theta A + \epsilon$. Now $\langle I_j \cap H_x\rangle_{j\in\mathbb{N}}$ and $\langle I_j \setminus H_x\rangle_{j\in\mathbb{N}}$ are sequences of half-open intervals and $A \cap H_x \subseteq \bigcup_{j\in\mathbb{N}}(I_j \cap H_x)$, $A \setminus H_x \subseteq \bigcup_{j\in\mathbb{N}}(I_j \setminus H_x)$. So

$$\begin{aligned}\theta(A \cap H_x) + \theta(A \setminus H_x) &\le \sum_{j=0}^{\infty} \lambda(I_j \cap H_x) + \sum_{j=0}^{\infty} \lambda(I_j \setminus H_x)\\ &= \sum_{j=0}^{\infty} \lambda I_j \le \theta A + \epsilon.\end{aligned}$$

Because ϵ is arbitrary, $\theta(A \cap H_x) + \theta(A \setminus H_x) \le \theta A$; because A is arbitrary, H_x is measurable, as remarked in 113D.

114G Proposition All Borel subsets of $\mathbb{R}$ are Lebesgue measurable; in particular, all open sets, and all sets of the following classes, together with countable unions of them:

(i) open intervals $]a, b[$, $]-\infty, b[$, $]a, \infty[$, $]-\infty, \infty[$, where $a < b \in \mathbb{R}$;

(ii) closed intervals $[a, b]$, where $a \le b \in \mathbb{R}$;

(iii) half-open intervals $[a, b[$, $]a, b]$, $]-\infty, b]$, $[a, \infty[$, where $a < b$ in $\mathbb{R}$.

We have moreover the following formula for the measures of such sets, writing μ for Lebesgue measure:

$$\mu\,]a, b[= \mu[a, b] = \mu\,[a, b[= \mu\,]a, b] = b - a$$

whenever $a \le b$ in $\mathbb{R}$, while all the unbounded intervals have infinite measure. It follows that every countable subset of $\mathbb{R}$ is measurable and of zero measure.

proof (a) I show first that all open subsets of $\mathbb{R}$ are measurable. **P** Let $G \subseteq \mathbb{R}$ be open. Let $K \subseteq \mathbb{Q} \times \mathbb{Q}$ be the set of pairs (q, q') of rational numbers such that $[q, q'[\subseteq G$. Now by the remarks in 111E-111F – specifically, 111Eb, showing that $\mathbb{Q}$ is countable, 111F(b-iii), showing that products of countable sets are countable, and 111F(b-i), showing that subsets of countable sets are countable – we see that K is countable. Also, every $[q, q'[$ is measurable, being $H_{q'} \setminus H_q$ in the language of 114F. So, by 111Fa, $G' = \bigcup_{(q,q')\in K} [q, q'[$ is measurable.

By the definition of K, $G' \subseteq G$. On the other hand, if $x \in G$, there is an $\epsilon > 0$ such that $]x - \epsilon, x + \epsilon[\subseteq G$. Now there are rational numbers $q \in]x - \epsilon, x]$ and $q' \in]x, x + \epsilon]$, so that $(q, q') \in K$ and $x \in [q, q'[\subseteq G'$. As x is arbitrary, $G = G'$ and G is measurable. **Q**

(b) Now the family Σ of Lebesgue measurable sets is a σ-algebra of subsets of $\mathbb{R}$ including the family of open sets, so must contain every Borel set, by the definition of Borel set (111G).

(c) Of the types of interval considered, all the open intervals are actually open sets, so are surely Borel. The complement of a closed interval is expressible as the union of at most two open intervals, so is Borel, and the closed interval, being the complement of a Borel set, is Borel. A bounded half-open interval is expressible as the intersection of an open interval with a closed interval, so is Borel; and finally an unbounded interval of the form $]-\infty, b]$ or $[a, \infty[$ is the complement of an open interval, so is also Borel.

(d) To compute the measures, we already know from 114Db that

$$\mu\,[a, b[= \theta\,[a, b[= b - a$$

if $a \le b$. For the other types of bounded interval, it is enough to note that $\mu\{a\} = 0$ for every $a \in \mathbb{R}$, as the different intervals differ only by one or two points; and this is so because $\{a\} \subseteq [a, a + \epsilon[$, so $\mu\{a\} \le \epsilon$, for every $\epsilon > 0$.

As for the unbounded intervals, they include arbitrarily long half-open intervals, so must have infinite measure.

(e) As just remarked, $\mu\{a\} = 0$ for every a. If $A \subseteq \mathbb{R}$ is countable, it is either empty or expressible as $\{a_n : n \in \mathbb{N}\}$. In the former case $\mu A = \mu\emptyset = 0$; in the latter, $A = \bigcup_{n\in\mathbb{N}}\{a_n\}$ is Borel and $\mu A \le \sum_{n=0}^{\infty} \mu\{a_n\} = 0$.

114X Basic exercises >(a) Let $g : \mathbb{R} \to \mathbb{R}$ be any non-decreasing function. For half-open intervals $I \subseteq \mathbb{R}$ define $\lambda_g I$ by setting

$$\lambda_g\emptyset = 0, \quad \lambda_g\,[a,b[= \lim_{x\uparrow b} g(x) - \lim_{x\uparrow a} g(x)$$

if $a < b$. For any set $A \subseteq \mathbb{R}$ set

$$\theta_g A = \inf\{\sum_{j=0}^{\infty} \lambda_g I_j : \langle I_j\rangle_{j\in\mathbb{N}} \text{ is a sequence of half-open intervals such that } A \subseteq \bigcup_{j\in\mathbb{N}} I_j\}.$$

Show that θ_g is an outer measure on $\mathbb{R}$. Let μ_g be the measure defined from θ_g by Carathéodory's method; show that $\mu_g I$ is defined and equal to $\lambda_g I$ for every half-open interval $I \subseteq \mathbb{R}$, and that every Borel subset of $\mathbb{R}$ is in the domain of μ_g.

(μ_g is the **Lebesgue-Stieltjes** measure associated with g.)

(b) At which point would the argument of 114Xa break down if we wrote $\lambda_g\,[a,b[= g(b) - g(a)$ instead of using the formula given?

>(c) Write θ for Lebesgue outer measure and μ for Lebesgue measure on $\mathbb{R}$. Show that $\theta A = \inf\{\mu E : E$ is Lebesgue measurable, $A \subseteq E\}$ for every $A \subseteq \mathbb{R}$. (*Hint*: Consider sets E of the form $\bigcup_{j\in\mathbb{N}} I_j$, where $\langle I_j\rangle_{j\in\mathbb{N}}$ is a sequence of half-open intervals.)

(d) Let X be a set, $\mathcal{I}$ a family of subsets of X such that $\emptyset \in \mathcal{I}$, and $\lambda : \mathcal{I} \to [0,\infty[$ a function such that $\lambda\emptyset = 0$. Define $\theta : \mathcal{P}X \to [0,\infty]$ by writing

$$\theta A = \inf\{\textstyle\sum_{j=0}^{\infty} \lambda I_j : \langle I_j\rangle_{j\in\mathbb{N}} \text{ is a sequence in } \mathcal{I} \text{ such that } A \subseteq \bigcup_{j\in\mathbb{N}} I_j\},$$

interpreting $\inf\emptyset$ as ∞, so that $\theta A = \infty$ if A is not covered by any sequence in $\mathcal{I}$. Show that θ is an outer measure on X.

(e) Let $E \subseteq \mathbb{R}$ be a set of finite measure for Lebesgue measure μ. Show that for every $\epsilon > 0$ there is a disjoint family $I_0, \dots, I_n$ of half-open intervals such that $\mu(E \triangle \bigcup_{j\le n} I_j) \le \epsilon$. (*Hint*: let $\langle J_j\rangle_{j\in\mathbb{N}}$ be a sequence of half-open intervals such that $E \subseteq \bigcup_{j\in\mathbb{N}} J_j$ and $\sum_{j=0}^{\infty} \mu J_j \le \mu E + \frac{1}{2}\epsilon$. Now take a suitably large m and express $\bigcup_{j\le m} J_j$ as a disjoint union of half-open intervals.)

>(f) Write θ for Lebesgue outer measure and μ for Lebesgue measure on $\mathbb{R}$. Suppose that $c \in \mathbb{R}$. Show that $\theta(A + c) = \theta A$ for every $A \subseteq \mathbb{R}$, where $A + c = \{x + c : x \in A\}$. Show that if $E \subseteq \mathbb{R}$ is measurable so is $E + c$, and that in this case $\mu(E + c) = \mu E$.

(g) Write θ for Lebesgue outer measure and μ for Lebesgue measure on $\mathbb{R}$. Suppose that $c > 0$. Show that $\theta(cA) = c\theta(A)$ for every $A \subseteq \mathbb{R}$, where $cA = \{cx : x \in A\}$. Show that if $E \subseteq \mathbb{R}$ is measurable so is cE, and that in this case $\mu(cE) = c\mu E$.

114Y Further exercises (a) In (a-iv) of the proof of 114D, show that $\sum_{m=0}^{\infty} \lambda I_{k_m,l_m}$ is actually equal to $\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \lambda I_{nj}$.

(b) Let g, $h : \mathbb{R} \to \mathbb{R}$ be two non-decreasing functions, with sum $g + h$; let μ_g, μ_h, μ_{g+h} be the corresponding Lebesgue-Stieltjes measures (114Xa). Show that

$$\operatorname{dom}\mu_{g+h} = \operatorname{dom}\mu_g \cap \operatorname{dom}\mu_h, \quad \mu_{g+h}E = \mu_g E + \mu_h E \text{ for every } E \in \operatorname{dom}\mu_{g+h}.$$

(c) Let $\langle g_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $g(x) = \sum_{n=0}^{\infty} g_n(x)$ is defined and finite for every $x \in \mathbb{R}$. Let μ_{g_n}, μ_g be the corresponding Lebesgue-Stieltjes measures. Show that

$$\operatorname{dom}\mu_g = \textstyle\bigcap_{n\in\mathbb{N}} \operatorname{dom}\mu_{g_n}, \quad \mu_g E = \sum_{n=0}^{\infty} \mu_{g_n} E \text{ for every } E \in \operatorname{dom}\mu_g.$$

(d)(i) Show that if $A \subseteq \mathbb{R}$ and $\epsilon > 0$, there is an open set $G \supseteq A$ such that $\theta G \le \theta A + \epsilon$, where θ is Lebesgue outer measure. (ii) Show that if $E \subseteq \mathbb{R}$ is Lebesgue measurable and $\epsilon > 0$, there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \le \epsilon$, where μ is Lebesgue measure. (*Hint*: consider first the case of bounded E.) (iii) Show that if $E \subseteq \mathbb{R}$ is Lebesgue measurable, there are Borel sets H_1, H_2 such that $H_1 \subseteq E \subseteq H_2$ and $\mu(H_2 \setminus E) = \mu(E \setminus H_1) = 0$. (*Hint*: use (ii) to find H_2, and then consider the complement of E.)

(e) Write θ for Lebesgue outer measure on $\mathbb{R}$. Show that a set $E \subseteq \mathbb{R}$ is Lebesgue measurable iff $\theta([-n,n] \cap E) + \theta([-n,n] \setminus E) = 2n$ for every $n \in \mathbb{N}$. (*Hint*: Use 114Yd to show that for each n there are measurable sets F_n, H_n such that $F_n \subseteq [-n,n] \cap E \subseteq H_n$ and $H_n \setminus F_n$ is negligible.)

(f) Repeat 114Xc and 114Yd-114Ye for the Lebesgue-Stieltjes measures of 114Xa.

(g) Write $\mathcal{B}$ for the σ-algebra of Borel subsets of $\mathbb{R}$, and let $\nu : \mathcal{B} \to [0,\infty]$ be a measure. Let g, λ_g, θ_g and μ_g be as in 114Xa. Show that if $\nu I = \lambda_g I$ for every half-open interval I, then $\nu E = \mu_g E$ for every $E \in \mathcal{B}$. (*Hint*: first consider open sets E, and then use 114Yd(i) as extended in 114Yf.)

(h) Write $\mathcal{B}$ for the σ-algebra of Borel subsets of $\mathbb{R}$, and let $\nu : \mathcal{B} \to [0,\infty]$ be a measure such that $\nu[-n,n] < \infty$ for every $n \in \mathbb{N}$. Show that there is a function $g : \mathbb{R} \to \mathbb{R}$ which is non-decreasing, continuous on the left and such that $\nu E = \mu_g E$ for every $E \in \mathcal{B}$, where μ_g is defined as in 114Xa. Is g unique?

(i) Write $\mathcal{B}$ for the σ-algebra of Borel subsets of $\mathbb{R}$, and let ν_1, ν_2 be measures with domain $\mathcal{B}$ such that $\nu_1 I = \nu_2 I < \infty$ for every half-open interval $I \subseteq \mathbb{R}$. Show that $\nu_1 E = \nu_2 E$ for every $E \in \mathcal{B}$.

(j) Let $\mathcal{E}$ be any family of half-open intervals in $\mathbb{R}$. Show that (i) there is a countable $\mathcal{C} \subseteq \mathcal{E}$ such that $\bigcup \mathcal{E} = \bigcup \mathcal{C}$ (definition: 1A1F) (ii) that $\bigcup \mathcal{E}$ is a Borel set, so is Lebesgue measurable (iii) that there is a disjoint sequence $\langle I_n \rangle_{n \in \mathbb{N}}$ of half-open intervals in $\mathbb{R}$ such that $\bigcup \mathcal{E} = \bigcup_{n \in \mathbb{N}} I_n$.

(k) Show that for almost every $x \in \mathbb{R}$ (as measured by Lebesgue measure) the set

$$\{(m,n) : m \in \mathbb{Z},\, n \in \mathbb{N} \setminus \{0\},\, |x - \frac{m}{n}| \le \frac{1}{n^3}\}$$

is finite. (*Hint*: estimate the outer measure of $\bigcup_{n \ge n_0} \bigcup_{|m| \le kn} [\frac{m}{n} - \frac{1}{n^3}, \frac{m}{n} + \frac{1}{n^3}]$ for n_0, $k \ge 1$.) Repeat with $2 + \epsilon$ in the place of 3.

(l) Write μ for Lebesgue measure on $\mathbb{R}$. Show that there is a countable family $\mathcal{F}$ of Lebesgue measurable subsets of $\mathbb{R}$ such that whenever μE is defined and finite, and $\epsilon > 0$, there is an $F \in \mathcal{F}$ such that $\mu(E \triangle F) \le \epsilon$. (*Hint*: in 114Xe, show that we can take the I_j to have rational endpoints.)

114 Notes and comments My own interests are in 'abstract' measure theory, and from the point of view of the structure of this treatise, the chief object of this section is the description of a non-trivial measure space to provide a focus for the general theorems which follow. Let me enumerate the methods of constructing measure spaces already available to us. (a) We have the point-supported measures of 112Bd; in some ways, these are trivial; but they do occur in applications, and, just because they are generally easy to deal with, it is often right to test any new ideas on them. (b) We have Lebesgue measure on $\mathbb{R}$; a straightforward generalization of the construction yields the Lebesgue-Stieltjes measures (114Xa). (c) Next, we have ways of building new measures from old, starting with subspace measures (113Yb), image measures (112Xf) and sums of measures (112Yf). Perhaps the most important of these is 'Lebesgue measure on $[0,1]$', I call it μ_1 for the moment, where the domain of μ_1 is $\{E : E \subseteq [0,1]$ is Lebesgue measurable$\} = \{E \cap [0,1] : E \subseteq \mathbb{R}$ is Lebesgue measurable$\}$, and $\mu_1 E$ is just the Lebesgue measure of E for each $E \in \operatorname{dom} \mu_1$. In fact the image measures of Lebesgue measure on $[0,1]$ include a very large proportion of the probability measures (that is, measures giving measure 1 to the whole space) of importance in ordinary applications.

Of course Lebesgue measure is not only the dominant guiding example for general measure theory, but is itself the individual measure of greatest importance for applications. For this reason it would be possible – though in my view narrow-minded – to read chapters 12-13 of this volume, and a substantial proportion of Volume 2, as if they applied only to Lebesgue measure on $\mathbb{R}$. This is, indeed, the context in which most of these results were first developed. I believe, however, that it is often the case in mathematics, that one's understanding of a particular construction is deepened and strengthened by an acquaintance with related objects, and that one of the ways to an appreciation of the nature of Lebesgue measure is through a study of its properties in the more abstract context of general measure theory.

For any proper investigation of the applications of Lebesgue measure theory we must wait for Volume 2. But I include 114Yk as a hint of one of the ways in which this theory can be used.

Version of 21.7.05

115 Lebesgue measure on $\mathbb{R}^r$

Following the very abstract ideas of §§111-113, there is an urgent need for non-trivial examples of measure spaces. By far the most important examples are the Euclidean spaces $\mathbb{R}^r$ with Lebesgue measure, and I now proceed to a definition of these measures (115A-115E), with a few of their basic properties. Except at one point (in the proof of the fundamental lemma 115B) this section does not rely essentially on §114; but nevertheless most students encountering Lebesgue measure for the first time will find it easier to work through the one-dimensional case carefully before embarking on the multi-dimensional case.

115A Definitions (a) For practically the whole of this section (the exception is the proof of Lemma 115B) r will denote a fixed integer greater than or equal to 1. I will use Roman letters a, b, c, d, x, y to denote members of $\mathbb{R}^r$, and Greek letters for their coordinates, so that $a = (\alpha_1, \dots, \alpha_r)$, $b = (\beta_1, \dots, \beta_r)$, $x = (\xi_1, \dots, \xi_r)$.

(b) For the purposes of this section, a **half-open interval** in $\mathbb{R}^r$ is a set of the form $[a, b[= \{x : \alpha_i \le \xi_i < \beta_i \ \forall\ i \le r\}$, where a, $b \in \mathbb{R}^r$. Observe that I allow $\beta_i \le \alpha_i$ in this formula; if this happens for any i, then $[a, b[= \emptyset$.

(c) If $I = [a, b[\subseteq \mathbb{R}^r$ is a half-open interval, then either $I = \emptyset$ or

$$\alpha_i = \inf\{\xi_i : x \in I\}, \quad \beta_i = \sup\{\xi_i : x \in I\}$$

for every $i \le r$; in the latter case, the expression of I as a half-open interval is unique. We may therefore define the r**-dimensional volume** λI of a half-open interval I by setting

$$\lambda\emptyset = 0, \quad \lambda\, [a, b[= \prod_{i=1}^r \beta_i - \alpha_i \text{ if } \alpha_i < \beta_i \text{ for every } i.$$

115B Lemma If $I \subseteq \mathbb{R}^r$ is a half-open interval and $\langle I_j \rangle_{j\in\mathbb{N}}$ is a sequence of half-open intervals covering I, then $\lambda I \le \sum_{j=0}^\infty \lambda I_j$.

proof The proof is by induction on r. For this proof only, therefore, I write λ_r for the function defined on the half-open intervals of $\mathbb{R}^r$ by the formula of 115Ac.

(a) The argument for $r = 1$, starting the induction, is similar to the inductive step; but rather than establish a suitable convention to set up a trivial case $r = 0$, or ask you to work out the details yourself, I refer you to 114B, which is exactly the case $r = 1$.

(b) For the inductive step to $r + 1$, where $r \ge 1$, take a half-open interval $I \subseteq \mathbb{R}^{r+1}$ and $\langle I_j \rangle_{j\in\mathbb{N}}$ a sequence of half-open intervals covering I. If $I = \emptyset$ then of course $\lambda_{r+1} I = 0 \le \sum_{j=0}^\infty \lambda_{r+1} I_j$. Otherwise, express I as $[a, b[$, where $\alpha_i < \beta_i$ for $i \le r + 1$, and each I_j as $\left[a^{(j)}, b^{(j)}\right[$. Write $\zeta = \prod_{i=1}^r \beta_i - \alpha_i$, so that $\lambda_{r+1} I = \zeta(\beta_{r+1} - \alpha_{r+1})$. Fix $\epsilon > 0$. For each $\xi \in \mathbb{R}$ let H_ξ be the half-space $\{x : \xi_{r+1} < \xi\}$, and consider the set

$$A = \{\xi : \alpha_{r+1} \le \xi \le \beta_{r+1},\ \zeta(\xi - \alpha_{r+1}) \le (1 + \epsilon)\sum_{j=0}^\infty \lambda_{r+1}(I_j \cap H_\xi)\}.$$

(Note that $I_j \cap H_\xi = \left[a^{(j)}, \tilde{b}^{(j)}\right[$, where $\tilde{\beta}_i^{(j)} = \beta_i^{(j)}$ for $i \le r$ and $\tilde{\beta}_{r+1}^{(j)} = \min(\beta_{r+1}^{(j)}, \xi)$, so $\lambda_{r+1}(I_j \cap H_\xi)$ is always defined.) We have $\alpha_{r+1} \in A$, because

$$\zeta(\alpha_{r+1} - \alpha_{r+1}) = 0 \le (1 + \epsilon)\sum_{j=0}^\infty \lambda_{r+1}(I_j \cap H_{\alpha_{r+1}}),$$

and of course $A \subseteq [\alpha_{r+1}, \beta_{r+1}]$, so $\gamma = \sup A$ is defined, and belongs to $[\alpha_{r+1}, \beta_{r+1}]$.

(c) We find now that $\gamma \in A$.

$$\mathbf{P}\ \zeta(\gamma-\alpha_{r+1}) = \sup_{\xi\in A}\zeta(\xi-\alpha_{r+1})$$
$$\le (1+\epsilon)\sup_{\xi\in A}\sum_{j=0}^{\infty}\lambda_{r+1}(I_j\cap H_\xi) \le (1+\epsilon)\sum_{j=0}^{\infty}\lambda_{r+1}(I_j\cap H_\gamma).\ \mathbf{Q}$$

(d) **?** Suppose, if possible, that $\gamma < \beta_{r+1}$. Then $\gamma \in [\alpha_{r+1}, \beta_{r+1}[$. Set

$$J = \{x : x \in \mathbb{R}^r,\ (x,\gamma) \in I\} = [a', b'[,$$

where $a' = (\alpha_1, \dots, \alpha_r)$, $b' = (\beta_1, \dots, \beta_r)$, and for each $j \in \mathbb{N}$ set

$$J_j = \{x : x \in \mathbb{R}^r,\ (x,\gamma) \in I_j\}.$$

Because $I \subseteq \bigcup_{j\in\mathbb{N}} I_j$, we must have $J \subseteq \bigcup_{j\in\mathbb{N}} J_j$. Of course both J and the J_j are half-open intervals in $\mathbb{R}^r$. (This is one of the places where it is helpful to count the empty set as a half-open interval.) By the inductive hypothesis, $\zeta = \lambda_r J \le \sum_{j=0}^{\infty}\lambda_r J_j$. As $\zeta > 0$, there is an $m \in \mathbb{N}$ such that $\zeta \le (1+\epsilon)\sum_{j=0}^{m}\lambda_r J_j$. Now for each $j \le m$, either $J_j = \emptyset$ or $\alpha_{r+1}^{(j)} \le \gamma < \beta_{r+1}^{(j)}$; set

$$\xi = \min(\{\beta_{r+1}\} \cup \{\beta_{r+1}^{(j)} : j \le m,\ J_j \ne \emptyset\}) > \gamma.$$

Then

$$\lambda_{r+1}(I_j \cap H_\xi) \ge \lambda_{r+1}(I_j \cap H_\gamma) + (\xi-\gamma)\lambda_r J_j$$

for every $j \le m$ such that J_j is non-empty, and therefore for every j. Consequently

$$\zeta(\xi-\alpha_{r+1}) = \zeta(\gamma-\alpha_{r+1}) + \zeta(\xi-\gamma)$$
$$\le (1+\epsilon)\sum_{j=0}^{\infty}\lambda_{r+1}(I_j\cap H_\gamma) + (1+\epsilon)(\xi-\gamma)\sum_{j=0}^{m}\lambda_r J_j$$
$$\le (1+\epsilon)\sum_{j=m+1}^{\infty}\lambda_{r+1}(I_j\cap H_\gamma) + (1+\epsilon)\sum_{j=0}^{m}\lambda_{r+1}(I_j\cap H_\xi)$$
$$\le (1+\epsilon)\sum_{j=0}^{\infty}\lambda_{r+1}(I_j\cap H_\xi),$$

and $\xi \in A$, which is impossible. **X**

(e) We conclude that $\gamma = \beta_{r+1}$, so that $\beta_{r+1} \in A$ and

$$\lambda_{r+1}I = \zeta(\beta_{r+1}-\alpha_{r+1}) \le (1+\epsilon)\textstyle\sum_{j=0}^{\infty}\lambda_{r+1}(I_j\cap H_{\beta_{r+1}}) \le (1+\epsilon)\sum_{j=0}^{\infty}\lambda_{r+1}I_j.$$

As ϵ is arbitrary,

$$\lambda_{r+1}I \le \textstyle\sum_{j=0}^{\infty}\lambda_{r+1}I_j,$$

as claimed.

Remark This proof is hard work, and not everybody makes such a mouthful of it. What is perhaps a more conventional approach is sketched in 115Ya, using the Heine-Borel theorem to reduce the problem to one of finite covers, and then (very often) saying that it is trivial. I do not use this method, partly because we do not need the Heine-Borel theorem elsewhere in this volume (though we shall certainly need it in Volume 2, and I write out a proof in 2A2F), and partly because I do not agree that the lemma is trivial when we have a finite sequence $I_0, \dots, I_m$ covering I. I invite you to consider this for yourself. It seems to me that any rigorous argument must involve an induction on the dimension, which is what I provide here. Of course dealing throughout with an infinite sequence makes it a little harder to keep track of what we are doing, and I note that in fact there is a crucial step which necessitates truncation of the sequence; I mean the formula

$$\xi = \min(\{\beta_{r+1}\} \cup \{\beta_{r+1}^{(j)} : j \le m,\ J_j \ne \emptyset\})$$

in part (d) of the proof. We certainly cannot take $\xi = \inf\{\beta_{r+1}^{(j)} : j \in \mathbb{N}, J_j \neq \emptyset\}$, since this is very likely to be equal to γ. Accordingly I need some excuse for truncating, which is in the sentence

As $\zeta > 0$, there is an $m \in \mathbb{N}$ such that $\zeta \le (1+\epsilon)\sum_{j=0}^m \lambda_r J_j$.

And that step is the reason for introducing the slack ϵ into the definition of the set A at the beginning of the proof. Apart from this modification, the structure of the argument is supposed to reflect that of 114B; so I hope you can use the simpler formulae of 114B as a guide here.

115C Definition Now, and for the rest of this section, define $\theta : \mathcal{P}(\mathbb{R}^r) \to [0,\infty]$ by writing

$$\theta A = \inf\{\sum_{j=0}^{\infty} \lambda I_j : \langle I_j\rangle_{j\in\mathbb{N}} \text{ is a sequence of half-open intervals}$$
$$\text{such that } A \subseteq \bigcup_{j\in\mathbb{N}} I_j\}.$$

Observe that every A can be covered by some sequence of half-open intervals – e.g., $A \subseteq \bigcup_{n\in\mathbb{N}} [-\mathbf{n},\mathbf{n}[$, writing $\mathbf{n} = (n,n,\dots,n) \in \mathbb{R}^r$; so that if we interpret the sums in $[0,\infty]$, as in 112Bc above, we always have a non-empty set to take the infimum of, and θA is always defined in $[0,\infty]$.

This function θ is called **Lebesgue outer measure** on $\mathbb{R}^r$; the phrase is justified by (a) of the next proposition.

115D Proposition (a) θ is an outer measure on $\mathbb{R}^r$.

(b) $\theta I = \lambda I$ for every half-open interval $I \subseteq \mathbb{R}^r$.

proof (a)(i) θ takes values in $[0,\infty]$ because every θA is the infimum of a non-empty subset of $[0,\infty]$.

(ii) $\theta\emptyset = 0$ because (for instance) if we set $I_j = \emptyset$ for every j, then every I_j is a half-open interval (on the convention I am using), $\emptyset \subseteq \bigcup_{j\in\mathbb{N}} I_j$ and $\sum_{j=0}^\infty \lambda I_j = 0$.

(iii) If $A \subseteq B$ then whenever $B \subseteq \bigcup_{j\in\mathbb{N}} I_j$ we have $A \subseteq \bigcup_{j\in\mathbb{N}} I_j$, so θA is the infimum of a set at least as large as that involved in the definition of θB, and $\theta A \le \theta B$.

(iv) Now suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}^r$, with union A. For any $\epsilon > 0$, we can choose, for each $n \in \mathbb{N}$, a sequence $\langle I_{nj}\rangle_{j\in\mathbb{N}}$ of half-open intervals such that $A_n \subseteq \bigcup_{j\in\mathbb{N}} I_{nj}$ and $\sum_{j=0}^\infty \lambda I_{nj} \le \theta A_n + 2^{-n}\epsilon$. (You should perhaps check that this formulation is valid whether θA_n is finite or infinite.) Now by 111F(b-ii) there is a bijection from $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$; express this in the form $m \mapsto (k_m, l_m)$. Then we find that

$$\sum_{m=0}^\infty \lambda I_{k_m,l_m} = \sum_{n=0}^\infty \sum_{j=0}^\infty \lambda I_{nj}.$$

(To see this, note that because every λI_{nj} is greater than or equal to 0, and $m \mapsto (k_m,l_m)$ is a bijection, both sums are equal to

$$\sup_{K\subseteq\mathbb{N}\times\mathbb{N} \text{ is finite}} \sum_{(n,j)\in K} \lambda I_{nj}.$$

Or look at the argument written out in 114D.) But now $\langle I_{k_m,l_m}\rangle_{m\in\mathbb{N}}$ is a sequence of half-open intervals and

$$A = \bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} I_{nj} = \bigcup_{m\in\mathbb{N}} I_{k_m,l_m},$$

so

$$\theta A \le \sum_{m=0}^\infty \lambda I_{k_m,l_m} = \sum_{n=0}^\infty\sum_{j=0}^\infty \lambda I_{nj}$$
$$\le \sum_{n=0}^\infty (\theta A_n + 2^{-n}\epsilon) = \sum_{n=0}^\infty \theta A_n + \sum_{n=0}^\infty 2^{-n}\epsilon = \sum_{n=0}^\infty \theta A_n + 2\epsilon.$$

Because ϵ is arbitrary, $\theta A \le \sum_{n=0}^\infty \theta A_n$ (again, you should check that this is valid whether or not $\sum_{n=0}^\infty \theta A_n$ is finite). As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, θ is an outer measure.

(b) Because we can always take $I_0 = I$, $I_j = \emptyset$ for $j \ge 1$, to obtain a sequence of half-open intervals covering I with $\sum_{j=0}^{\infty} \lambda I_j = \lambda I$, we surely have $\theta I \le \lambda I$. For the reverse inequality, use 115B; if $I \subseteq \bigcup_{j\in\mathbb{N}} I_j$, then $\lambda I \le \sum_{j=0}^{\infty} \lambda I_j$; as $\langle I_j\rangle_{j\in\mathbb{N}}$ is arbitrary, $\theta I \ge \lambda I$ and $\theta I = \lambda I$, as required.

115E Definition Because Lebesgue outer measure (115C) is indeed an outer measure (115Da), we may use it to construct a measure μ, using Carathéodory's method (113C). This measure is **Lebesgue measure on** $\mathbb{R}^r$. The sets E for which μE is defined (that is, for which $\theta(A \cap E) + \theta(A \setminus E) = \theta A$ for every $A \subseteq \mathbb{R}^r$) are called **Lebesgue measurable**.

Sets which are negligible for μ are called **Lebesgue negligible**; note that these are just the sets A for which $\theta A = 0$, and are all Lebesgue measurable (113Xa).

115F Lemma If $i \le r$ and $\xi \in \mathbb{R}$, then $H_{i\xi} = \{y : \eta_i < \xi\}$ is Lebesgue measurable.

proof Write H for $H_{i\xi}$.

(a) The point is that $\lambda I = \lambda(I \cap H) + \lambda(I \setminus H)$ for every half-open interval $I \subseteq \mathbb{R}^r$. **P** If either $I \subseteq H$ or $I \cap H = \emptyset$, this is trivial. Otherwise, I must be of the form $[a, b[$, where $\alpha_i < \xi < \beta_i$. Now $I \cap H = [a, x[$ and $I \setminus H = [y, b[$, where $\xi_j = \beta_j$ for $j \ne i$, $\xi_i = \xi$, $\eta_j = \alpha_j$ for $j \ne i$, $\eta_i = \xi$, so both are half-open intervals, and

$$\begin{aligned}\lambda(I \cap H) + \lambda(I \setminus H) &= (\xi - \alpha_i)\prod_{j\ne i}(\beta_j - \alpha_j) + (\beta_i - \xi)\prod_{j\ne i}(\beta_j - \alpha_j)\\ &= (\beta_i - \alpha_i)\prod_{j\ne i}(\beta_j - \alpha_j) = \lambda I. \ \mathbf{Q}\end{aligned}$$

(b) Now suppose that A is any subset of $\mathbb{R}^r$, and $\epsilon > 0$. Then we can find a sequence $\langle I_j\rangle_{j\in\mathbb{N}}$ of half-open intervals such that $A \subseteq \bigcup_{j\in\mathbb{N}} I_j$ and $\sum_{j=0}^{\infty} \lambda I_j \le \theta A + \epsilon$. In this case, $\langle I_j \cap H\rangle_{j\in\mathbb{N}}$ and $\langle I_j \setminus H\rangle_{j\in\mathbb{N}}$ are sequences of half-open intervals, $A \cap H \subseteq \bigcup_{j\in\mathbb{N}}(I_j \cap H)$ and $A \setminus H \subseteq \bigcup_{j\in\mathbb{N}}(I_j \setminus H)$. So

$$\begin{aligned}\theta(A \cap H) + \theta(A \setminus H) &\le \sum_{j=0}^{\infty} \lambda(I_j \cap H) + \sum_{j=0}^{\infty} \lambda(I_j \setminus H)\\ &= \sum_{j=0}^{\infty} \lambda I_j \le \theta A + \epsilon.\end{aligned}$$

Because ϵ is arbitrary, $\theta(A \cap H) + \theta(A \setminus H) \le \theta A$; because A is arbitrary, H is measurable, as remarked in 113D.

115G Proposition All Borel subsets of $\mathbb{R}^r$ are Lebesgue measurable; in particular, all open sets, and all sets of the following classes, together with countable unions of them:

open intervals $]a, b[= \{x : x \in \mathbb{R}^r,\ \alpha_i < \xi_i < \beta_i\ \forall\ i \le r\}$, where $\alpha_i,\ \beta_i \in \mathbb{R} \cup \{-\infty, \infty\}$ for each $i \le r$;

closed intervals $[a, b] = \{x : x \in \mathbb{R}^r,\ \alpha_i \le \xi_i \le \beta_i\ \forall\ i \le r\}$, where $\alpha_i,\ \beta_i \in \mathbb{R} \cup \{-\infty, \infty\}$ for each $i \le r$.

We have moreover the following formula for the measures of such sets, writing μ for Lebesgue measure:

$$\mu\,]a, b[= \mu[a, b] = \textstyle\prod_{i=1}^{r} \beta_i - \alpha_i$$

whenever $a \le b$ in $\mathbb{R}^r$. Consequently every countable subset of $\mathbb{R}^r$ is measurable and of zero measure.

proof (a) I show first that all open subsets of $\mathbb{R}^r$ are measurable. **P** Let $G \subseteq \mathbb{R}^r$ be open. Let $K \subseteq \mathbb{Q}^r \times \mathbb{Q}^r$ be the set of pairs (c, d) of r-tuples of rational numbers such that $[c, d[\subseteq G$. Now by the remarks in 111E-111F – specifically, 111Eb, showing that $\mathbb{Q}$ is countable, 111F(b-iii), showing that the product of two countable sets is countable, and 111F(b-i), showing that subsets of countable sets are countable – we see, inducing on r, that $\mathbb{Q}^r$ is countable, and that K is countable. Also, every $[c, d[$ is measurable, being

$$\textstyle\bigcap_{i\le r} H_{i\delta_i} \setminus H_{i\gamma_i},$$

in the language of 115F, if $c = (\gamma_1, \dots, \gamma_r)$ and $d = (\delta_1, \dots, \delta_r)$. So, by 111Fa, $G' = \bigcup_{(r,s)\in K} [r, s[$ is measurable.

By the definition of K, $G' \subseteq G$. On the other hand, if $x \in G$, there is an $\epsilon > 0$ such that $y \in G$ whenever $\|y - x\| < \epsilon$. Now for each i there are rational numbers γ_i, δ_i such that $\gamma_i \le \xi_i < \delta_i$ and $\delta_i - \gamma_i \le \frac{\epsilon}{\sqrt{r}}$. If $y \in [c, d[$ then $|\eta_i - \xi_i| < \frac{\epsilon}{\sqrt{r}}$ for every i so $\|y - x\| < \epsilon$ and $y \in G$. Accordingly $(c, d) \in K$ and $x \in [c, d[\subseteq G'$. As x is arbitrary, $G = G'$ and G is measurable. **Q**

(b) Now the family Σ of Lebesgue measurable sets is a σ-algebra of subsets of $\mathbb{R}^r$ including the family of open sets, so must contain every Borel set, by the definition of Borel set (111G).

(c) Of the types of interval considered, all the open intervals are actually open sets, so are surely Borel. A closed interval $[a, b]$ is expressible as the intersection $\bigcap_{n\in\mathbb{N}}]a - 2^{-n}\mathbf{1}, b + 2^{-n}\mathbf{1}[$ of a sequence of open intervals, so is Borel.

(d) To compute the measures, we already know from 115Db that $\mu\,[a, b[= \prod_{i=1}^r \beta_i - \alpha_i$ if $a \le b$ in $\mathbb{R}^r$. For the other types of bounded interval, it is enough to note that if $-\infty < \alpha_i < \beta_i < \infty$ for every i, then

$$[a + \epsilon\mathbf{1}, b[\subseteq]a, b[\subseteq [a, b] \subseteq [a, b + \epsilon\mathbf{1}[$$

whenever $\epsilon > 0$ in $\mathbb{R}$. So

$$\mu\,]a, b[\le \mu[a, b] \le \inf_{\epsilon>0} \mu\,[a, b + \epsilon\mathbf{1}[= \inf_{\epsilon>0} \prod_{i=1}^r (\beta_i - \alpha_i + \epsilon) = \prod_{i=1}^r \beta_i - \alpha_i.$$

If $\beta_i = \alpha_i$ for any i, then we must have

$$\mu\,]a, b[= \mu[a, b] = 0 = \prod_{i=1}^r \beta_i - \alpha_i.$$

If $\beta_i > \alpha_i$ for every i, then set $\epsilon_0 = \min_{i\le r} \beta_i - \alpha_i > 0$; then

$$\begin{aligned}\mu[a, b] \ge \mu\,]a, b[&\ge \sup_{0<\epsilon\le\epsilon_0} \mu\,[a + \epsilon\mathbf{1}, b[\\ &= \sup_{0<\epsilon\le\epsilon_0} \prod_{i=1}^r (\beta_i - \alpha_i - \epsilon) = \prod_{i=1}^r \beta_i - \alpha_i.\end{aligned}$$

So in this case

$$\prod_{i=1}^r \beta_i - \alpha_i \le \mu\,]a, b[\le \mu[a, b] \le \prod_{i=1}^r \beta_i - \alpha_i$$

and

$$\mu\,]a, b[= \mu[a, b] = \prod_{i=1}^r \beta_i - \alpha_i.$$

(e) By (d), $\mu\{a\} = \mu[a, a] = 0$ for every $a \in \mathbb{R}^r$. If $A \subseteq \mathbb{R}^r$ is countable, it is either empty or expressible as $\{a_n : n \in \mathbb{N}\}$. In the former case $\mu A = \mu\emptyset = 0$; in the latter, $A = \bigcup_{n\in\mathbb{N}} \{a_n\}$ is Borel and $\mu A \le \sum_{n=0}^\infty \mu\{a_n\} = 0$.

115X Basic exercises If you skipped §114, you should now return to 114X and assure yourself that you can do the exercises there as well as those below.

(a) Show that if I, J are half-open intervals in $\mathbb{R}^r$, then $I \setminus J$ is expressible as the union of at most $2r$ disjoint half-open intervals. Hence show that (i) any finite union of half-open intervals is expressible as a finite union of disjoint half-open intervals (ii) any countable union of half-open intervals is expressible as the union of a disjoint sequence of half-open intervals.

>(b) Write θ for Lebesgue outer measure, μ for Lebesgue measure on $\mathbb{R}^r$. Show that $\theta A = \inf\{\mu E : E$ is Lebesgue measurable, $A \subseteq E\}$ for every $A \subseteq \mathbb{R}^r$. (*Hint*: consider sets E of the form $\bigcup_{j\in\mathbb{N}} I_j$, where $\langle I_j\rangle_{j\in\mathbb{N}}$ is a sequence of half-open intervals.)

(c) Let $E \subseteq \mathbb{R}^r$ be a set of finite measure for Lebesgue measure μ. Show that for every $\epsilon > 0$ there is a disjoint family $I_0, \dots, I_n$ of half-open intervals such that $\mu(E \triangle \bigcup_{j\le n} I_j) \le \epsilon$. (*Hint*: let $\langle J_j\rangle_{j\in\mathbb{N}}$ be a sequence of half-open intervals such that $E \subseteq \bigcup_{j\in\mathbb{N}} J_j$ and $\sum_{j=0}^\infty \mu J_j \le \mu E + \frac{1}{2}\epsilon$. Now take a suitably large m and express $\bigcup_{j\le m} J_j$ as a disjoint union of half-open intervals.)

>(d) Suppose that $c \in \mathbb{R}^r$. (i) Show that $\theta(A+c) = \theta A$ for every $A \subseteq \mathbb{R}^r$, where $A+c = \{x+c : x \in A\}$. (ii) Show that if $E \subseteq \mathbb{R}^r$ is measurable so is $E+c$, and that in this case $\mu(E+c) = \mu E$.

(e) Suppose that $\gamma > 0$. (i) Show that $\theta(\gamma A) = \gamma^r \theta A$ for every $A \subseteq \mathbb{R}^r$, where $\gamma A = \{\gamma x : x \in A\}$. (ii) Show that if $E \subseteq \mathbb{R}^r$ is measurable so is γE, and that in this case $\mu(\gamma E) = \gamma^r \mu E$

115Y Further exercises (a) (i) Suppose that M is a strictly positive integer and k_i, l_i are integers for $1 \le i \le r$. Set $\alpha_i = k_i/M$ and $\beta_i = l_i/M$ for each i, and $I = [a, b[$. Show that $\lambda I = \#(J)/M^r$, where J is $\{z : z \in \mathbb{Z}^r, \frac{1}{M}z \in I\}$. (ii) Show that if a half-open interval $I \subseteq \mathbb{R}^r$ is covered by a *finite* sequence $I_0, \dots, I_m$ of half-open intervals, and all the coordinates involved in specifying the intervals $I, I_0, \dots, I_m$ are rational, then $\lambda I \le \sum_{j=0}^m \lambda I_j$. (iii) Assuming the Heine-Borel theorem in the form

> whenever $[a, b]$ is a closed interval in $\mathbb{R}^r$ which is covered by a sequence $\langle]a^{(j)}, b^{(j)}[\rangle_{j\in\mathbb{N}}$ of open intervals, there is an $m \in \mathbb{N}$ such that $[a, b] \subseteq \bigcup_{j\le m}]a^{(j)}, b^{(j)}[$,

prove 115B. (*Hint*: if $[a, b[\subseteq \bigcup_{j\in\mathbb{N}} [a^{(j)}, b^{(j)}[$, replace $[a, b[$ by a smaller closed interval and each $[a^{(j)}, b^{(j)}[$ by a larger open interval, changing the volumes by adequately small amounts.)

(b)(i) Show that if $A \subseteq \mathbb{R}^r$ and $\epsilon > 0$, there is an open set $G \supseteq A$ such that $\theta G \le \theta A + \epsilon$, where θ is Lebesgue outer measure. (ii) Show that if $E \subseteq \mathbb{R}^r$ is Lebesgue measurable and $\epsilon > 0$, there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \le \epsilon$, where μ is Lebesgue measure. (*Hint*: consider first the case of bounded E.) (iii) Show that if $E \subseteq \mathbb{R}^r$ is Lebesgue measurable, there are Borel sets H_1, H_2 such that $H_1 \subseteq E \subseteq H_2$ and $\mu(H_2 \setminus E) = \mu(E \setminus H_1) = 0$. (*Hint*: use (ii) to find H_2, and then consider the complement of E.)

(c) Write θ for Lebesgue outer measure on $\mathbb{R}^r$. Show that a set $E \subseteq \mathbb{R}^r$ is Lebesgue measurable iff $\theta([-\mathbf{n}, \mathbf{n}] \cap E) + \theta([-\mathbf{n}, \mathbf{n}] \setminus E) = (2n)^r$ for every $n \in \mathbb{N}$, writing $\mathbf{n} = (n, \dots, n)$. (*Hint*: use 115Yb to show that for each n there are measurable sets F_n, H_n such that $F_n \subseteq [-\mathbf{n}, \mathbf{n}] \cap E \subseteq H_n$ and $H_n \setminus F_n$ is negligible.)

(d) Assuming that there is a set $A \subseteq \mathbb{R}$ which is not a Borel set, show that there is a family $\mathcal{E}$ of half-open intervals in $\mathbb{R}^2$ such that $\bigcup \mathcal{E}$ is not a Borel set. (*Hint*: consider $\mathcal{E} = \{[\xi, 1+\xi[\times [-\xi, 1-\xi[: \xi \in A\}$.)

(e) Let X be a set and $\mathcal{A}$ a **semiring** of subsets of X, that is, a family of subsets of X such that

> $\emptyset \in \mathcal{A}$,
>
> $E \cap F \in \mathcal{A}$ for all $E, F \in \mathcal{A}$,
>
> whenever $E, F \in \mathcal{A}$ there are disjoint $E_0, \dots, E_n \in \mathcal{A}$ such that $E \setminus F = E_0 \cup \ldots \cup E_n$.

Let $\lambda : \mathcal{A} \to [0, \infty]$ be a functional such that

> $\lambda\emptyset = 0$,
>
> $\lambda E = \sum_{i=0}^\infty \lambda E_i$ whenever $E \in \mathcal{A}$ and $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$ with union E.

Show that there is a measure μ on X extending λ. (*Hint*: use the method of 113Yi.)

115 Notes and comments In the notes to §114 I ran over the methods so far available to us for the construction of measure spaces. To the list there we can now add Lebesgue measure on $\mathbb{R}^r$.

If you look back at §114, you will see that I have deliberately copied the exposition there. I hope that this duplication will help you to see the essential elements of the method, which are three: a primitive concept of volume (114A/115A); countable subadditivity (114B/115B); and measurability of building blocks (114F/115F).

Concerning the 'primitive concept of volume' there is not much to be said. The ideas of length of an interval, area of a rectangle and volume of a cuboid go back to the beginning of mathematics. I use 'half-open intervals', as defined in 114Aa/115Ab, for purely technical reasons, because they fit together neatly (see 115Xa and 115Ye); if we started with 'open' or 'closed' intervals the method would still work. One thing is perhaps worth mentioning: the blocks I use are all upright, with edges parallel to the coordinate axes. It is in fact a non-trivial exercise to prove that a block in any other orientation has the right Lebesgue measure, and I delay this until Chapter 26. For the moment we are looking for the shortest safe path to a precise definition, and the fact that rotating a set doesn't change its Lebesgue measure will have to wait.

The big step is 'countable subadditivity': the fact that if one block is covered by a sequence of other blocks, its volume is less than or equal to the sum of theirs. This is surely necessary if blocks are to be measurable with the right measures, by 112Cd. (What is remarkable is that it is so nearly sufficient.) Here we have some work to do, and in the r-dimensional case there is a substantial hill to climb. You can do the climb in two stages if you look up the Heine-Borel theorem (115Ya); but as I try to explain in the remarks following 115B, I do not think that this route avoids any of the real difficulties.

The third thing we must check is that blocks are measurable in the technical sense described by Carathéodory's theorem. This is because they are obtainable by the operations of intersection and union and complementation from half-spaces, and half-spaces are measurable for very straightforward reasons (114F/115F). Now we are well away, and I do very little more, only checking that open sets, and therefore Borel sets, are measurable, and that closed and open intervals have the right measures (114G/115G). Some more properties of Lebesgue measure can be found in §134. But every volume, if not quite every chapter, of this treatise will introduce further features of this extraordinary construction.

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Concordance for Chapter 11

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

112E-112F Image measures These paragraphs, referred to in the 2001 and 2003 editions of Volume 2, and the 2003 and 2006 editions of Volume 4, have been moved to 234C-234D in Volume 2.

112Ya Sums of measures This material, referred to in the 2001 and 2003 editions of Volume 2, has been moved to 234G in Volume 2.

Version of 31.5.03

References for Volume 1

In addition to those (very few) works which I have mentioned in the course of this volume, I list some of the books from which I myself learnt measure theory, as a mark of grateful respect, and to give you an opportunity to sample alternative approaches.

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